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ELEMENTARY ALGEBRA

ELEMENTARY ALGEBRA

BY

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PREFACE

In many recent text-books of Elementary Mathematics, in which there has been a considerable departure from the traditional treatment of the subject, it is evident that the authors have been faced by a serious dilemma. If on the one hand they omit a detailed exposition of the ideas underlying their special mode of treatment, their plan is unintelligible or at least obscure; if on the other hand they insert all necessary explanations in the text, the book becomes unsuitable for the use of pupils.

In the early stages of such a subject as Algebra, teaching must be almost entirely of an oral character. Discussions in the text are of little or no value to the ordinary boy. All that he requires to see in print are a few simple cautions, certain definite instructions and illustrative examples: these are sufficient to enable him to use his text-book for reference and revision purposes. The moral effect produced by a book whose size has been diminished through absence of "talk" is of definite value; and discussions hinder the pupil in achieving the very necessary object of learning his way about the book.

The authors have accordingly placed in an Introduction, which is included only in the Teacher's Edition, all explanations which they consider unsuitable for the pupil. They are fully conscious that these introductory remarks may be considered too diffuse; but they have preferred to risk this charge rather than one of obscurity to which a briefer exposition might have laid them open.

In selecting their material the authors have chiefly kept in mind the needs of the non-specialist, for whom it is contended the real function of Algebra is to furnish an equipment which will enable anyone of average ability before leaving school to become acquainted with the fundamental ideas of Trigonometry, Mechanics and the Calculus. If this equipment is made too elaborate, little time remains for dealing with subsequent applications and the pupil is left with an impression of Algebra as a useless "tyranny of x and y." Its essential elements and the ends it should serve have been the subject of many debates at the meetings of the *Mathematical Association*, and the authors have derived much assistance both from those discussions and the published Reports of the Association.

Chapters I.–X. (Part I.) contain what in their opinion is the most suitable normal school course as far as quadratic equations: Chapters XII.–XV. (Part II.) deal with indices, logarithms, ratio, variation and functionality: then follow four sections which are independent of one another, (*a*) Chapters XVI.–XVIII. on Calculus Ideas, (*b*) Chapter XIX. on Series, (*c*) Chapters XX., XXI. on Permutations, Combinations and the Binomial Theorem, (*d*) Chapters XXII., XXIII. on Empirical Formulae and Nomography: these can be taken in any order, and any one or more can be omitted without prejudice to the others.

For some pupils, however, a more elaborate equipment is required; the last chapter in each part (Chapter XI. in Part I. and Chapter XXIV. in Part II.) accordingly contains such further developments of the previous work as seem advisable for those who are in some degree specialists.

The inclusion of a chapter on Nomography may invite criticism; it is justified partly by its growing utility in practical mathematics, but also by the educational value which all such general methods possess. The authors have felt more hesitation in admitting a chapter on Series, a subject which often loses its value in an elementary course by being reduced to mere rule of thumb. It is hoped that the treatment adopted obviates or at least minimises this danger.

The revision papers (220 in number) are arranged to suit the variations of order in which different sections of the book may be taken.

Every effort has been made to introduce as much variety as possible into the examples and to infuse them with practical interest. Repetitions of special types may eventually induce some kind of facility in handling them, but this method dulls the mind: it is variety alone which makes the pupil "keep thinking."

As there is little explanatory matter and few definitions in the text, the Glossary has been made rather fuller than usual, and it is desirable that its use should be encouraged. Pages are numbered consecutively through the two Parts, but the diagrams in Parts I. and II. are numbered independently; where any reference is made to a diagram on a different page, the number of this page is also added; but such references are rare.

The death of Mr. G. W. Palmer shortly after the final preparation of Part I. for the press, but before any further progress had been made, necessitated new arrangements: and the authors of Part II. must bear the sole responsibility both for the choice of material in this Part and its manner of presentation.

Acknowledgment is due to The Controller of H.M. Stationery Office and to the Oxford and Cambridge Joint Board for kind permission to include questions set in recent examinations, and to Messrs. Macmillan and Co., who have sanctioned the use of their method of arrangement of logarithm tables. Special mention must also be made of Molesworth's *Pocket Hand-book for Engineers*, which contains in a concise form an admirable collection of formulae which has provided excellent material for examples. The authors also wish to take this opportunity of expressing their indebtedness to Mr. R. C. Fawdry, Head of the Military and Engineering Side, Clifton College; to Mr. D. B. McQuistan, Head of the Mathematical Department, Allan Glen's School; to Mr. A. E. Broomfield of Winchester College; and to Mr. H. K. Marsden of Eton College, whose criticisms have been most helpful.

C. V. D.
R. M. W.

April, 1921.

In response to requests for the provision of additional drill exercises in certain sections of Part I., there has now been added (after page 251) a selection from the Extra Practice Exercises in my *Practical School Algebra.* Reference to these additional exercises is made by a footnote on the pages concerned. The Extra Practice Exercises can also be used as a revision course.

C. V. D.

March 1928.

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INTRODUCTION

In this book Algebra is introduced as an extension of Arithmetic, and the subject has been developed in the early chapters with the idea of enabling the pupil to pass from one to the other as easily and naturally as possible. There is a wide gulf between the use of numbers and the use of letters. Until this has been bridged, the pupil is unable to attach any real meaning to his work. Initially, therefore, every effort must be concentrated on this object. Concrete examples must take the place of abstract conceptions. It is not enough to point out to the pupil, however often, that the letters he is using represent numbers : obviously this has to be done : but the work must be so arranged that in the early stages he is being continually forced to replace letters by numbers for himself. While using letters, he must be compelled by the nature of the work he is doing to "think numerically." Only by this means will the notation of Algebra become tangible and intelligible. The authors have tried to suggest various ways of carrying out this principle—in the first three chapters, the chief features of which are (i) the construction of formulae, (ii) the interpretation of formulae, (iii) the generalisation of arithmetical work. This course of action also serves the important purpose of enabling the pupil to form some idea of the object of Algebra. He is able to see for himself that the new notation is a kind of mathematical shorthand, in that it gives him a means of crystallising into a single statement an unlimited number of arithmetical results which obey a common law by an explicit statement of what that law is. Further, by continual comparison with the similar processes in Arithmetic, the elementary operations of Algebra become intelligible and reasonable.

The course of work indicated in the early chapters will be more effective for this purpose if, in the initial stages, the teaching is mainly oral. This is discussed more fully on pp. xiv and xv. It is assumed that the pupil has at this stage a thorough knowledge of the more elementary parts of Arithmetic and some acquaintance with the following further sections of the subject: (i) Areas and Volumes, (ii) Averages, (iii) Percentage and Simple Interest. He is also supposed to know that the adjacent angles formed by two straight lines are supplementary, and that the sum of the angles of a triangle is two right angles. This gives a wider range and greater variety of examples. But it is quite possible for anyone to use this book, though unacquainted, for example, with Percentage and Simple Interest; it is only necessary to omit one section of Chapter III. (which is not essential) and a few examples elsewhere.

In the early chapters the analysis is of the simplest kind possible: practice in the use of letters to represent number and in the generalisation of arithmetical processes is regarded as more important than dexterity in handling elaborate algebraical expressions. This does not mean that ability to perform the ordinary algebraic operations is regarded as unnecessary, but that in the early stages only such expressions as may reasonably be expected to occur in a natural manner should be handled. Attention is thereby focussed on the group of new ideas which accompany the introduction to analysis.

CHAPTER I.—THE USE OF LETTERS IN ALGEBRA.

The chief object of this chapter is to familiarise the pupil with the use of letters to represent numbers. It contains three sections.

I. Generalisation. A number of particular statements which obey a common law are taken, and letters are then used to formulate the general statement in which the various special cases are all included.

II. Notation. A short account is given of the notation of Algebra, in so far as it is required in the early stages.

III. Substitution. Particular statements are deduced from a given general statement by the substitution of definite numbers in the place of the letters which occur in it.

Throughout the early stages of the Algebra course, a great deal of *viva-voce* work is most desirable, and it is suggested that nearly the whole of this chapter should be taken orally. There is no attempt to represent such work by discussions in the text, but every endeavour has been made to select examples which are suitable for oral treatment, to vary their character, and reduce manipulation to a minimum. Thus, after a preliminary discussion on some such lines as the illustrative examples indicate, it is suggested that the examples in Exercise I. a. should be taken orally. In each question taken, the whole class should be told to write down the generalised answer; and the correct solution should then be explained immediately. An example, explanation included, need not occupy, as a rule, more than two minutes, and it will be found that the ground is covered very rapidly. It is inadvisable at this stage to set boys to work through the whole or part of an exercise by themselves: this tends to produce mechanical and unintelligent work. The constant intervention of the teacher, the immediate dealing with the various questions that arise, and the opportunity it affords of correcting mistakes and explaining difficulties on the spot more than compensate for the fact that the pace of the stronger has to be accommodated to that of the weaker. It will increase the interest of the subject if, to the examples given in the book, are added others of a topical character: *e.g.* find a formula giving the number of days to the end of the term from the n^{th} of June; it is t o'clock now, what would it be if there were no Daylight Saving Bill? Pupils should be encouraged to suggest and invent formulae for themselves.

In connection with Exercise I. a., a warning should be given that a statement is not necessarily true in general, merely because it is true in three or four special cases. For example, if we investigate the statement that the sum of the first $2n+2$ prime numbers, counting 1 as a prime number, where n is a positive integer, ends in either 1 or 9, we find that it is true for 4, 6, 8, 10, 12, 14, 16, 18, 20 primes, but it is not true for 22. It also follows that, even if a general statement is true, it has not been proved to be so, when it has only been verified in a few cases. This caution is one that requires frequent repetition throughout the elementary course.

There is a natural tendency on the part of most or all pupils to divide their mathematical work, at any rate mentally, into disconnected groups: and it is worth while devoting considerable time to breaking down the barriers they erect between the various subjects. Unless this is done, there arises, for example, an idea that there is one system of principles and rules for Arithmetic and a totally different set for Algebra. Merely to point out the real continuity subsisting between them is not enough: what is needed is the formation of a definite mental habit, and this can only come from a repeated insistence on the idea, illustrated again and again by examples. It is important therefore to remove at the outset the possibility of confusion due to notation, arising in the comparison of $23 = 2 \times 10 + 3$ and $2\frac{1}{4} = 2 + \frac{1}{4}$ with $ab = a \times b$. But when this has been done. every opportunity should be taken of comparing the analogous methods of expressing quantities in Arithmetic and Algebra. This will correct such mistakes as imagining that $2x^2$ means $(2x)^2$, or that the square of $3y$ is $3y^2$, or that the result of multiplying $a \times b$ by 4 is $4a \times 4b$, etc.

From the first the fact should be emphasised that letters are not used to represent numbers-of-things, but simply numbers. For example x may represent the *number* of pence that a boy has in his pocket, but not the sum of money itself. It needs constant attention to secure that, in written work, there is no confusion of this kind. It is, of course, a matter that becomes especially prominent in connection with the problem work of Chapter VI., but frequent occasions occur in the early symbolical work, when this point can be made.

In the first four chapters, the existence of negative numbers is ignored: in all the examples letters are used to represent positive numbers or zero. As a matter of fact, in the majority of cases they are taken to represent whole numbers. This has been done to avoid arithmetical difficulties, at a time when the pupil is fully occupied with new algebraical ideas. Fractional values should, however, be introduced, from time to time, in *viva-voce* work, to remove the impression that a letter always stands for a whole number.

As has been said above, the chief purpose of this chapter is to teach the pupil to think numerically, while working with letters. If he can attach a definite meaning to such statements as "the sum of the first x odd integers is x^2," the arguments he hears and the operations he is taught to perform cease to be mysterious abstractions and rather tedious conjuring tricks. He has, in fact, begun to see what it is all about. The reality of the transition from the one subject to the other can be brought home vividly by making members of the class supply simple examples for each other. One boy states any simple arithmetical question he can think of, such as "a golf ball costs two shillings, what do six golf balls cost?" Another boy then states it in a more general form, and a third boy may be able to generalise it still further. The mere fact that they build up for themselves the complete generalisation from a simple particular case means that they understand it, when it is obtained.

CHAPTERS II.—III.—GENERALISED ARITHMETIC.

The work which the pupil has done in Arithmetic is here repeated in some detail and extended to algebraical expressions. By this means the pupil is made to realise the connection between Arithmetic and Algebra, and he is less likely to fall into the error of imagining that rules and principles of the one are inapplicable to the other. The examples have been arranged with this end in view, but it is doubtful whether written work, on whatever plan, can be as effective as a *viva-voce* treatment.

It has been convenient to divide this work into two chapters :

Chapter II. Processes, *i.e.* The Four Rules, H.C.F., L.C.M., Fractions, etc.

Chapter III. Problems, *i.e.* applications of the rules and processes of Chapter II. to various kinds of problems, such as occur in connection with Unitary Method, Areas and Volumes, Percentages, etc. There is also a section on the further use of formulae.

CHAPTER II.

Some teachers may be surprised at the simplicity of the majority of the examples in this chapter, and it is the case that there are comparatively few to which the answer cannot be given without any intermediate work. The authors, however, believe that it is simplicity alone that enables the principles under consideration to be illustrated effectively. No apology is therefore made for including examples of so ordinary a character ; if the line of the argument has been made clear and meets with the teacher's approval, it is obvious that he can manufacture any number of suitable examples without the smallest difficulty. But in class work it often saves time and enables the pupils to follow the work more easily if the examples taken are printed in the book in front of them.

It will be noticed that frequently the working in the illustrative examples has been set out in much greater detail than would be expected or desired from the pupil in his own work (see, for instance, Example X. page 16). If, however, his work is intelligent and not merely mechanical he should be able—and indeed should be occasionally asked—to expand his work on the lines there indicated. He will probably not like doing it, but there are few more effective ways of testing his grasp of the subject.

The definitions of H.C.F. and L.C.M. on p. 16 will require some explanation. It should be pointed out that in Algebra we cannot say generally of two *independent* numbers a and b which is the greater in the same way that we can of two numbers in Arithmetic, for a and b may have many different values. Nor even, of two *related* numbers, such as a and a^2, can we say which is the greater. Later on we shall find that we cannot even say that $2a$ is greater than a, without some qualification.

But if we take any algebraical expressions, *e.g.* $12xy^2$ and $18xyz$ and write down their common factors, we shall find that there is one, of which all the others are factors. In this case it is $6xy$, and all the other common factors, x, $2x$, $6x$, y, $2y$, $3xy$, etc., are factors of $6xy$. This factor we agree to call the Highest Common Factor (or more shortly the H.C.F.) of $12xy^2$ and $18xyz$. It is not necessarily greater than the other common factors, when we give x and y numerical values. *E.g.* if $y=\frac{1}{2}$, $6x$ is greater than $6xy$.

Similarly, if we take any algebraical expressions and write down common multiples of them we shall find that there is one of which all the others are multiples. This we agree to call the Lowest Common Multiple of the given expression.

It should be pointed out that the algebraical and arithmetical H.C.F. (or L.C.M.) are not necessarily the same. For example, the algebraical H.C.F. of $2x$ and $4y$ is 2; but if x represents 15 and y represents 21, the arithmetical H.C.F. is not 2, but 6.

The only other section in the chapter which is likely to present difficulties is that on Brackets. Attention has been confined to such simple cases as may reasonably and naturally occur in the work that follows; and within this limited range every endeavour has been made to illustrate the advantages resulting from the employment of brackets and to encourage an intelligent use of them. Except in one exercise at the end of the chapter, no example contains more than one kind of bracket. The course suggested will be evident from the examples given; it falls into three parts: (i) the interpretation of expressions containing brackets, (ii) the use of brackets to express operations with numbers and quantities, (iii) the simplification of brackets.

Although it is an easy matter to teach the pupil the rules for the removal of brackets and to secure a fairly accurate application, it is very far from easy to make him understand the justification for these rules. If anything beyond mechanical accuracy is desired, there must be a large measure of oral instruction, and the rules must be illustrated by suitable examples. It is, therefore, suggested that the questions in Exercise II. 1. should be considered in careful detail: in this way instances will be taken of all the various cases which arise in the simplification of $a \pm (b \pm c)$. A discussion of each example in turn, supplemented by others the teacher may devise or the pupil suggest, will clear away any mystery that surrounds the rules which are afterwards enunciated.

CHAPTER III.

It is not necessary to work completely through this chapter before going on to the succeeding chapters. The amount that can be taken conveniently in a first course will depend on the pupil's knowledge of Arithmetic. The chapter contains examples which vary very much in difficulty, and some of them will be unsuitable at the stage of the work indicated by the position of the chapter; it is suggested consequently that parts of it should be taken at intervals, throughout the period of time allotted to the whole book (Part I.). Its object is partly to give further practice in the use of letters and partly to improve the pupil's ideas of arithmetical method by generalising the work. As an instance of this, take the example on p. 40: "If a train is travelling at the rate of a miles an hour, how long will it take to go x miles?"

It is important that the pupil should realise that the answer is $\frac{x}{a}$ hours in all cases; it is often necessary to correct the false impression that this is only true if x is greater than a; there is also the tendency to imagine that this result holds only if x and a are integers. Such special cases as "he travels half a mile in $\frac{\frac{1}{2}}{a}$ hours" should be mentioned. This work gives an opportunity for considering such questions as, (i) is xy greater than x? (ii) is $\frac{x}{y}$ less than x? (iii) is the square of x greater than x? etc., which smooth away kindred difficulties that have occurred in Arithmetic.

CHAPTER IV.—GRAPHICAL WORK.

The introduction of graphical work at this stage is not due to any belief that it is in itself of much value so early in the course but because, from this point onwards, it is useful in various places as a means of illustration. Much later on, in Part II., graphical methods will be employed to develop the idea of a function and to explain some of the properties of simple functions; but for the present the work is confined to such simple ideas as will be found helpful in illustrating the work of Part I. It is therefore necessary to deal only with the plotting of functions of a single variable.

The graphical representation of an algebraical relation between two variables, *e.g.* the representation of $2x+3y-1=0$ by a straight line, has been deliberately omitted, because it embodies an idea which is far too difficult for pupils at this stage. They can, of course, be taught to draw the graphs corresponding to linear relations between x and y, but unless a great deal of time is devoted to this work and the intellectual ability of the pupil is out of the ordinary, the process will be merely mechanical, and not, in any true sense of the word, educative. The authors do not believe that it is wise to trespass at this stage on the province of co-ordinate Geometry. This limitation involves the omission of the graphical solution of simultaneous equations. Those teachers who do not agree with this point of view will find examples that can be used for graphical solution in Exercise VII. a. 11-34, and Exercise X. f. For similar reasons, applications of graphical methods to Geometry, such as finding the distance between two points of given co-ordinates, the area of a triangle whose vertices are given, the equation of a line joining two given points, etc., have been omitted.

The first set of examples deals merely with the meaning of axes of reference and of co-ordinates and with the plotting of points. The necessary instruction should be given orally, and will naturally include the following considerations:

(i) The meaning of the terms employed.

(ii) The representation of numbers and quantities by lines of suitable length.

(iii) The representation of a pair of numbers or quantities by a point in the plane.

(iv) The difference between a variable and a function of that variable.

(v) The difference between diagrams containing only isolated points and those which involve an unlimited number of points and so lead to a continuous curve.

(vi) The clear indication on the figure of the axes and the scales that have been chosen.

(vii) The choice of a convenient scale. In this connection two things are important: (*a*) The scale must be a simple one, *i.e.* such that any number or quantity can be plotted with only the simplest calculation, *e.g.* if 3 large divisions represent 10 lb. it is not easy to represent 7 lb. or 13·8 or to read off the weight represented by any length chosen at random; but 3 large divisions = 1 hour is quite convenient. (*b*) To secure as great a degree of accuracy as possible, the scales should be so chosen that the graph extends across the greater part of the given piece of squared paper, if it can be done with a convenient scale. If OA and OB are the axes, it is not necessary to make the point O correspond to zero for either axis (see Example I. p. 56).

(viii) The importance of using decimals, instead of vulgar fractions, in any table of values.

(ix) The importance of neatness and accuracy.

In the second set of examples (Exercise IV. b.) groups of values are represented graphically. In this way the changes in a varying quantity may be seen more readily than by inspecting a list of numbers. For example, from the diagram on p. 56 the changes in the average annual price of a quarter of barley between 1900 and 1910 can be seen at a glance. In most of these examples we are plotting isolated points, and it is advisable to join the points by straight lines, but it is important to point out that these straight lines have no quantitative meaning.

The next set of examples (Exercise IV. c.) deals with the plotting of simple algebraical functions of a variable. The primary object of this is to show how such functions change as the variable changes. For example, on p. 58 our object is to show how $\frac{x}{2}+3$ changes as x changes; not how to represent graphically the relation $y=\frac{x}{2}+3$ or the equation $x-2y+6=0$.

Probably at a first reading it is advisable to take only the easier examples in Exercise IV. c., and to leave the remainder for a second course.

In tabulating values of such functions as $3x^2-7x+2$, the pupil should be required to write down all the numerical work in tabular form, and should not be allowed to write miscellaneous computations on another sheet or at the side. It should be pointed out that the work can be easily checked when arranged in this way.

The tendency to imagine that $f(0)$ is always equal to 0 needs a special word of correction.

At a later stage, after the pupils have had considerable practice in drawing graphs of simple algebraical functions, their attention should be directed to some of their more elementary properties, such as the following:

(1) The graph of ax, where a is a number and x the variable, is a straight line through the origin (0, 0).

(2) The graph of $ax+b$, when a and b are numbers and x the variable, is a straight line.

(3) The graph of ax^2 is a curve symmetrical about the axis OB; it is concave upwards, and its lowest point is the origin (0, 0) if a is positive; it is convex upwards, and its highest point is the origin if a is negative.

CHAPTER V.—NEGATIVE NUMBERS.

Until the beginning of the seventeenth century mathematicians dealt exclusively with positive quantities. This historical fact is of itself sufficient evidence of the intrinsic difficulties involved in the idea of a negative number; and it is not surprising, therefore, that at the present time there exists much divergence of opinion as to the way in which this subject should be handled in the elementary course. On the one hand, some teachers believe in a thorough and elaborate investigation of the laws regulating the use of negative numbers. On the other hand, many are content with making their pupils learn by heart a few rules dealing with the combination of such numbers. The authors do not consider the first method practicable or justifiable at this stage, but they are reluctant to follow the line of least resistance and adopt the second. The course adopted in the present chapter is consequently a compromise, and, as such, is liable to be attacked from both sides. It will probably not be acceptable to the rigorous school of thought, and it may appear unduly protracted to those who believe in covering as much ground as possible in the shortest time. Nor is it maintained that any startling results can be expected. The main purpose is to enable the pupil to judge of the "reasonableness" of the rules which after a time will become mechanical.

One of the first steps is to explain why negative numbers are employed. A good example of this is furnished by the history of the thermometer. Fahrenheit took as the zero of his scale the lowest temperature with which he was acquainted. When lower temperatures than 0° F. were obtained, the introduction of negative numbers on the scale avoided the necessity of a re-arrangement. But Celsius, who invented the Centigrade scale, made use of negative numbers from the first, and so obtained a more convenient system of measurement.

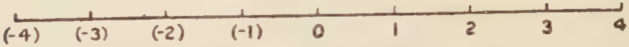
The advantage of using negative numbers in Algebra is seen by considering such an expression as $(a-b)-(a-c)$. Without them, this expression is meaningless, unless $a>b$ and $a>c$ and $(a-b)>(a-c)$. Clearly it would be extremely laborious to make such a series of restrictions of this nature as would be required in any piece of algebraical manipulation. And in any protracted analysis, the accumulation of such restrictions would, for all practical purposes, make progress impossible. It would lead to the consideration of endless special cases, and would permit of very few general statements being made. We could not say, for example, that if a man climbs a hill x feet high and descends into a valley y feet deep, he is $x-y$ feet above his starting-point; nor that an equation of the first degree must always have one root, since it would not be true of such an equation as $x+1=0$.

One of the greatest difficulties the pupil experiences is that of distinguishing between the number " -3 " and the operation "subtract 3." This is no doubt largely due to his being accustomed to use the same symbol to represent these two distinct ideas. At the risk of appearing to adopt an artificial treatment and of seeming to introduce unnecessary difficulties, the authors have come to the conclusion that the best way of illustrating the difference is to employ at first separate symbols. The new numbers are therefore represented, in descending order, by (-1), (-2), (-3), etc., with intermediate numbers such as $(-1{\cdot}2)$, $(-2\frac{3}{4})$, etc. Later on this special notation will be discarded.

It is well to point out that new symbols might have been used, entirely different from those for positive numbers; compare, for example, the use of *t* and *e* for the digits ten and eleven in the duodecimal system of numeration. But all notation should, as a matter of convenience, be as suggestive as possible in its form, in order to avoid causing any unnecessary burden to the memory.

We have now an extended scale of numbers, such as is indicated by a thermometer scale, which may be imagined to extend without limit in either direction. This provides us with a series of symbols arranged in a definite order, and this fact is referred to as the ordinal property which is possessed by all numbers, positive or negative.

Scale of centimetres.



We say that the symbol a represents a *greater* number than the symbol b, if it occurs in this scale to the right of b, while it represents a smaller number than b if it occurs to the left of b. Thus

$$4 \text{ is greater than } (-1) \quad \text{or} \quad 4 > (-1),$$
$$(-3) \text{ is less than } (-1) \quad \text{or} \quad (-3) < (-1).$$

So far we have given a meaning to 0, (-1), (-2), $(-1{\cdot}3)$, etc.; it now remains to discuss the operations of addition, multiplication, etc., for such numbers. In this connection such statements as $a-(-b)=a+b$ or $(-a)\times(-b)=ab$ should be regarded as extensions of the meaning of the operations of subtraction and multiplication rather than as enunciations of new truths. In fact what we have to do is to find meanings for $+$, $-$, $\times$, $\div$ when we are dealing with negative numbers.

The following points may properly form subjects of discussion:

(i) If you invent new symbols, you can lay down any laws you like (provided that they are consistent) which those symbols are to obey.

(ii) Some systems of laws are more convenient than others.

(iii) In the particular matter in hand, it is obviously desirable to select, if possible, a system which will enable us to operate with the symbols a, b, c, etc., in such a manner that it is immaterial whether a, b, c, etc., represent positive or negative numbers. In this way we avoid special cases and secure generality. This last consideration is most important.

Addition. *Case* I. $(-a)+b$. In this and all the following cases a and b represent positive numbers. In dealing with positive numbers the operation $+b$, *i.e.* "add b," applied to a number means that we count up b from that number or move b units to the right on the scale above. This is consequently, in accordance with the principles in (iii) above, our statement of what we mean by addition *or* our definition of addition, whether the number to which we are adding b is positive or zero or negative. Thus $(-3)+4$ means that we start from the number (-3) and count up 4, or start from (-3) on the scale and move 4 units to the right.

It can be shown that this agrees with general practice when dealing with negative numbers on the thermometer scale, a contour map, etc.

Case II. $a+(-b)$. It is important to emphasise the fact that $+(-b)$ is an operation which so far is quite meaningless. As before, we shall use the principle in (iii), p. xxiv, to give it a meaning. Now, so long as m and n represent positive numbers, $m+n=n+m$. We shall assume that this is true for all values of m and n, positive or negative. Thus $a+(-b)=(-b)+a$, and the meaning of this is given in Case I. Of course we might have taken some other property of positive numbers instead of $a+b=b+a$ and extended it to include negative numbers. For example, we might have taken $a+(-b)+c=a+[(-b)+c]=[a+(-b)]+c$.

It will be found that the meaning that we have found for the operation $+(-b)$ gives $a+(-b)$ the same value as if we were to start from the number a and count b down, or on the scale start from a and move b units to the *left*. For example,

$$4+(-7)=(-7)+4=(-3).$$

And we obtain the same result (-3) if we start from 4 and move 7 units to the left.

Case III. $(-a)+(-b)$. We shall assume that, as in Case II., the effect of the operation $+(-b)$ on a number can be represented by starting from that number and moving b units to the left. This is again in accordance with the principle in (iii), p. xxiv. We can now say that $(-4)+(-3)=(-7)$, etc.

We have now settled the meaning of the operation of addition whether we are dealing with positive or negative numbers. It should be pointed out that we have not proved that, with these meanings, negative numbers satisfy the fundamental laws of Algebra that have been proved for positive numbers. For example, in Case II. we know that $a+b=b+a$, but we have not proved it in Case III. Of course we can easily do so for any given case; *e.g.* $(-4)+(-3)$ and $(-3)+(-4)$ both give (-7); or more generally, $(-a)+(-b)$ and $(-b)+(-a)$ both give $-(a+b)$. For further remarks on this see p. xxvii.

Subtraction. Subtraction can be made to depend on addition, It will be assumed that the symbol $a-b$, which represents the result of the operation of subtracting b from a, is equal to the number which, when added to b, gives a. For example, to find the value of $3-4$, we find the number which, added to 4, gives 3. Thus we see that the operation -4, or "subtract 4," gives the same result as the operation $+(-4)$, or "add (-4)." Again, to find the value of $3-(-5)$, we find the number which, added to (-5), gives 3. Then we see that the operation $-(-5)$ gives the same result as $+5$.

To put shortly the results that we have obtained :

(i) $+b$ means "move b units to the right on the scale."
(ii) $+(-b)$ „ „ „ left „ „
(iii) $-b$ „ „ „ „ „ „
(iv) $-(-b)$ „ „ „ right „ „

Multiplication. *Case* I. $(-a)\times b$. This may be made to depend on addition. *E.g.*

$$(-5)\times 4=(-5)+(-5)+(-5)+(-5)=(-20).$$

Generally $(-a)\times b=(-ab)$.

Case II. $a\times(-b)$. We assume that

$$m\times n=n\times m$$

for all values of m and n.

Thus $\quad a\times(-b)=(-b)\times a=(-ba) \quad \text{or} \quad (-ab).$

Case III. $(-a)\times(-b)$. We assume that

$$m(n+p)=mn+mp$$

for all values of m, n, p. For example, to find the value of $(-4)\times(-5)$, we put $m=(-4)$, $n=6$, $p=(-5)$.

$$(-4)\,[6+(-5)]=(-4)\times 6+(-4)\times(-5)=(-24)+(-4)\times(-5).$$

But $\quad (-4)\,[6+(-5)]=(-4)\times 1=(-4)=(-24)+20.$

Comparing the two results, we see that $(-4)\times(-5)=20$.

Generally $\quad (-a)\times(-b)=ab.$

We could also have used this method in Case II.

Division. The operation of division is made to depend on that of multiplication by the following definition :

The symbol $a\div b$ represents the number which, when multiplied by b, gives a.

This immediately gives

$$\text{(i)}\ (-20)\div 4=(-5),\quad \text{(ii)}\ 20\div(-4)=(-5),$$
$$\text{(iii)}\ (-20)\div(-4)=5.$$

The Symbol 0. So far nothing has been said as to operations involving the symbol 0.

From the scale of numbers we have

$$a-a=0\,;\quad a+(-a)=0.$$

And, by putting $c=(-b)$ in $(b+c)\,a=a(b+c)=ab+ac$, we get

$$0\times a=a\times 0=0.$$

Also, making division depend on multiplication, it follows from this that $0\div a=0$, if a is not equal to 0.

The operation $\div 0$ is excluded from our consideration, and no meaning is attached to it.

We have extended the meanings of the fundamental operations so as to include negative numbers. In each case we have obtained a meaning by extending either some fundamental law of the Algebra of positive numbers, *e.g.* $a+b=b+a$, or some previous interpretation of an operation, *e.g.* $+5$, which means count up 5 when applied to positive numbers. But we have not proved that all the laws of the Algebra of positive numbers, *e.g.*

$$(a+b)+c=a+(b+c) \text{ or } a\times b=b\times a, \text{ etc.,}$$

are now true for all numbers. It may be worth while doing this in some special cases. But probably the most convincing test is that of experience. It will be found that, in the work which follows, the results are consistent, and when this Algebra of negative numbers is applied to practical affairs, it leads to results which agree with those obtained by other methods.

Of course we might have been content with merely stating the extended meanings of $+$, $-$, $\times$, $\div$, etc., and then appealing to experience, in this way saving the work of pp. 69-77, but this would probably have appeared a very arbitrary proceeding to the pupil, who is unlikely to have any idea of the reasons which led to these particular meanings being adopted.

It is suggested that the sort of discussion indicated in these pages should be given frequently in small doses. It could be begun before the pupils have finished Chapter III. and continued while they are doing Chapter IV. It must obviously be left to each teacher, should he follow the line of thought here indicated, to decide, according to the ability of his class, at what point it is best (i) to drop the bracket notation $(-a)$ for negative numbers, and (ii) to permit the use of *rules* for operations with negative numbers. This is done in the text on pp. 81 and 82.

It seemed worth while to introduce a section on graphical work, partly as an exercise in work with negative numbers, partly as giving a practical application of the extension of the scale of numbers below zero, but partly also to convince pupils diagrammatically of the continuity of results in the extended scale. In the example given, p. 78, by plotting x for positive numbers a portion of a straight line is obtained, and it is shown that, corresponding to negative values of x, we get further portions of the same straight line.

CHAPTER VI.—PROBLEMS AND EQUATIONS.

As the solution of problems by means of algebraical equations usually presents considerable difficulties to beginners, an attempt has been made to sub-divide the work and, as far as possible, to face one difficulty at a time.

The following order has been adopted :

(i) The expression of a statement connecting *numbers* by means of an equation, without solving the equation.

(ii) Similar work with *quantities*, instead of *numbers*.

(iii) The solution of simple equations, firstly by means of the four axioms, and afterwards by the use of rules.

(iv) Problems relating to numbers.

(v) Problems relating to quantities.

It will be noticed that in the text the attention of the reader is continually drawn to the importance of the correct use of letters. The natural tendency to unintelligent working generally causes a dangerous unwillingness in the student to state explicitly what his unknown really represents. The confusion between numbers and quantities must be removed at the earliest stage possible : it is mainly due to a form of mental laziness. This is also, in part, the reason why at first the student will appear disinclined to adopt the "writing out of the problem" method suggested on pp. 102, 103, the chief purpose of which is to make him grasp clearly the data of the question. He should be made to work on these lines, in some very easy examples at least, even where he can state the equation immediately, because if he does not acquire the habit in simple cases, he will find it very difficult to adopt in harder problems.

Equations should at first be solved step by step with reference to the fundamental axioms. It is of little value for the student merely to quote the axiom by number or in any other way, but he should write down in his own words the operation he is performing, *e.g.* "add 3 to each side," or "divide each side by 5." The object of this is to ensure that he understands what is really happening : unless he does this he is very liable to make such mistakes as $ax=b$; $\therefore\ x=b-a$. At what stage the rules, given on p. 95, should be introduced is, of course, a matter for the individual judgment of the teacher.

In connection with problems, there is great difficulty in providing good examples in sufficient quantity. Some of the examples one sees are artificial and fanciful. Take the following (which has a classical sanction): "Of a swarm of bees clustered on a tree, the square root of half their number flew away. Eight-ninths of the original number then departed, leaving but two behind. How many were there at first?" Others, such as the following, seem a useless form of breakfast-table conundrum: "A woman, selling apples, sells half her stock and one more to A, one-third of the remainder and two more to B; one-third of what then remained and three more to C: 25 are left. How many had she originally?" Probably the inclusion of some artificial examples is inevitable, at least in the early stages, if a sufficient number of an elementary kind is to be provided; but it is curious, if this subject is as important as it is generally considered, that there should be such a poor supply of natural applications.

Literal equations have been postponed to Chapter IX., but it may be thought desirable to include some of the simplest type in the work of this chapter. Such examples will be found on p. 145.

CHAPTER VII.—SIMULTANEOUS EQUATIONS.

The following is suggested as one method of introducing Simultaneous Equations.

Consider the problem, "Find two numbers whose sum is 32." This gives the equation $x+y=32$, but it is evident that the problem is indeterminate. More information is needed, so we add "one of which is greater than the other by 6." This gives a second equation $x-y=6$.

The method of elimination can then be explained, by means of which, from two equations containing x and y, we obtain one equation containing, for example, x; in this way the value of x can be found, and so the value of y.

It is evident, from the method of solution, that we can as a rule determine two unknown numbers or quantities if we have two statements, each of which can be expressed as an equation of the form $ax+by=c$.

Later on it will be shown that, from three such equations, we can as a rule obtain three unknown numbers or quantities.

The examples on pp. 111, 112, are written as a pupil might be expected to write them after he has had some experience, and no detailed explanation is given. But at first it is necessary that he should be able to explain, at any rate orally, what is meant by such operations as adding the equations

$$10x+12y=32 \quad \text{and} \quad 9x-12y=6.$$

So also care is necessary in clearing an equation of fractions.

One or two hints may be useful to the pupils.

(i) After one unknown number has been found, the other may be found by substituting in either of the given equations or in any equation that has been deduced from them in the course of the work. The easiest should be chosen.

(ii) The equation selected for checking should not be the same as that used for substitution, nor directly connected with it. *E.g.* in Example II. p. 112, y is found by substituting in $10x-y=20\frac{5}{6}$. The equation $2x-\frac{y}{5}=4\frac{1}{6}$ should not be chosen for verification, as it is really the same.

(iii) In equations with fractional coefficients it is sufficient to clear the x and y terms of fractions. The constant terms may remain as fractions whether they are vulgar or decimal fractions.

The supply of good problems to be worked by simultaneous equations is even poorer than in the case of simple equations, as many of the former can be worked as well or even better with only one unknown number.

CHAPTER VIII.—FACTORS AND MULTIPLES OF MORE THAN ONE TERM.

So far, multiplication and division have been by single-term expressions. In this chapter they are extended to expressions of more than one term. As a rule, such work is divided into sections which are kept separate: one on long multiplication and division, one on factorisation, one on H.C.F., one on L.C.M., one on simplification of fractions (addition and subtraction), one on simplification of fractions (multiplication and division). The danger of this plan is the tendency it produces to render the work mechanical and imitative. The pupil may work through a dozen or more examples, all of similar character, with success, and so on through the various types; but a month later he will have forgotten all about them, and if faced with a miscellaneous exercise will break down badly, because his mind has not really been trained by the work he has done. The writers have endeavoured to obviate this by presenting each relation in as many ways as possible, and by using it for as many purposes as possible: *e.g.* the relation $a^2-b^2=(a+b)(a-b)$ gives the product of $a+b$ and $a-b$, the result of dividing a^2-b^2 by $a-b$, the factors of a^2-b^2, etc. It can then be used for multiplication, division, H.C.F., L.C.M., simplification of fractions, solution of equations, etc. All these applications are put together in one section of the chapter. Another point to which importance is attached is the close connection between the simplification of fractions and the solution of fractional equations. It is hoped that the emphasis laid on the close relation between Arithmetic and Algebra is a partial cure for the widespread disease of dropping denominators when simplifying fractions. Confusion of mind is liable to occur when the pupil sees the denominators disappearing in the solution of fractional equations. This source of confusion must be faced boldly: the two processes should go on side by side, and repeated explanations are needed to enable him to discriminate between the two operations and to understand the justification of what is done. Such work as

$$\frac{1}{x+1}+\frac{1}{x+4}=\frac{2}{x+2};$$

$$\therefore \frac{(x+4)(x+2)+(x+1)(x+2)=2(x+1)(x+4)}{(x+1)(x+4)(x+2)}$$

should never be allowed.

The only safe course is to make quite sure that the pupil understands what he is doing, and is not merely working by rule.

It may be said that such a scheme as has been adopted makes things more difficult for the pupil; and in one sense it is true, because it demands from him continuous mental labour. He is not working all the time by type, and the variation in the form of the question will compel him to use his brain to a greater degree; but it is this very mental exercise which will impress his work on him and will enable him to retain and reproduce what he has been doing after an interval which would obliterate the effect of merely mechanical processes. It makes a greater tax on his mind, compels him to reason and discriminate, and for that very reason leads to intelligent working.

CHAPTER IX.—LITERAL EQUATIONS. FORMULAE (CHANGE OF SUBJECT). IDENTITIES.

Literal equations are introduced by problems, parallel problems being given (see p. 144) which lead, one to a numerical, the other to a literal equation. Only those of a simple kind are given in this chapter, others (including simultaneous equations) will be found in Chapter XI. Simple examples furnish good opportunities of testing whether the pupil understands the process of solution of an equation, and on this account some of the examples may be taken concurrently with Chapter VI. Their chief value, however, lies in their connection with formulae, and the section on "change of subject" (see p. 147) should be taken carefully.

Only quite simple identities are considered in this chapter. The immediate object is merely to give the average boy an idea of the difference between an identity and an equation. More difficult examples, which tend to develop the sense of algebraic form, will be found in Chapter XI.

CHAPTER X.—PROBLEMS. QUADRATICS AND SIMULTANEOUS QUADRATICS.

In the solution of quadratics, it is advisable to insist on the pupil, throughout the first course, inserting all the steps in the argument. Failure to do this is responsible for a very common mistake. For example, in solving $x^2+4x-21=0$, the steps are:

$$(x-3)(x+7)=0;$$
$$\therefore x-3=0 \quad \text{or} \quad x+7=0; \quad \leftarrow$$
$$\therefore x=3 \quad \text{or} \quad -7.$$

The pupil is very likely to omit the step marked with an arrow, and if he does this he will soon start to write down, $\therefore x=-3$ or 7. And he will tend to act on an unintelligent rule, without being conscious of the reasoning which underlies his work.

It will be noted that the general method of solution is based on factors rather than the extraction of the square root of each side of the equation. It may be true that this is a slightly more difficult course, but firstly, it is continuous with the method of solution when the roots are rational: although a rather harder application of that method, because the factors cannot be guessed at sight. Secondly, it shows why there are two roots. Thirdly, it enables the pupil to avoid the mistake of obtaining only one root, which is likely to happen, owing to his omitting the ± sign when he takes the square root. The fact that the working is made slightly longer is comparatively unimportant, as eventually the formula would naturally be always employed.

In the solution of simultaneous quadratics (one being linear) it has been left to the teacher to point out the wisdom of considering, before substitution, whether it is better to express x in terms of y, or y in terms of x. It is worth while taking a few *viva-voce* questions on this point, without actually going through the working, so as to make it clear. For example, which is the better in the following cases and why?

$$\left.\begin{aligned} x-2y&=1,\\ x^2+3y^2&=7\,;\end{aligned}\right\} \qquad \left.\begin{aligned} 7x+2y&=10,\\ 3x^2-5y^2&=1\,;\end{aligned}\right\}$$

$$\left.\begin{aligned} 3x+y&=8.\\ 5xy-7y^2&=1\,;\end{aligned}\right\} \qquad \left.\begin{aligned} 3x-y&=5,\\ xy&=2.\end{aligned}\right\}$$

Examples, where both equations are quadratic, will be found in Chapter XI. The simple case given at the bottom of p. 165 is included here, because many problems can be solved either by a single quadratic equation or by simultaneous equations, and so equations of this type are likely to occur. See the train problem on p. 166.

The section on graphical work has been kept fairly short. Some of the topics, often discussed at this stage, are postponed till much later, when properties of simple functions of one variable will be considered more fully.

CHAPTER XI.—FURTHER DEVELOPMENTS.

The principles which have guided the choice of material for this chapter and the use to which it is intended to be put have been stated on p. vi.

REVISION PAPERS.

The later papers of set A are more difficult than the early ones, and are more suitable for revision at a later stage. The same thing is true of the later papers of sets B and C. Such papers are marked with an asterisk.

PART II.

CHAPTER XII.

Indices. The laws for positive integral indices are as follows :

If m, n are positive integers,

$$\text{(i)}\ a^m \times a^n = a^{m+n}; \qquad \text{(ii)}\ a^m \div a^n = a^{m-n}, \quad \text{if } m > n,$$

$$= \frac{1}{a^{n-m}}, \quad \text{if } m < n;$$

$$\text{(iii)}\ (a^m)^n = a^{mn}.$$

The proofs of these laws follow immediately from the definition of the symbol a^m,

$$a^m = a \times a \times a \times a \times \ldots \text{ to } m \text{ factors.}$$

Exercise XII. a. illustrates the use of these laws, and is intended to show the reader how they are proved. Many numerical examples are usually required to clear away the haziness of idea most pupils have about indices, and in particular about (iii) ; and it is essential this should be done before proceeding to the interpretation of fractional and negative indices. It is hopeless to attempt to tackle these, until the pupil uses his reason and does not merely work by rule in his handling of positive integral indices. But when this stage is reached, there are few who, although they may never have seen or heard of fractional or negative indices, will not answer successfully such series of questions as :

1. What is the square root of (i) x^6 ; (ii) x^8 ; (iii) x^7 ?
2. If $x^5 \div x^8 = x^n$, what is n ?
3. If $x^7 \div x^7 = x^n$, what is n ? and so on.

The examples in Exercise XII. b., which to a large extent should be taken orally, are intended to lead the pupil to make his own interpretations. The conclusions are summarised at the end of the exercise, but some teachers will no doubt prefer to ask for formal interpretations while the working of the exercise is in progress.

It is worth noting that most boys are slow to realise that $\sqrt[n]{a^m}$ and $(\sqrt[n]{a})^m$ are identical, or to see the reason for it, or to make use of it when evaluating such expressions as $\sqrt{4^5}$.

Exercise XII. c. may well be postponed to a second reading.

Irrational numbers. The *theory* of irrational number is outside the scope of this book: for that, the reader is referred to Nunn's *Algebra*, Part II., page 13, or Hardy's *Pure Mathematics*, Chapter I. The present section is purely practical, and aims at giving facility in handling irrational expressions by examples on the statements

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}, \quad (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b.$$

The name "surd" is not used.

The graphical section which concludes the chapter is the natural link between indices and logarithms.

CHAPTER XIII.—LOGARITHMS AND SLIDE RULE.

Far more examples are given in the text than will be required by any single student, the object being to enable each teacher to make the selection he prefers. The graphical section should be taken orally.

No explanation is given in the text of the use of the slide rule, as oral instruction is almost essential. It is suggested that this should start as follows:

Draw a line 5 inches long, preferably on a strip of cardboard, and call it A_1A_{10}. Mark points A_2, A_3, A_4, ... A_9 on it such that

(i) $A_1A_2 = 5\,(\log 2 - \log 1)$ inches $= 5 \log 2$ inches
$= 5 \times 0{\cdot}301'' = 1{\cdot}505''$.

(ii) $A_1A_3 = 5\,(\log 3 - \log 1)$ inches $= 5 \times 0{\cdot}477'' = 2{\cdot}385''$.

(iii) $A_1A_4 = 5\,(\log 4 - \log 1) = 5 \times 0{\cdot}602 = 3{\cdot}010''$,

and so on.

Thus $A_1A_7 = 5 \log 7 = 5 \times 0{\cdot}845 = 4{\cdot}225''$.

A part of this line is shown in Fig. 1, where the graduations are represented by 1, 2, 3, ... instead of A_1, A_2, A_3, ...

The line can now be subdivided by the same method.

To obtain the graduation 1·5 we have

$$5\,(\log 1\cdot5 - \log 1) = 5 \times 0\cdot176 = 0\cdot880'',$$

and so on.

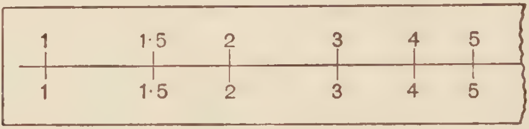


FIG. 1.

Now cut the cardboard along the straight line and you have the elements of a rough slide rule.

It is so graduated that the distance of any graduation n from the graduation 1 is proportional to $\log n$. Slide the lower half along until "1" on it comes below "2" on the upper half, then you will see that "3" on the lower half comes below 2×3 or 6 on the upper half. This is because

$$\log 2 + \log 3 = \log 6.$$

By this method any two numbers can be multiplied together: and by reversing the process we can find the quotient of any two numbers.

CHAPTER XIV.—RATIO AND PROPORTION.

Definition. If a, b are two numbers such that, if b is divided into an integral number q equal parts, a contains exactly p of these parts, a and b are said to be in the ratio $p : q$.

Or, If integers p, q exist such that $\frac{a}{b} = \frac{p}{q}$, then the ratio of a to b is said to be equal to the ratio of p to q.

For example, if $a = 2\frac{1}{3} = \frac{7}{3}$ and $b = 2\frac{1}{4} = \frac{9}{4}$, b can be divided into 27 equal parts, each equal to $\frac{1}{12}$, since $b = \frac{9}{4} = \frac{27}{12}$.

a can be divided into 28 equal parts, each equal to $\frac{1}{12}$, since $a = \frac{7}{3} = \frac{28}{12}$;

$$\therefore\ a : b = 28 : 27.$$

This process amounts to finding a common sub-multiple of the two numbers.

Next consider a square of side 3 inches; the length of its diagonal is $\sqrt{18}$ inches.

In this case it is impossible to find an integer q such that if the diagonal is divided into q equal parts, the side will contain an integral number of these parts.

Such related numbers as 3 and $\sqrt{18}$ are said to be *incommensurable.* The term 'ratio' can only be applied to such numbers if the definition given above is extended in scope. The necessary discussion is regarded as beyond the limits of this volume.

In addition to ratios of numbers, we may have ratios of *similar quantities*, *e.g.* the ratio of the populations of two towns or of the rents of two houses, etc. But it is meaningless to compare different kinds of magnitudes with each other, *e.g.* 5 inches with 2 hours.

The statement that the ratio of the populations of two towns X, Y is $a : b$ may be expressed in any of the following ways:

(i) The number of people in $X = \frac{a}{b} \times$ the number in Y.

(ii) $\dfrac{\text{The number of people in } X}{\text{The number of people in } Y} = \dfrac{a}{b}.$

(iii) If the people in X are arranged in groups of a and the people in Y are arranged in groups of b, there will be the same number of such groups in X as in Y.

(iv) If the population of X is az, the population of Y is bz.

It has been said that the uneducated mind compares two numbers or quantities by noting their difference, the educated mind by noting their ratio. Ratio provides the best test of comparison in general, because the absolute value of the difference shows nothing of its importance relative to the quantities concerned. A difference between the heights of two men of $\frac{1}{8}$ inch is considered trifling in our world; in Lilliput it would be noteworthy. Again, a rise of income of £1 a week is a big rise to a labourer earning £2 a week, but is insignificant to the millionaire.

Ratios appear in Mathematics in a variety of forms. Percentage is a special form of ratio; specific gravity, sines and cosines, coefficients of friction, of expansion, of elasticity, etc., are all examples of the use of ratio and furnish material for oral instruction.

Variation. The idea of variation is best explained by taking orally a number of simple examples, such as

(i) In uniform motion, distance $\propto$ time.

(ii) The number of Kms. between any two places $\propto$ the number of miles between those places.

(iii) The length of a shadow at any given moment $\propto$ the height of the object.

(iv) For substances of equal weight, the specific gravity varies inversely as the volume.

(v) For spheres of the same material, the weight varies as the cube of the diameter.

Oral questions should be taken of such types as the following (expressed as concrete examples):

(i) If $A \propto B$, what is the effect on A of doubling B or on B of doubling A?

(ii) If $A \propto \frac{1}{B}$, what happens to A if B is halved?

(iii) If $A \propto B^2$, in what ratio is A increased if B is increased in the ratio 3 : 2?

A boy cannot answer such questions unless he understands what he is talking about.

Graphical work should be taken side by side with each form of variation, and attention should be directed to the graphical tests for variation. Thus, to test whether A varies inversely as B, the values of A should be plotted against the corresponding values of $\frac{1}{B}$.

It helps also to clear up difficulties if pupils are made to examine or construct tables which follow a variation law, and some examples of functionality which are not variation laws should be taken to establish the distinction.

The obvious danger is unintelligent working. A boy easily acquires the habit of saying that if $y \propto x^3$, then $y = kx^3$, without thinking what it means. It is a much harder matter to make him realise that this variation-relation really implies that if x is altered in any ratio, then y is altered in the cube of that ratio. But his power to do this is the real measure of his understanding. Mere juggling with k is of little value.

It is left to the teacher to discuss and expand the statements given in the summary of results: this will naturally be done during the course of the exercises which precede it.

The discussion in the text on joint variation is rather fuller than usual, in order to indicate one way of introducing this idea. It is useful to point out to the student the difference in value of λ, according as $V = \lambda . h$ or $V = \lambda . r^2$. The construction of a double-entry table of values drives home the meaning of the double variation.

CHAPTER XV.—FUNCTIONS OF ONE VARIABLE.

The idea of functionality is both difficult and important. It is fairly easy to learn the mechanical process of computing special values of a function and plotting them on squared paper. It is much harder to grasp the meaning of the "equation of a curve" and to form a mental picture of the graph of a simple function from observation of the manner in which the function changes. In this chapter we are mainly concerned with the general characteristics of simple functions; *e.g.* $(x-3)(x-2)(x-1)$ is large when x is large, decreases to zero as $x \to 3$, is negative for $3 > x > 2$, increases to zero as $x \to 2$, is positive for $2 > x > 1$, decreases to zero as $x \to 1$, is negative for $1 > x$ and tends to $-\infty$ as $x \to -\infty$. This is the *life history* of the function, and from it the reader should be able to picture its graph. Conversely, given the graph of a function, its life history can be outlined. The plotting of special values is irrelevant and should be discouraged in this connection, except for verification, and squared paper should not be used. Whether it is advisable to include the harder examples in XV. a. at a first reading depends on the capacity of the class, but many of the difficulties can be lightened by oral treatment.

Exercise XV. b. gives some examples on the construction of functions corresponding to various geometrical properties.

The use of accurate plotting of functions is illustrated in XV. c., applications being made to maxima and minima problems and the solution of equations—a type of question which appears to be a favourite with many examining bodies. Exercise XV. d. provides practice in functional notation, which is essential for the Calculus; but the second half of the exercise should only be attempted by students of special ability.

CHAPTER XVI.—LIMITS AND GRADIENTS.

No attempt has been made in the text to define and discuss exhaustively the meaning of a limit; the analytical idea is too subtle and delicate for most students at this stage ; when desired, reference should be made to the excellent expositions in Nunn's *Algebra*, Part II., Section VIII., and Hardy's *Pure Mathematics*.

The subject-matter of this chapter is arranged on the assumption that the work will be accompanied by oral instruction and general discussion, the extent of which must depend on the ability of the class. Example I. and Exercise XVI. a. are designed to bring out two fundamental ideas :

(i) That the statement $\underset{x \to a}{\mathrm{Lt}} f(x) = L$ implies that given any positive quantity ϵ, a positive quantity δ exists such that, for *all* values of x included in

$$0 < |x - a| < \delta, \quad |f(x) - L| < \epsilon.$$

(ii) That the statement $\underset{x \to a}{\mathrm{Lt}} f(x) = L$ does not necessarily require that a value of x exists for which $f(x) = L$.

The notation $\underset{x = a}{\mathrm{Lt}} f(x)$ should be avoided : it is misleading, and may easily produce inaccurate ideas on the subject.

The explanation of the phrases $n \to \infty$ or $\underset{x \to c}{\mathrm{Lt}} f(x) \to \infty$ is left for oral treatment. As regards the first, the ordinary boy does not find much difficulty in forming a reasonably correct idea of what is represented : for example, suppose n takes in succession the values of the prime numbers 1, 2, 3, 5, 7, 11, 13, 17, ... arranged in a steadily increasing sequence. However large a number your opponent cares to select, you can always find a stage in the sequence beyond which *all* the numbers of the sequence exceed the number chosen by your opponent. In such a case you say n tends to infinity. The second expression may be treated similarly. The important idea for the pupil to grasp at this stage is that *infinity is not a number* and that expressions such as $\frac{1}{0}$ or $\frac{0}{0}$ are meaningless.

Kinematics probably furnishes the best illustration of the use of limits, and the exact meaning of the sentence " a car was travelling at 20 miles an hour and its speed was then gradually reduced to 15 miles an hour " should be discussed in detail and illustrated graphically. This leads on to the idea of the gradient of the graph, regarding the slope of the curve measured by the tangent at a special point as the limit of the average slope represented by a chord.

Although it is easier to grasp the meaning of rate of change referred to the time, it is important that some examples should be taken to illustrate the use of the word 'rate' referred to any independent variable. The rate at which a piece of ground on the side of a hill rises provides a simple and familiar example; but it should be pointed out that the measure of the slope or gradient is at variance with the ordinary usage of surveyors, who measure the gradient of a road by the sine of the angle of slope and not by the tangent of that angle.

The work on calculation of gradients is divided into two stages: the first deals with examples in which rates of change are estimated from observed data or statistics, and the second mainly with examples on formulae which permit of exact calculation.

CHAPTER XVII.—DIFFERENTIATION.

The notation of the Calculus is best explained orally, and might follow some such line as the following:

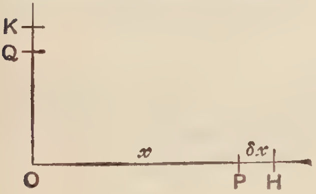


FIG. 2.

FIG. 3.

If x is the distance of a point P from O, and H is a point close to P on the line OP, the length of PH is represented by the symbol δx, which means "a small increment of the variable x," or colloquially "a little bit of x," and is positive if $OH > OP$ and negative if $OH < OP$.

Similarly, if $OQ = y$ and K is near Q on the line OQ, QK may be denoted by δy. Unless some condition is imposed, δx and δy may be any small quantities at all. But if x and y are connected by an equation, although δx can still be any small quantity whatever, δy can be calculated in terms of δx.

For example, suppose $xy = 36$.

Then δy is the change of y caused by a change δx in x, and is computed from the equation

$$(x + \delta x)(y + \delta y) = 36.$$

Again, suppose a train starts at O (Fig. 3), and arrives at N, where $ON = s$ feet, after t secs. and passes M after $t + \delta t$ secs., then the extra distance NM travelled in δt secs. is denoted by δs feet. Here δt might be anything at all. But if we know δt and know the equation connecting s and t, *i.e.* how the train travels, we can calculate δs in terms of δt.

It must be emphasised that in the symbol δs, the 'δ' and the 's' cannot be separated, that it is simply a shorthand notation for the phrase "a small increase in s," and does not mean δ multiplied by s.

Exercises XVII. a., b. are dull work ; but the reader must acquire facility in differentiating x^n (at any rate in simple cases) and handling easy polynomial expressions before proceeding to applications. A formal proof of the general formula $\frac{d}{dx}x^n = nx^{n-1}$ is excluded as unsuitable at this stage, but sufficient special cases are given as exercises to make its assumption plausible.

Exercise XVII. c. deals with the interpretation of the sign of $\frac{dy}{dx}$: this is not really difficult, but when the work is extended to cover the sign of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, there is liable to be at first some confusion of idea. This line of thought is, however, so instructive that it is worth while devoting time to it.

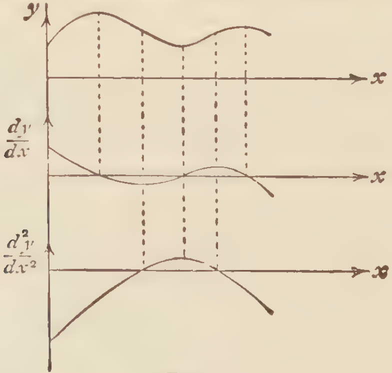


FIG. 4

The reader has already had a good deal of practice in the graphic representation of functions. This exercise carries the method a stage further forward, as it involves examining a figure to see whether the rate of increase is itself an increasing or decreasing function. The examples are intended to make the progress slow, and at first should be taken orally. It is useful and illuminating to exhibit, one below the other, the graphs of y and $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for some simple functions, as, for example, in Figure 4, basing the work on an appeal to the eye.

Exercise XVII. e. is introduced partly because methods of approximation have an importance of their own, but mainly as leading up to the integration work in the next chapter; the last six questions in it introduce the method for the differentiation of $F[\phi(x)]$. Anyone can be taught the process as a mere rule of thumb; but the understanding of the method is difficult and should at this stage be reserved for the clever boy who gets ahead of the class.

The work on calculation of rates of change in Exercise XVII. f. always presents difficulty, and may well be deferred to a second reading.

CHAPTER XVIII.—INTEGRATION.

INTEGRATION is treated as the reverse process of differentiation: the idea of summation is more difficult and may well be deferred. The investigation of the analytical basis upon which Dufton's and Simpson's approximate rules depend is omitted as beyond the scope of this volume. For an account of the former, the reader is referred to *Nature*, Vol. CV., pp. 354, 455.

The temptation to introduce more physical applications into the examples was great, but was resisted on the ground that it would disturb the proportions of the book, which aims merely at a general introduction. Those who intend to go further will naturally turn to a special text-book. Problems on centres of gravity, pressure, moments of inertia, etc., are therefore omitted, except in cases where the formulae are supplied in the question itself.

CHAPTER XIX.—SIMPLE SERIES.

THE inclusion of a chapter on this subject is open to criticism. Certainly there is little educational value in memorising the formulae for the sums of arithmetical and geometrical progressions, and in getting up the stereotyped questions which some examination papers still contain. But, examinations apart, there is a definite value in the appreciation of any kind of algebraic form and in the cultivation of the power to generalise ideas and to utilise formulae. If the reader can be led to construct formulae for himself and to understand the idea of order, he will gain definite advantage from the work of this chapter. It is therefore suggested that formulae should not be allowed to be used in Exercises XIX. a., b., c. In order that the pupil may not attach too much importance to the ordinary progressions, as much variety as possible has been introduced into Exercise XIX. a., which is intended to give training in the general idea of a law.

CHAPTER XX.—PERMUTATIONS AND COMBINATIONS.

A SHORT section on this subject is introduced as a necessary preliminary to a treatment of the Binomial Theorem. The illustrative examples have been made as brief as possible, and the main idea in each exercise has been to lead up through a large number of special cases to the general formula; the general formula should not be given until the pupil begins to see that there must be one, and that, if he knows it, it will simplify his work. A brief mention has been made of probability because of the intrinsic interest of the subject even, or perhaps especially, to the boy whose main interests are non-mathematical. The real importance of the subject lies in its application to Insurance and the Theory of Statistics; an excellent introductory account is given in Nunn's *Algebra*, Part II., where the idea of relative frequencies is clearly developed. A discussion of the Probability Curve is reluctantly omitted from this chapter as being disproportionate to the scheme of the book.

CHAPTER XXI.—BINOMIAL THEOREM.

IT was considered that an elementary book on Algebra would not be complete unless the Binomial Theorem, at any rate for a positive integral index, was included. Probably few teachers will take this chapter before doing the chapters on Functions and Differentiation, but there is nothing in the scheme of the book to prevent their doing so if they wish. The educational value of the chapter lies primarily in its giving the pupil a sense of algebraical form. If he sees that he can write down the expansion of $(x+a)(x+b)(x+c)(x+d)$ or of $(x+a)^4$ without any laborious multiplication, he obtains a sense of mastery of his subject and a better appreciation of the value of algebraic method. On the practical side the chief value of the Binomial is its application to approximate calculations, and a short section on this subject has therefore been included. The proof of the Binomial Theorem for any index is obviously unsuitable at this stage. It seems the best plan to follow the historical method, obtaining, as Newton did, the expansion for $(1+x)^{-1}$ by long division and for $(1+x)^{\frac{1}{2}}$ by the square-root rule. And it is of interest to point out to the reader Newton's brilliant induction of the general law from the special cases he considered. In the same way here, the general fact that, if x is small, $1+nx$ is an approximation for $(1+x)^n$ is an inference the student should be encouraged to draw from the examples he works out in Exercise XXI. c.

CHAPTER XXII.—EMPIRICAL FORMULAE.

Graphical work so far has consisted of :

(i) The representation of statistics or experimental results for which no functional relation is known.

(ii) The representation of functions of one variable.

We now proceed to consider a complementary problem : given a series of observations, is it possible to find an analytical formula which governs them to a fair degree of approximation ?

When this can be done, the result is called an *empirical* formula. Before investigating the problem, two cautionary remarks may be made : (i) however closely the formula may fit the data within the observed range, it is impossible to rely upon deductions from the formula *outside* this range In other words, inferences based on *interpolation* are trustworthy, but those based on *extrapolation* are of little value. (ii) Unless the *form* of the expression has been determined from theoretical considerations, there is no reason to suppose that the functional character of the formula corresponds to any fundamental physical property.

We shall deal in the present section with results that either conform to or are reducible by various methods to a linear law. For practical work, a student should have either a fine black thread or a piece of transparent celluloid on which is cut a fine line running along its length down the middle from end to end.

The first example in the text (page 416) shows the results of an experiment plotted in the ordinary way. The points do not lie precisely on a straight line ; their divergence may be due to errors of observation, imperfect conditions or faulty apparatus ; but the divergence is sufficiently small to suggest that a straight line can be drawn which will represent the data within a reasonable degree of accuracy. The black thread, or better the celluloid, should now be used to determine that straight line which *best fits* the observations. Mark two points on this "best fit line" at *opposite* ends of it and read off their coordinates. Substitute them in the formula $\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}$, and reduce this to the form $y=ax+b$, where a, b are expressed correct to as many significant figures as the methods and data employed justify. The formula so obtained is only approximate, and therefore a, b should not be given in fractional form, and in no case to more figures than the data warrant.

In general, when experimental results are plotted, the graph is a curve instead of a straight line. In such cases it may be possible by some method to transform the curve into a straight line. The following list gives a few of the means which may be tried.

If y is a function of x, plot

(i) $\log y$ against x or $\log x$ against y ;
(ii) $\log y$ against $\log x$;
(iii) xy against x or xy against y ;
(iv) y against $\frac{1}{x}$ or x against $\frac{1}{y}$;
(v) $\frac{x}{y}$ against x.

Unless theoretical considerations furnish some kind of a clue, nothing but experience can teach the student how best to proceed.

When trying (i) and (ii), much time is saved by using specially ruled logarithmic or semi-logarithmic paper. [This can be obtained from scientific instrument makers, *e.g.* C. Baker, 244 High Holborn.]

For further information, the reader is advised to consult an excellent monograph by Professor Running. [*Empirical Formulae*: T. R. Running. Chapman & Hall.]

CHAPTER XXIII.—NOMOGRAPHY.

The object of this section is to give the reader some idea of the use now being made of nomograms by engineers. There is great variety of method, special problems requiring distinctive treatment, and only a brief outline is given here. Those who are interested in the subject, which is still in its infancy, will naturally turn to books specially devoted to it. The classics on the subject are d'Ocagne's *Calcul Graphique et Nomographie* and *Traité de Nomographie*, and it is d'Ocagne who is the real inventor of the idea.

The following English works provide an easier introduction :

S. Brodetsky, *First Course in Nomography.* (Bell. 10s.)
E. S. Andrews, *Alignment Charts.* (Chapman & Hall. 1s. 3d.)
Capt. R. K. Hezlet, R.A., *Nomography.* (R.A. Institution. 3s.)
E. W. Tipple, *Line Charts.* (By application to the author, 5 Cross Flatts Row, Beeston, Leeds. 2s.)

The reader may be sceptical of the practical value of a method which at first sight appears exceedingly laborious and leads, like all graphical methods, only to approximate results. But once the nomogram has been constructed in a durable form by a skilled draughtsman, practically no skill is required in reading from it any number of numerical results that may be required for engineering or commercial purposes. And that astonishing accuracy can be attained by skilled draughtsmen using graphical methods is shown by the results obtained by sound-ranging sections in France in locating the position of hostile guns from the intersections of hyperbolas.

The theoretical side of the subject, which is given here, is of educational value. The idea of non-uniform graduation gives an opportunity for discussing or revising the principle of the slide rule. In practice it will be found that the examples occupy a great deal of time; logarithmic graduation is a slow process until experience gives facility. How much of this work should be done by rule is open to question, but in any event there is abundant intellectual exercise to be obtained in such matters as the choice of units, the best positions for the axes and suitable methods for checking the work.

The illustrative examples in this section are worked out more fully than has been customary in earlier chapters owing to the comparative novelty of the subject; and a great deal of assistance is given in some of the exercises. This makes them appear longer than they really are, due to the fact that the statement of the problem is often almost a skeleton solution. Nothing has been said in the text of the use made in Nomography of the graduation of curves; but the outline of d'Ocagne's solution of the quadratic, which is given in the last example of XXI. c., indicates the use that may be made of them. The skeleton method has in fact made it possible to introduce into the examples a greater variety of method than could be included in the text.

The main point to be grasped is that the simple operation of laying a straight-edge between two points on graduated lines and noting the value of the graduation at which the straight-edge cuts a third line or curve also suitably graduated leads to approximate numerical solutions of an almost inexhaustible variety of formulae and equations.

CHAPTER XXIV.—FURTHER DEVELOPMENTS FOR THE SPECIALIST.

Although one of the sections in this chapter is explicitly described as Algebraic Form, the subject-matter of the whole chapter and the general characteristics of the exercises have been chosen with the idea of developing that analytical sense which is the true hall-mark of the specialist. There is not a great deal of bookwork, and the variety of methods that must be employed diminishes the usefulness of illustrative examples and increases proportionately the value of work of this type. The importance of insisting on the use of symmetrical methods, wherever possible, can hardly be exaggerated.

CHAPTER I.

THE USE OF LETTERS IN ALGEBRA.

GENERALISATION. NOTATION. SUBSTITUTION.

THE possibility of generalised statements by means of letters must be explained orally and should be fully illustrated. For this and other matters connected with this chapter, refer to the Full Introduction (see Preface).

GENERALISATION.

What is an odd number ?

This question can be answered in several ways.

(i) We might give a large number of instances, *e.g.* 1, 3, 5, 7, 11, 13 .... .

(ii) We might give a rule for testing whether a number is odd, for example : A whole number is odd if there is a remainder when it is divided by 2.

(iii) We might proceed as follows :

$2 \times 1 + 1$ is an odd number,
$2 \times 2 + 1$,, ,,
$2 \times 3 + 1$,, ,,
$2 \times 4 + 1$,, ,,
$2 \times 5 + 1$,, ,,
and so on.

$2 \times 73 + 1$ is an odd number,
and so on.

We can put all these facts into a single sentence, as follows:

$2 \times x + 1$ *is an odd number, if* x *is any whole number.*

In (i) we give a number of special cases; we cannot write down all odd numbers in this way. In (ii) we give a general rule which applies to all odd numbers.

In (iii) we first give a number of special cases, and then deduce from them a general expression, *i.e.* a way of writing all odd numbers.

Such a statement as that given in italics is called a **generalisation.** One of the advantages of Algebra is that it enables us to generalise arithmetical statements.

Example I. Give a general statement to include the following statements:

8 exceeds 5 by $8-5$.
10 exceeds 2 by $10-2$.
140 exceeds 37 by $140-37$.

Answer. Any number x exceeds any other number y (less than x) by $x-y$.

Example II. Give a general statement to include the following statements:

$2 \times 3 = 3 \times 2$; $5 \times 9 = 9 \times 5$; $4 \times 7 = 7 \times 4$; $6 \times 8 = 8 \times 6$.

In each case we have two numbers, and we state that when the first is multiplied by the second, we get the same result as when the second is multiplied by the first.

Assuming that this is always true, we may state it as follows:

If x and y are any two numbers, $x \times y = y \times x$.

Example III. Give a general statement to include the following statements:

The sum of the first 2 odd numbers is 2^2.
" " 3 " 3^2.
" " 7 " 7^2.

[*N.B.*—These statements should be tested; *e.g.* the sum of the first 3 odd numbers $= 1 + 3 + 5 = 9 = 3^2$.]

Answer. The sum of the first n odd numbers (where n is a whole number) is n^2.

This result can be tested for other values of n. It can also be used as a short way of finding the sum of such numbers; *e.g.* the sum of the first 10 odd numbers is 100.

EXERCISE I. a.*

(NOTE.—*The majority of these examples should be taken orally, especially the more difficult.*)

Give a general statement, by means of letters, to include all the statements in each of the examples 1–18.

1. $3+3=$ twice 3.
$4+4=$ twice 4.
$7+7=$ twice 7.

2. $2-2=0$.
$3-3=0$.
$8-8=0$.
$21-21=0$.

3. $3\times 3=3^2$.
$7\times 7=7^2$.
$19\times 19=19^2$.

4. $2+3=3+2$.
$5+9=9+5$.
$7+4=4+7$.
$12+73=73+12$.
$96+1=1+96$.

5. $2\times 5-1$ is an odd number.
$2\times 7-1$ is an odd number.
$2\times 12-1$ is an odd number.

6. $5\times\frac{1}{5}=1$.
$6\times\frac{1}{6}=1$.
$11\times\frac{1}{11}=1$.
$42\times\frac{1}{42}=1$.

7. $\frac{3\times 2}{5\times 2}=\frac{3}{5}$.
$\frac{3\times 7}{5\times 7}=\frac{3}{5}$.
$\frac{3\times 10}{5\times 10}=\frac{3}{5}$.

8. $5^2\times 5=5^3$.
$6^2\times 6=6^3$.
$7^2\times 7=7^3$.
$10^2\times 10=10^3$.
$26^2\times 26=26^3$.

9. What must be added to 3 to obtain 19? $19-3$.
,, ,, 3 ,, 24? $24-3$.
,, ,, 3 ,, 36? $36-3$.

10. By what must 7 be multiplied to obtain 28? $\frac{28}{7}$.
,, 7 ,, ,, 35? $\frac{35}{7}$.
,, 7 ,, ,, 40? $\frac{40}{7}$.
,, 7 ,, ,, 11·4? $\frac{11{\cdot}4}{7}$.

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* For additional examples, see E.P. 1, page 251/2.

11. $7-1, 7, 7+1$ are three consecutive whole numbers.
$5-1, 5, 5+1$,, ,, ,,
$97-1, 97, 97+1$,, ,, ,,

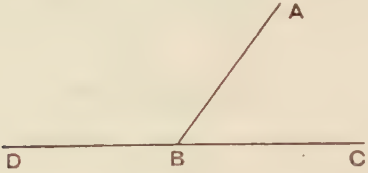

FIG. 1.

12. If the angle *ABC* is 60 degrees, the angle *ABD* is $180-60$ degrees.

If the angle *ABC* is 75 degrees, the angle *ABD* is $180-75$ degrees.

If the angle *ABC* is 113 degrees, the angle *ABD* is $180-113$ degrees.

13. The number of shillings in 4 guineas is 4×21.
,, ,, 11 ,, 11×21.
,, ,, $2\frac{1}{2}$,, $2\frac{1}{2}\times 21$.

14. There are $\frac{8}{2}$ even numbers up to and including 8.
,, $\frac{20}{2}$,, ,, 20.
,, $\frac{18}{2}$,, ,, 18.

15. $0\times 4=0$.
$0\times 7=0$.
$0\times 13=0$.

16. 3 times $5+5=4$ times 5.
3 times $6+6=4$ times 6.
3 times $11+11=4$ times 11.

17. Since $5-3=2$, $\therefore\ 5=2+3$.
Since $11-7=4$, $\therefore\ 11=4+7$.
Since $20-6=14$, $\therefore\ 20=14+6$.

18. 3 per cent. of a number is $\frac{3}{100}$ of that number.
7 ,, ,, $\frac{7}{100}$,,
12 ,, ,, $\frac{12}{100}$,,
$2\frac{1}{2}$,, ,, $\frac{2\frac{1}{2}}{100}$,,

19. What must be subtracted from 20 to leave 4? The answer is $20-4$. What must be subtracted from 20 to leave 7? What must be subtracted from 20 to leave a? What must be subtracted from b to leave a?

20. The next even number above the even number 6 is $6+2$. What is the next even number above x, if x is an even number? What is the next even number above x, if x is an odd number?

21. The number of feet in 5 yards is 5×3. What is the number of feet in 28 yards? What is the number of feet in a yards? What is the number of yards in b feet?

22. 6 per cent. of 85 is $\frac{6}{100}\times 85$; 11 per cent. of 247 is $\frac{11}{100}\times 247$. What is x per cent. of 85? What is x per cent. of y?

23. Prove that the sum of the first 3 even numbers is 3^2+3. Prove that the sum of the first 4 even numbers is 4^2+4. What is the sum of the first 5 even numbers? Prove it. What is the sum of the first n even numbers?

NOTATION.

The symbols $+$, $-$, $\times$, $\div$ have the same meaning in Algebra as in Arithmetic.

The product of 2 and 3 is represented in Arithmetic by 2×3 or $2 \,.\, 3$; so in Algebra the product of a and b may be represented by $a\times b$ or $a \,.\, b$; it is, however, generally represented by ab. Similarly $3x$ represents the product of 3 and x. We know that the product of 1 and x is x, so that the expression $1x$ is not used.

In a product we write a number before a letter, *e.g.* we write $3x$, not $x3$.

In other respects the signs used in Algebra have the same meaning as in Arithmetic.

7^4 means $7\times 7\times 7\times 7$.	a^4 means $a\times a\times a\times a$.
$\sqrt{8}$ means the square root of 8.	$\sqrt{x}$ means the square root of x.

$\frac{3}{4}$ is the same as $3 \div 4$. $\frac{a}{b}$ is the same as $a \div b$.

$4 . 5^2 . 7^3$ means $4a^2b^3$ means

$4 \times 5 \times 5 \times 7 \times 7 \times 7$. $4 \times a \times a \times b \times b \times b$.

For other symbols, see page 252.

Do not think of Arithmetic and Algebra as separate subjects. In the following chapters you will repeat in Algebra most of the processes to which you have been accustomed in Arithmetic.

Example IV. What is the meaning of $2x + 5$?

To obtain $2x + 5$, we multiply 2 by x (or x by 2) and add 5 to the result.

Example V. If x stands for 5, what does $2x^3$ stand for?

$2x^3$ stands for $2 \times x \times x \times x$,

i.e. for $2 \times 5 \times 5 \times 5$, if x stands for 5, *i.e.* for 250.

Example VI. If a stands for 3 and b for 2, what does $a^2 + 5ab$ stand for?

a^2 stands for $a \times a$, *i.e.* 3×3, *i.e.* 9.

$5ab$ stands for $5 \times a \times b$, *i.e.* $5 \times 3 \times 2$, *i.e.* 30.

$\therefore$ $a^2 + 5ab$ stands for $9 + 30$, *i.e.* for 39.

Example VII. If $a = 9$ and $b = 16$, what are the values of $\sqrt{a}$, $\sqrt{b}$, $\sqrt{a} + \sqrt{b}$?

$\sqrt{a} = \sqrt{9} = 3$, $\sqrt{b} = \sqrt{16} = 4$, $\sqrt{a} + \sqrt{b} = 3 + 4 = 7$.

EXERCISE I. b.*

1. Give the meanings of the following (see Example **IV.** above):

(i) $2a$. (ii) $3a + b$. (iii) x^3. (iv) $x^3 + x$.

(v) $\sqrt{2a}$. (vi) $\sqrt{x} + 1$. (vii) $4ab^2$. (viii) $ab - b$.

2. How do you write more shortly $x \times x \times x \times x \times x$? How would you write it if there were thirteen factors, each equal to x?

3. In what other ways can you write $2 \times x^2$?

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* For additional examples, see E.P. 2, page 251/4.

4. If x stands for 9, what do the following stand for ?

(i) $2x$. (ii) $2+x$. (iii) $x+2$.
(iv) x^2. (v) $x+3x$. (vi) $\sqrt{x}$.

5. If a stands for 2 and b stands for 3, what do the following stand for ?

(i) $a+b$. (ii) $b+a$. (iii) ab. (iv) ba.
(v) a^2+2ab. (vi) $3a-2b$. (vii) $5a+2b$. (viii) $10ab$.
(ix) a^2b. (x) ab^2. (xi) b^2-a^2. (xii) $\frac{b}{a}$. (xiii) $\frac{b+5}{a}$.

6. If $x=5$ and $y=2$, what are the values of the following ?

(i) y^3. (ii) $x-2y$. (iii) $2xy$.
(iv) xy^2. (v) $x+\sqrt{2y}$. (vi) x^2-y^3.

7. If $x=\frac{1}{2}$, what are the values of the following ?

(i) $1-x$. (ii) x^2. (iii) $2x$.
(iv) $x+2$. (v) $\frac{1}{x}$. (vi) $2x^3$.

8. If $x=2$, $y=5$, $z=3$, what are the values of the following ?

(i) xyz. (ii) $3xyz$. (iii) x^3yz. (iv) xyz^2.
(v) $x^2+y^2+z^2$. (vi) $x+y+z$. (vii) $y-x-z$.

9. If $x=3$, find the values of the following :

(i) x^2-1. (ii) $3x^2-5x+2$. (iii) $2x^3$.
(iv) $\frac{x+1}{2x}$. (v) x^2-3x. (vi) $\frac{x}{12}$. (vii) $\frac{24}{x}$.

10. If $x=\frac{1}{3}$, $y=\frac{1}{5}$, find the values of the following :

(i) $x-y$. (ii) $2xy$. (iii) $10xy$.
(iv) $\frac{x}{y}$. (v) $x+1$. (vi) $6x+15y$.

11. If $x=0$, $y=2$, find the values of the following :

(i) $x+1$. (ii) $7x$. (iii) $x+y$. (iv) xy.
(v) x^2y. (vi) $3xy^2$. (vii) x^2+y^2.

---

12. If $x=3$ and $y=2x$, what is the value of y ?

13. If $x+y=10$ and $y=2$, what is the value of x ?

14. If $y = 2x^2 + x$ and $x = 3$, what is the value of y ?

15. If $ab = 48$ and $b = 8$, what is the value of a ?

16. If $a = 3b$ and $b = 2$, what is the value of ab ?

17. If $x + 2y = z$, what is the value of z, when $x = 1$ and $y = 3$?

18. If x is the square of y and $y = 3z$ and $z = 2$, what is the value of xyz ?

SUBSTITUTION.

We have seen that it is often possible to combine in a single algebraical statement a number of different arithmetical results.

We shall now consider the reverse process, *i.e.* from a general algebraical statement to obtain various arithmetical results.

Example VIII. Give two numerical examples of the following statement :

If x is a whole number, $x + 1$ is a factor of $x^2 - 1$.

(i) Let x stand for 6. The statement becomes :

$6 + 1$ is a factor of $6^2 - 1$,

or, 7 is a factor of 35.

(ii) Let x stand for 10. The statement now becomes :

$10 + 1$ is a factor of $10^2 - 1$,

or, 11 is a factor of 99.

Each of these two statements is seen to be true.

Example IX. If a room is l feet long, w feet wide and h feet high, the area of the walls is $2hl + 2hw$ sq. ft. Use this statement to find the area of the walls of a room 20 ft. long, 15 ft. wide and 9 ft. high.

In this case $l = 20$, $w = 15$, $h = 9$.

$\therefore$ the area of the walls is $2 \times 9 \times 20 + 2 \times 9 \times 15$ sq. ft.,

i.e. $18 \times 20 + 18 \times 15$ sq. ft.,

i.e. 18×35 sq. ft.,

i.e. **630 sq. ft.**

EXERCISE 1. c.*

[Oral.]

Deduce two arithmetical results from each of the following statements:

1. The sum of x and $3x$ is $4x$.

2. The difference between $10x$ and x is $9x$.

3. The product of a and $a+1$ is equal to a^2+a.

4. The square of $3y$ is $9y^2$.

5. If the square of $x-1$ is subtracted from the square of $x+1$, the result is $4x$.

6. If a side of a square is x ft., the perimeter is $4x$ ft.

7. If the diameter of a circle is d inches, its circumference may be taken to be $\frac{22d}{7}$ inches.

8. If coal costs xs. a ton, you get $\frac{560}{3x}$ lb. for a penny at the same rate.

9. In a right-angled triangle, if one acute angle is x degrees, the other is $90-x$ degrees.

10. If x is less than y (and they are whole numbers), there are $y-1-x$ whole numbers between x and y.

[Written.]

11. If you multiply $x+5$ by 3, the result is $3x+15$. Show that this is the case when x stands for (i) 4, (ii) 0.

12. If x is multiplied by $\frac{y}{x}$ the product is y. Show that this is true when $x=5$ and $y=10$.

13. $x+1$ is the square root of x^2+2x+1. Show that this is true when $x=6$.

14. If you subtract $3x$ from $11x$, the result is $8x$. Show that this is true when $x=7$.

15. If x and y are two consecutive whole numbers, x being the smaller, then y^2-x^2 is equal to $y+x$. Show that this is true when $x=8$

---

* For additional examples, see E.P. 3, page 251/5.

16. After the xth day of November there are $61-x$ days to the end of the year. How many days are there after the 25th of November?

17. If the temperature is x degrees Centigrade, it is $\frac{9x}{5}+32$ degrees Fahrenheit. Express 15 degrees Centigrade in degrees Fahrenheit.

18. If a rectangle is a inches long and b inches wide, its area is ab sq. in. Find the area of a rectangle 10 inches long and $3\frac{1}{2}$ inches wide.

19. x miles may be taken as equal to $\frac{8x}{5}$ kilometres. Express 100 miles in kilometres.

20. The sum of the first x whole numbers (1. 2, 3, 4, ...) is $\frac{1}{2}x^2+\frac{1}{2}x$. Show that this is true when $x=6$.

21. If n of the numbers 1, 4, 7, 10, 13, 16, 19, ... are written down, the last number is $3n-2$. Show that this is true when (i) $n=4$, (ii) $n=8$.

22. If a polygon has n sides, the sum of its angles is $2n-4$ right angles. What is the sum of the angles of (i) a quadrilateral (four sides), (ii) a figure of eight sides?

23. If we assume that the middle of the day is at 12 o'clock and the sun rises at x o'clock, it sets at $12-x$ o'clock. If on April 19th it rises at 5 o'clock, when does it set?

24. If the diameter of a circle is d inches, its area may be taken to be $\frac{11d^2}{14}$ sq. in. Find the area of a circle of diameter 10 inches, to the nearest square inch.

25. If n post cards cost $(n+1)$ half-pennies, when n is less than 12, find the cost of eight post cards.

CHAPTER II.

GENERALISED ARITHMETIC. PROCESSES.

ADDITION AND SUBTRACTION.

LIKE TERMS.

EXERCISE II. a.*

The value of $3+3+3+3+3$ *is* 3×5.
The value of $x+x+x+x+x$ *is* $x\times 5$, *i.e.* $5x$.

In a similar manner, write down the value of

1. $5+5+5+5$.
2. $a+a+a+a$.
3. $x^2+x^2+x^2$.
4. $xy+xy$.
5. $x+x+x+x+x+x$.
6. $a+a+a+\ldots$ to thirty terms.

The value of $3\times 19+7\times 19$ *is* 10×19.
The value of $3x+7x$ *is* $10x$.
The value of $6x-x$ *is* $5x$.

In a similar manner, write down the value of

7. $5\times 17+3\times 17$.
8. $5x+3x$.
9. $7\times 9-2\times 9$.
10. $7x-2x$.
11. $5\times 9+9$.
12. $5b+b$.
13. $2ab+3ab$.
14. $7x^2-2x^2$.
15. $8a-a$.
16. $2x^2-x^2$.
17. $12xy+9xy$.
18. $15a^3-10a^3$.

UNLIKE TERMS.

Example I. Add x and x. *Answer* $\mathbf{2x}$.
Example II. Add x and y. *Answer* $\mathbf{x+y}$.

---

* For additional examples, see E.P. 4, Nos. 1-24, page 251/8.

Example III. Simplify $2x + 3x + y$. *Answer* **$5x + y$.**
Example IV. Simplify $5a - 2a + 1$. *Answer* **$3a + 1$.**

EXERCISE II. b.*

Simplify:

1. $a + a + b$.
2. $x + x + x + y + y$.
3. $b + b + b + 7$.
4. $x + y + x + y + x$.
5. $x + 3 + x + 5 + x$
6. $a + b + 5 + b + 1$.
7. $a + a + b - a$.
8. $3x + 2x + y$.
9. $8a + b - 3a$.
10. $3x + y - 2x + y$.
11. $a + b - a$.
12. $a + b + c + a + b + c$.
13. $x + 2y + 2x + y$.
14. $x + y - z + x - y + z$.
15. $3x + a + x - a$.
16. $4a + b + 2c - 2a - b$.
17. $x + y + 5 + x + 3$.
18. $2a + 3b + 4 - a - 2b - 3$.
19. $3xy + 2yz - xy$.
20. $2x^2 - y^2 + 2y^2 - z^2 + 2z^2 - x^2$.
21. $x^2 + 2xy + 2xz - xy - xz$.
22. $4xy + 2x + 2y - x - y$.
23. $10a^2 + 7b^2 - 2c^2 + 3b^2 - 3c^2$.
24. $5yz + 2zy - yz - 6xy$.

POWERS.

Example V. Simplify $2x^2 + 3x + 2 - x^2 + 2x + 1$.

Taking like terms together, *i.e.* $2x^2 - x^2$, then $3x + 2x$, then $2 + 1$, we get **$x^2 + 5x + 3$.**

An expression such as $2x^5 + 7x^4 + x^3 + 2x^2 + 8$ cannot be simplified. Each term contains a power of x multiplied by a number, which is called the **coefficient** of that power. The term $2x^5$ is said to be of **degree 5** in x (or of the 5th degree) and the coefficient of x^5 is 2. The term x^3 is said to be of degree 3 in x, and the coefficient of x^3 is 1. The x term, or term of degree 1, is missing. The last term, 8, is called the **constant term,** or term independent of x.

It is convenient to write such expressions in an orderly way. They are consequently usually written either (i) *in descending powers of x*, *i.e.* beginning with the highest power, then the next highest, and so on; or (ii) *in ascending powers of* x, *i.e.* beginning with the constant term, then the x term, then the x^2 term, and so on.

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* For additional examples see E.P. 4, Nos. 25-56, page 251/8.

For example,

$2x^5+7x^4+x^3+2x^2+13x+8$ is in descending powers,
and $8+13x+2x^2+x^3+7x^4+2x^5$ is in ascending powers.

EXERCISE II. c.*

1. The number 347 may be written $3 \cdot 10^2+4 \cdot 10+7$. In a similar way write 265. In your answer write x for 10, x^2 for 100. Can you simplify $2x^2+6x+5$?

2. The number 2083 may be written $2 \cdot 10^3+8 \cdot 10+3$. In a similar way write 4609. In your answer write x^2, x^3 for 10^2, 10^3. Can you simplify $4x^3+6x^2+9$?

3. Can you simplify x^2+x?

4. Can you simplify $2x^2-x$? **5.** Add $2x^2$ and $3x$.

6. Subtract 1 from x^2. **7.** Subtract x^2 from x^3.

8. What is the value of x^2+2x+9 when $x=10$?

Simplify:

9. $6x^2-4x^2$. **10.** $x^2+3x-2x$.

11. $a^3+3a^2+3a-a^2$. **12.** $x^2-2x+1+x^2+2x+1$.

13. $5x^2+8x-4x^2-8$. **14.** $x^2-1+x-1$.

15. a^4+2a^4. **16.** $a^2+a+2+2-a$.

17. $x^4+9x^2+1-5x^2$. **18.** $4-4a+a^2-a^2-2a-1$.

19. $a^3+a^2+a+a^2+a+1$.

20. $5a^3+10a^2+10a+5-5a^2-5a$.

21. Write down (i) the term of degree 4, (ii) the coefficient of x^3, (iii) the constant term, in

$$7x^4+2x^3+3x^2+x+5.$$

22. Write down (i) the term of highest degree, (ii) the coefficient of x^2, (iii) the constant term, in

$$x^2+5x^3+3x+2.$$

23. Arrange $2+7x^2+x^3+9x^4+2x$ in (i) descending powers of x, (ii) ascending powers of x.

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* For additional examples, see E.P. 4, Nos. 57-90, page 251/8.

24. Arrange $x + x^8 + 3x^2 + 1 + 4x^6$ in (i) descending powers of x, (ii) ascending powers of x.

25. Simplify $x^3 + 2x^2 + 3x + 4 - x - 2x^2$, arranging the answer in ascending powers.

26. Simplify $x^3 + 1 + 3x^2 + 3x + 1 - 3x + 3x^2 - x^3$, arranging the answer in descending powers.

MULTIPLICATION AND DIVISION.

Example VI. To prove that $x^3 \times x^2 = x^5$.

$$x^3 = x \times x \times x \quad \text{and} \quad x^2 = x \times x.$$

$$\therefore\ x^3 \times x^2 = x \times x \times x \times x \times x = x^5.$$

Example VII. Multiply $3a^2$ by $2ab$.

$$3a^2 = 3 \times a^2 \quad \text{and} \quad 2ab = 2 \times a \times b.$$

$$\begin{aligned}\therefore\ 3a^2 \times 2ab &= 3 \times a^2 \times 2 \times a \times b\\ &= 3 \times 2 \times a^2 \times a \times b\\ &= 6 \times a^3 \times b = \mathbf{6a^3b}. \quad \textit{Answer.}\end{aligned}$$

This is the same method as is used in Arithmetic for multiplying two numbers together, each of which is given in prime factors.

Thus, putting 5 for a and 7 for b,

$$3 . 5^2 \times 2 . 5 . 7 = 2 . 3 . 5 . 5^2 . 7 = 6 . 5^3 . 7.$$

Example VIII. Divide $2x^3yz$ by x^2z.

$$\begin{aligned}2x^3yz \div x^2z = \frac{2x^3yz}{x^2z} &= \frac{2 \times x \times x \times x \times y \times z}{x \times x \times z}\\ &= 2 \times x \times y = \mathbf{2xy}. \quad \textit{Answer.}\end{aligned}$$

After a little practice the answers to such questions as Ex. VII. or Ex. VIII. can be written down without any working.

EXERCISE II. d.*

In a manner similar to that of Examples VI.–VIII. above, prove the statements in Exx. 1–6.

1. $2^3 \times 2^4 = 2^7$.

2. $a^3 \times a^3 = a^6$.

3. $2x^3 \times x = 2x^4$.

4. $x^6 \div x^2 = x^4$.

5. $6x^4 \div 2x = 3x^3$.

6. $2xy \times 3x^2y = 6x^3y^2$.

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* For additional examples, see E.P. 5, page 251/9.

Write down the values of the following :

7. $6^3 \times 6^2$.
8. $a^3 \times a^2$.
9. $y^3 \times y^4$.
10. $2 \,.\, 5^2 \times 5$.
11. $2a^2 \times a$.
12. $3c^2 \times 2c^3$.
13. $2ab \times a$.
14. $3xy \times xz$.
15. $3a^3 \times 3a^3$.
16. $5^4 \div 5^3$.
17. $a^4 \div a^3$.
18. $11^3 \div 11$.
19. $x^3 \div x$.
20. $2a^3 \div a^2$.
21. $6a^3 \div a$.
22. $6x^6 \div 2x^2$.
23. $3y \div y$.
24. $xy^2 \div xy$.
25. $4a^2b \div ab$.
26. $12abc \times a^2b$.
27. $3ab \times abc$.
28. $a^2 \times 2bc^2$.
29. $6abc^3 \div 6bc^3$.
30. $4p^2q^2r^2 \div 4pqr$.

SQUARES, SQUARE ROOTS, ETC.

Example IX. To prove that $\sqrt{x^6} = x^3$.

$$x^3 \times x^3 = x^6.$$

$\therefore$ the square of x^3 is x^6.

$\therefore$ a square root of x^6 is x^3.

[*N.B.*—We shall find later that a number has more than one square root, but at present we shall consider only one.]

EXERCISE II. e.

In a manner similar to that of Example IX. above, prove the following statements :

1. $\sqrt{10^4} = 10^2$.
2. $\sqrt{a^8} = a^4$.
3. $\sqrt{4x^2y^2} = 2xy$.
4. A square root of 49^3 is 7^3.
5. A cube root of x^6 is x^2.
6. A cube root of $8a^3b^3$ is $2ab$.

Write down the square of :

7. $3x$.
8. a^3.
9. ab^2.
10. $5x^2y$.
11. $4ab^2c$.
12. $2a^2b^3c^5$.

Write down a square root of :

13. 3^4.
14. a^{16}.
15. $4x^4$.
16. $9y^8$.
17. $a^2b^2c^2$.
18. $16x^2y^4$.
19. Write down the cube of (i) $2x$, (ii) $3ab^2$.
20. Write down a cube root of $64x^{27}$.

HIGHEST COMMON FACTOR. LOWEST COMMON MULTIPLE.

Of the two expressions, $4x^2y$ and $6xy^2$, there are several common factors, viz.: $\mathbf{2xy}$, 2, $2x$, $2y$, xy. But all the rest of these are factors of the first one, $2xy$.

$\mathbf{2xy}$ is called the **Highest Common Factor** (H.C.F.) of $4x^2y$ and $6xy^2$.

Of the two expressions $4x^2y$ and $6xy^2$, we can write down as many common multiples as we like, viz.: $\mathbf{12x^2y^2}$, $240x^3y^2$, $12x^3y^3$, $60x^2y^2z^2$, etc. But all the rest of these are multiples of the first one, $12x^2y^2$.

$\mathbf{12x^2y^2}$ is called the **Lowest Common Multiple** (L.C.M.) of $4x^2y$ and $6xy^2$.

Example X. Find (i) the H.C.F., (ii) the L.C.M. of $6a^2b^2$ and $15ab^2c$.

(i) 3 is the H.C.F. of the coefficients; of the powers of a, viz.: a^2 and a, we take the lowest, $\mathbf{a}$; of the powers of b, viz.: b^2 and b^2, we take the lowest, $\mathbf{b^2}$; we leave out c, as it is not a factor of $6a^2b^2$.

Multiplying together the factors we have taken, viz.: 3, a, b^2, we get the H.C.F. $\mathbf{3ab^2}$.

(ii) 30 is the L.C.M. of the coefficients; of the powers of a we take the highest, $\mathbf{a^2}$; of the powers of b the highest, $\mathbf{b^2}$; of the powers of c the highest, $\mathbf{c}$.

Multiplying these together we get the L.C.M. $\mathbf{30a^2b^2c}$.

EXERCISE II. f.*

1. Of which of the expressions 2, $2x$, $4x^3$, $2xy$, $6xy^2$, $8x^4y$, $2x^2y^2$ is $2x^2$ a factor ?

2. Of which of the expressions $3x$, $2x^2$, $3xy^2$, $6y^2$, $6x^2y$ is $6x^2$ a multiple ?

3. Of which of the pairs of expressions (i) $12x^2$ and $6x^3$; (ii) $2xy$ and $4x^2y$; (iii) $10x^4y^4$ and $12x^5y^2$ is $2x^2$ a common factor ?

4. Is $20x^2y^3$ a multiple of $5y^2$? Why ?

5. Is $3xyz$ a factor of $x^3y^3z^3$? Why ?

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* For additional examples, see E.P. 6, Nos. 1-30, page 251/10.

6. Is $4a^2b^3$ a common factor of $4a^2$ and b^3? Why?

7. Is $3x^2y^2z^2$ a common multiple of $3xy$ and yz? Why?

8. Of the expressions, 2, $2x$, $4x^2$, $3x$, $12x^3y^3$, $6x^2y^2$, $2xy$, $6x^2y$, $18x^4$, x, xy, $6xy$, $3x^2y$, $2xz$, $6x^2z$, $6xyz^2$, write down those which are (i) factors of $2x^2$, (ii) factors of $3xy$, (iii) common factors of $2x^2$ and $3xy$, (iv) multiples of $2x^2$, (v) multiples of $3xy$, (vi) common multiples of $2x^2$ and $3xy$.

9. Of the expressions 6, a^2b, $a^2b^2c^2$, $3a^2$, b, $6a^4$, $6a^3bc$, abc^2, a, $3bc$, $2a^3$, $12a^3bc$, bc, write down those which are factors of (i) abc, (ii) $6a^3$.

10. Find the H.C.F. of $2a$ and $6ab$.

11. Find the L.C.M. of $2a$ and $6ab$.

12. Find the H.C.F. of $4xy$ and $4xz$.

13. Find the H.C.F. of b^2c^2 and $4bc$.

14. Find the L.C.M. of $2ab$ and $3a^2b$.

15. Find the H.C.F. of ab and abc.

16. Find the L.C.M. of a^2, $6ab$, $3b^3$.

17. Find the L.C.M. of a^3, $5abc$, $2a^2$: write down the quotients obtained by dividing the L.C.M. by each expression.

18. Find the L.C.M. of $4x^2y$, $6x^3z^3$, $3xyz^2$, $2x^4$: write down the quotients obtained by dividing the L.C.M. by each expression.

19. Find the H.C.F. and L.C.M. of $10x^2$ and $15x^3$.

20. Find the H.C.F. and L.C.M. of 24, $3x$, $3xy$, $6xz$.

FRACTIONS.

Example XI. Simplify $\frac{4x^3y}{6x^2yz}$.

The H.C.F. of the numerator and denominator is $2x^2y$. Divide the numerator and denominator by $2x^2y$.

$$\frac{4x^3y}{6x^2yz} = \frac{2x}{3z}. \quad \textit{Answer.}$$

EXERCISE II. g.*

Simplify the following fractions :

1. $\frac{2x}{2y}$. **2.** $\frac{a}{a^2}$. **3.** $\frac{2x}{2}$. **4.** $\frac{6x}{9x}$.

5. $\frac{ab}{ac}$. **6.** $\frac{3ab}{a}$. **7.** $\frac{x^2}{xy}$. **8.** $\frac{xyz}{xyz}$.

9. $\frac{5a^2}{15ab^2}$. **10.** $\frac{2a}{2ab}$. **11.** $\frac{3xyz}{30z^3}$. **12.** $\frac{a}{abc}$.

13. $\frac{cd}{cbd}$. **14.** $\frac{6a^3}{6a^2b}$. **15.** $\frac{4x^3yz}{4yz}$. **16.** $\frac{6a^2b^2}{8a^2bc}$.

17. $\frac{a^2x}{xa^2}$. **18.** $\frac{6a^2cd^2}{9abc^2}$. **19.** $\frac{x^2yxz^3}{xy^2z}$. **20.** $\frac{a^2}{a^3b}$.

21. $\frac{a^2b^3c^2}{a^2b^3c^2}$. **22.** $\frac{18b^3c^3}{12bac^2}$. **23.** $\frac{2ac}{a^2c^2}$. **24.** $\frac{x^{10}y^8z^4}{x^5y^4z^2}$.

Example XII. Express $\frac{2a}{3b}$ as a fraction of the same value, having $6b$ as its denominator.

Multiply numerator and denominator by 2.

$$\frac{2a}{3b}=\frac{2a\times 2}{3b\times 2}=\frac{4a}{6b}.\quad \textit{Answer.}$$

Example XIII. Express $\frac{x^2}{5yz}$ as a fraction of the same value, having $10x^2y^2z^2$ as its denominator.

If $10x^2y^2z^2$ is divided by $5yz$, the quotient is $2x^2yz$.
Multiply numerator and denominator by $2x^2yz$.

$$\frac{x^2}{5yz}=\frac{x^2\times 2x^2yz}{5yz\times 2x^2yz}=\frac{2x^4yz}{10x^2y^2z^2}.\quad \textit{Answer.}$$

Example XIV. Add the fractions $\frac{2}{x}$ and $\frac{1}{2x}$.

The L.C.M. of the denominators x, $2x$ is $2x$.
Each fraction is so expressed that its denominator is $2x$.

$$\frac{2}{x}+\frac{1}{2x}=\frac{4}{2x}+\frac{1}{2x}=\frac{4+1}{2x}=\frac{5}{2x}.\quad \textit{Answer.}$$

Compare this with $\frac{2}{7}+\frac{1}{14}=\frac{4}{14}+\frac{1}{14}=\frac{5}{14}$.

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* For additional examples, see E.P. 6, Nos. 31-60, page 251/11.

Example XV. Subtract $\frac{b}{a}$ from $\frac{a}{b}$.

The L.C.M. of the denominators a, b is ab.

$$\frac{a}{b}-\frac{b}{a}=\frac{a^2}{ab}-\frac{b^2}{ab}=\frac{\mathbf{a}^2-\mathbf{b}^2}{\mathbf{ab}}. \quad \textit{Answer.}$$

Compare this with $\frac{5}{3}-\frac{3}{5}=\frac{5^2}{3\times5}-\frac{3^2}{3\times5}=\frac{5^2-3^2}{3\times5}$.

Example XVI. Simplify $a-\frac{b^2}{a}$.

The L.C.M. of the denominators 1, a is a.

$$a-\frac{b^2}{a}=\frac{a^2}{a}-\frac{b^2}{a}=\frac{\mathbf{a}^2-\mathbf{b}^2}{\mathbf{a}}. \quad \textit{Answer.}$$

EXERCISE II. h. *

1. Express $\frac{2}{3}$ as a fraction of the same value, having (i) 6, (ii) 30, (iii) 42 as its denominator.

2. Express $\frac{x}{y}$ as a fraction of the same value, having (i) $2y$, (ii) ay, (iii) y^2 as its denominator.

3. Express $\frac{2x}{5y}$ as a fraction of the same value, having (i) $10y$, (ii) $5yz$, (iii) $15y^2$ as its denominator.

4. Express $\frac{a}{b}$ and $\frac{b}{a}$ as fractions of the same value, having equal denominators.

5. The fractions $\frac{5}{2x}$, $\frac{x}{6y^2}$, $\frac{y}{3x^2}$ are to be so expressed that all their denominators are equal. What is the simplest way in which this can be done?

6. Fill in the gaps in the following:

(i) $\frac{2x}{y}=\frac{\quad}{3y^2}$; (ii) $\frac{3a^2b}{6ab^2}=\frac{\quad}{12a^2b^2}=\frac{\quad}{4b}$;

(iii) $a^3=\frac{\quad}{b^2}$; (iv) $\frac{1}{x}=\frac{y^2}{\quad}$; (v) $\frac{a^2}{x}=\frac{a^3b^3}{\quad}$.

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* For additional examples, see E.P. 6, Nos. 61-90, page 251/11.

Simplify the following fractions :

7. (i) $\frac{2}{7}+\frac{3}{7}$; (ii) $\frac{2}{a}+\frac{3}{a}$; (iii) $\frac{3}{4a}+\frac{1}{4a}$.

8. (i) $\frac{1}{7}+\frac{1}{2\times 7}$; (ii) $\frac{1}{x}+\frac{1}{2x}$; (iii) $\frac{y}{x}+\frac{3y}{5x}$.

9. (i) $\frac{2}{5}+\frac{1}{10}$; (ii) $\frac{2}{a}+\frac{1}{2a}$; (iii) $\frac{3b}{2a}+\frac{b}{a}$.

10. $\frac{1}{2x}+\frac{1}{3x}$. **11.** $\frac{x}{y}+\frac{y}{x}$. **12.** $1+\frac{2}{3}$.

13. $1+\frac{a}{b}$. **14.** $\frac{1}{x}+3$. **15.** $\frac{1}{ab}+\frac{2}{a}$.

16. (i) $\frac{5}{7}-\frac{2}{7}$; (ii) $\frac{5}{a}-\frac{2}{a}$; (iii) $\frac{x}{2a}-\frac{y}{2a}$.

17. (i) $\frac{1}{7}-\frac{1}{3\times 7}$; (ii) $\frac{1}{x}-\frac{1}{3x}$; (iii) $\frac{y}{x}-\frac{2y}{3x}$.

18. (i) $\frac{2}{5}-\frac{1}{10}$; (ii) $\frac{2}{a}-\frac{1}{2a}$; (iii) $\frac{bc}{a}-\frac{b}{3a}$.

19. (i) $1-\frac{2}{7}$; (ii) $1-\frac{a}{b}$; (iii) $1-\frac{1}{xy}$.

20. (i) $\frac{1}{3}-\frac{3}{11}$; (ii) $\frac{1}{x}-\frac{x}{11}$; (iii) $\frac{1}{xy}-\frac{xy}{z}$.

21. (i) $7-\frac{2}{5}$; (ii) $7-\frac{2}{x}$; (iii) $ab-\frac{c}{d}$.

22. Subtract $\frac{a^2}{3}$ from $\frac{a^2}{2}$. **23.** Add $\frac{x}{y}$ to 1.

24. Add $\frac{a}{b}$ to $\frac{c}{b}$. **25.** Subtract $\frac{1}{3x}$ from $\frac{1}{2x}$.

26. Subtract $\frac{b}{a}$ from $\frac{c}{a}$. **27.** Subtract $\frac{x}{2}$ from $\frac{1}{x}$.

Simplify the following :

28. (i) $\frac{2}{7}\times 3$; (ii) $\frac{a}{b}\times 3$; (iii) $\frac{x}{y}\times z$.

29. (i) $\frac{2}{7}\div 3$; (ii) $\frac{a}{b}\div 3$; (iii) $\frac{x}{y}\div z$.

30. (i) $3\times\frac{2}{7}$; (ii) $3\times\frac{a}{b}$; (iii) $z\times\frac{x}{y}$.

31. (i) $\frac{2}{3}\times\frac{5}{7}$; (ii) $\frac{2}{3}\times\frac{a}{b}$; (iii) $\frac{a}{b}\times\frac{2}{3}$.

32. (i) $\frac{2}{3}\div\frac{5}{7}$; (ii) $\frac{2}{3}\div\frac{a}{b}$; (iii) $\frac{a}{b}\div\frac{2}{3}$.

33. (i) $3\div\frac{2}{7}$; (ii) $3\div\frac{a}{b}$; (iii) $z\div\frac{x}{y}$.

34. (i) $\dfrac{a}{\frac{b}{c}}$; (ii) $\dfrac{\frac{a}{b}}{c}$; (iii) $\dfrac{\frac{a}{b}}{\frac{c}{d}}$.

35. (i) $\frac{a}{b}\times\frac{c}{a}$; (ii) $\frac{x}{y}\div\frac{x}{z}$; (iii) $\frac{a^2}{bc}\times\frac{b^2}{ac}$.

36. (i) $x\div\frac{x}{y}$; (ii) $xz\div\frac{x}{z}$; (iii) $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{a}$.

37. (i) $x^3\div\frac{x^2}{y^2}$; (ii) $\frac{ab}{c^2}\times\frac{ac}{b^2}$; (iii) $\frac{a^2b^2}{c}\times\frac{c^2}{ab}$.

38. (i) $\frac{10a}{b^2}\div\frac{2a}{3}$; (ii) $\frac{12bc}{35a^2}\times\frac{14ab^2}{24b^3}$; (iii) $\frac{27x^2y^2}{5z^3}\div\frac{18xy^2}{10z^2}$.

39. (i) $a-\frac{a}{4}$; (ii) $x-\frac{7}{100}x$; (iii) $y+\frac{2}{9}y$.

40. (i) $x+\frac{8}{100}x$; (ii) $\frac{2\frac{1}{2}}{100}x$; (iii) $x+\frac{x}{y}$.

41. (i) $\frac{a}{b}+\frac{a}{c}$; (ii) $\frac{x}{y}+1+\frac{y}{x}$; (iii) $a+\frac{b^2}{a}$.

42. (i) $\frac{1}{xy}+\frac{1}{x}$; (ii) $\frac{a}{a^2}+\frac{2}{a}$; (iii) $\frac{pq}{q^2}+\frac{r^2}{qr}$.

43. What fraction is x pence of y pence? By substituting in your answer, find what fraction (i) 3 pence is of 5 pence, (ii) 7 pence is of 2 pence. In the first part of the question, does it matter whether x is greater or less than y?

44. Express (i) x yards in inches, (ii) x inches in yards.

45. Express a shillings $+\, b$ pence (i) in pence, (ii) in shillings as a single fraction.

46. What fraction is x pence of y shillings ?

47. What fraction is $3x$ inches of x feet ?

48. If $\frac{x}{100}$ of a school are boys, what fraction are girls ?

49. If $\frac{1}{x}$ of a stick is cut off, what fraction remains ?

50. Does the value of the fraction $\frac{3x}{4x}$ alter, if different values are given to x ?

51. Does the value of $\frac{x+10}{5}-\frac{x}{5}$ depend on the value of x ?

52. Simplify $\frac{x}{2\frac{1}{2}}+\frac{x}{1\frac{2}{3}}$.

53. Simplify $\frac{a}{\frac{1}{3}}-\frac{b}{\frac{1}{4}}$.

54. If 8 Km. $=5$ mi., what fraction is x Km. of x mi.?

55. If 8 Km. $=5$ mi., what fraction is $8x$ Km. of $5x$ mi. ?

56. ABC is a straight line ; if $AB=a$, $BC=b$, what fraction is (i) AB of AC, (ii) AC of BC ?

BRACKETS.

EXERCISE II. k.*

To obtain the value of $2(x+y)$, *we add* y *to* x *and multiply the result by* 2.

To obtain the value of $2x+y$, *we multiply* x *by* 2 *and add* y *to the result.*

State in *words*, in a similar manner, *how to obtain* the value of the following. Do not try to simplify them.

1. $3(a-b)$. **2.** $3a \div b$. **3.** $x+2(y+z)$.
4. $x+2y+z$. **5.** $(x+2)y+z$. **6.** $\overline{x+2}y+z$.
7. $(x+2)(y+z)$. **8.** $(a+b)-c$. **9.** $c-(a+b)$.
10. $c-\overline{a+b}$. **11.** $c-a+b$. **12.** $\overline{a+b}-\overline{a+c}$.
13. $a-2(b+c)$. **14.** $a-(2b+c)$. **15.** $(a-2)(b+c)$.
16. $a-2b+c$. **17.** $\sqrt{a+b}$. **18.** $\sqrt{2(x-y)}$.

Example XVII. Express by means of brackets the remainder after subtracting x times $y-z$ from x^2.

x times $y-z$ is represented by $x(y-z)$.

$\therefore$ the remainder $=\mathbf{x^2-x(y-z)}$. *Answer.*

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* For additional examples, see E.P. 7, page 251/12.

Express *by means of brackets* the following: do *not* simplify them.

19. The remainder after $b+c$ is subtracted from $2a$.

20. The remainder after $c-d$ is subtracted from $a-b$.

21. Twice the remainder after y is subtracted from x.

22. The product of $2x$ and $x-y$.

23. Half the sum of b and c.

24. The quotient when a is divided by $x+y$.

25. The square of the sum of x and y.

26. The sum of the squares of $x+y$ and $x-y$.

27. The square of the product of 2 and $x+y$.

28. The sum of half of $x+y$ and three-quarters of $x-y$.

29. The square root of the difference of x and y. [$x>y$.]

30. The difference of the square roots of x and y. [$x>y$.]

31. The excess of 1 over $x+y$.

32. The product of two numbers, one less than x by 2 and the other the sum of a and b.

33. The product of three consecutive integers, of which a is the least.

Example XVIII. A man walks 34 miles in 3 days. He walks x miles the first day, y miles the second day; how far does he walk on the third day?

He walks $x+y$ miles the first two days.
$\therefore$ he walks $\mathbf{34-(x+y)}$ miles the third day.

Example XIX. A box of sugar weighs p lb.; the box when empty weighs q lb. How many boxes are used for despatching x lb. of sugar?

Each box contains $p-q$ lb. of sugar.

$\therefore$ $\mathbf{x\div(p-q)}$ or $\mathbf{\dfrac{x}{p-q}}$ boxes are used.

Express the answers to the following questions by means of *brackets*. Do not try to simplify the result.

34. How many pence are there in $x+y$ shillings ?

35. From a stick a inches long a part $b+c$ inches long is cut off ; how much is left ?

36. A man has walked $x+y$ miles of a journey of $a+b$ miles ; how far has he still to walk ?

37. A book costs $x-\frac{x}{4}$ shillings and is sold for 15 shillings ; what is the profit ?

38. The length of a rectangle is $a+b$ feet and the width is c feet ; what is the area ?

39. The area of a square is x^2+xy sq. feet ; what is the length of a side ?

40. Rewrite the product 871×99, putting $100-1$ instead of 99.

41. Rewrite the product $59 \times 1\frac{6}{7}$, putting $2-\frac{1}{7}$ instead of $1\frac{6}{7}$.

42. A man's rent is £x a year and his food costs £y a year ; how much is left, after paying these, out of an income of £350 ?

43. A man's income is £$(x+y)$. What is it, after it has been increased by a tenth ?

44. A man buys x copies of a book and later y copies of the same book : the bill for the whole is 35 shillings ; what was the cost of each book ?

45. A tin when empty weighs p lb. and will contain q lb. of coffee ; 50 of these tins of coffee are packed in a box weighing r lb. ; what is the total weight ?

46. A man has to pay a bill of x shillings ; a discount of y shillings is taken off the bill. He pays it with a £5 bank-note. How many shillings change does he receive ?

47. A cask when empty weighs x lb. and when full weighs y lb. ; what is its weight when half full ?

SIMPLIFICATION OF BRACKETS.

ADDITION AND SUBTRACTION.

The method of simplifying brackets should be explained orally by examples, before formal rules are given. See Introduction, p. xviii.

Example XX. In a school of x boys, y are Scotch and z are Irish; how many are neither Scotch nor Irish?

First method. There are $y+z$ Scotch and Irish boys.

$\therefore x-(y+z)$ are neither Scotch nor Irish.

Second method. There are $x-y$ boys who are not Scotch, of these z are Irish.

$\therefore x-y-z$ are neither Scotch nor Irish.

Comparing these two results, we see that

$\mathbf{x-(y+z)}$ is the same as $\mathbf{x-y-z}$.

Example XXI. In Fig. 2, $OP=a$, $OQ=b$, $RQ=c$ inches. Show that $a-(b-c)=a-b+c$.

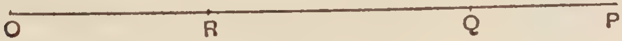

FIG. 2.

$OP=a$ inches, $OR=b-c$ inches;

$\therefore OP-OR=a-(b-c)$ inches,

i.e. $RP=a-(b-c)$ inches. ....(A)

Again, $QP=a-b$ inches;

$\therefore QP+RQ=a-b+c$ inches,

i.e. $RP=a-b+c$ inches. ....(B)

Comparing the two results (A) and (B), we see that

$$\mathbf{a-(b-c)=a-b+c}.$$

Examples 1–15 of Exercise II. 1. are intended as illustrations of the following results:

(i) $a+(b+c)=a+b+c$. (ii) $a+(b-c)=a+b-c$.

(iii) $a-(b+c)=a-b-c$. (iv) $a-(b-c)=a-b+c$.

EXERCISE II. 1.

(NOTE.—*Examples* 1–15 *should to a large extent be taken orally.*)

1. A boy divides his day into working, eating and playing. In a day of a hours he is working b hours and eating c hours.

How long is he (i) working or eating ?

,, ,, (ii) not working or eating ?

Again, how long is he (iii) not working ?

and by subtracting the numbers of hours he is eating, find how long he is (iv) not working nor eating.

Compare the two results (ii) and (iv).

2. A farmer has x horses and y cows ; he sells z cows.

(i) How many cows has he left ?

(ii) By adding this number to the number of horses, find the number of animals he has.

Again, (iii) find how many animals he had before he sold the cows.

(iv) By subtracting the number sold, find how many he had after he sold the cows.

Compare the two results (ii) and (iv).

3. $ORQP$ is a straight line, so that $OP=11''$, $OQ=6''$, $QR=2''$. What are the lengths of :

(i) $OP-OR$;

(ii) $OP-(OQ-QR)$;

(iii) $OP-OQ+QR$?

4. $ORQP$ is a straight line, so that $OP=a''$, $OQ=b''$, $QR=c''$.

(i) What line is of length $b-c$ inches ?

(ii) ,, ,, $a-(b-c)$ inches ?

(iii) ,, ,, $a-b$ inches ?

(iv) ,, ,, $a-b+c$ inches ?

(v) Express the length of RP in two different ways.

5. By a figure similar to those used in Questions 3 and 4 show that

(i) $11+(6+2)=11+6+2$.

(ii) $a+(b+c)=a+b+c$.

(iii) $11+(6-2)=11+6-2$.

(iv) $a+(b-c)=a+b-c$.

Work out Examples 6–10 in the same way as Example XX.

6. A boy has 2s. He receives from home two postal orders, one of 10s. and the other of 5s. He would have just the same money if he had received one postal order for $10+5$ shillings. How much money has he in each case ?

7. A stick is a inches long. A man cuts off a piece $b+c$ inches long. From another stick a inches long, a man first cuts off b inches and then another c inches. What length is left in each case ?

8. A man is standing at the third floor of a building in New York. He gets into the lift and goes up 6 floors, meets a friend, and they go up another 4 floors to his office. Next day, starting again from the third floor, he goes straight to his office, up $6+4$ floors. What floor does he reach in each case ?

9. Domodossola is 2155 ft. above sea-level. From Domodossola to the top of the Simplon Pass you go up 4425 ft., and from the top of the pass to Brigue you go down 4325 ft. But if you go from Domodossola to Brigue by the Simplon Tunnel you go up (4425-4325) ft. Give the height of Brigue in each case.

10. A man has £x. He buys a new camera, and pays £y for it. He then sells his old camera and gets £z for it. Instead of doing this, he might have exchanged his old camera for the new one, and paid the difference £$(y-z)$. How much would he have had left in each case ?

To show that $20+(13-6)=20+13-6$,

$$20+(13-6)=20+7=27,$$

$$20+13-6=33-6=27.$$

In a similar way show that :

11. $12+(8-7)=12+8-7$.

12. $100+(20+9)=100+20+9$ **13.** $15-(3+2)=15-3-2$.

14. $13-(12-1)=13-12+1$. **15.** $6-(5-3)=6-5+3$.

Example XXII. Simplify $(2a+5b)-(a-2b)$.

$$\begin{aligned}(2a+5b)-(a-2b)&=2a+5b-a+2b\\&=\mathbf{a+7b}.\quad \textit{Answer.}\end{aligned}$$

Simplify the following :

16. $x+(y+2z)$.
17. $x+(2y-z)$.
18. $x+(x+y)$.
19. $x+(y-x)$.
20. $x-(y-x)$.
21. $(x+1)+(x-1)$.
22. $(x+2)-(x+1)$.
23. $x^2-(2x-1)$.
24. $1-(x-1)$.
25. $(x^2-5x)-(x^2-5x)$.

The results numbered (i), (ii), (iii), (iv) on p. 25 could have been obtained from the following rules :

(A) If a bracket has a + sign in front of it and the bracket is removed, the sign of each term remains the same as it was in the bracket.

(B) If a bracket has a − sign in front of it and the bracket is removed, the sign of each term is different from what it was in the bracket.

You can use these rules to work out Exx. 26–45.

Simplify the following :

26. $a-(b+2c)$.
27. $a-(2b-c)$.
28. $2a-(a+b)$.
29. $(a+b+c)+(a-b-c)$.
30. $(a+b+c)-(a-b+c)$.
31. $(x^2+x-1)+(x^2-x+1)$.
32. $(2a+5b-c)-(a-b-c)$.
33. $x^3+x^2-(x+1)$.
34. $x^3+x-(x^2+1)$.
35. $1-3x+(3x^2-x^3)$.
36. $a^3-(a^2b-ab^2+b^3)$.
37. $(x^2-xy)-(xy-y^2)$.
38. $(x^2+2xy+y^2)-(x^2-2xy-y^2)$.
39. $(3x^2-4x+2)-(2x^2-x-3)$.
40. $x^3-2x^2+x-(x^2-2x+1)$.
41. $(x-z)-(y-z)$.
42. $(b-c)+(c-a)+(a-b)$.
43. $(2a-b-c)-(2b-c-a)$.
44. $x^2+3y^2+z^2-(3y^2-z^2)$.
45. $x^2+y^2+z^2-(2yz-2zx-2xy)$.

MULTIPLICATION AND DIVISION.

In Arithmetic, we often use such facts as

3 times 17 + 5 times 17 = 8 times 17,

or (3 + 5) times 17.

And, in the earlier part of this chapter,

$$3x + 5x = 8x \quad \text{or} \quad (3+5)x.$$

These are special cases of the general fact that

a times c + b times c = $(a+b)$ times c,

or $ac + bc = (a+b)c$.

This can be illustrated by a figure.

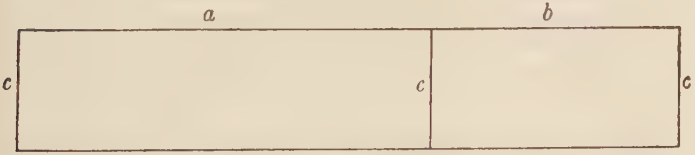

FIG. 3.

The diagram represents a rectangle of length $a+b$ inches and breadth c inches, divided into two smaller rectangles. The areas of the two smaller rectangles are ac square inches and bc square inches.

The area of the whole rectangle is $(a+b)c$ or $c(a+b)$;

$$\therefore\ ac + bc = (a+b)c = c(a+b).$$

The reader should draw a similar diagram showing that

$$ac - bc = (a-b)c = c(a-b).$$

Again, since $\qquad (a+b)c = ac + bc,$

$$(ac+bc) \div c = a + b.$$

Similarly $\qquad (ac-bc) \div c = a - b.$

We will now collect the results of this chapter.

(i) $\mathbf{a+(b+c)=a+b+c}$. (v) $\mathbf{(a+b)c}$ **or** $\mathbf{c(a+b)=ac+bc}$.

(ii) $\mathbf{a+(b-c)=a+b-c}$. (vi) $\mathbf{(a-b)c}$ **or** $\mathbf{c(a-b)=ac-bc}$.

(iii) $\mathbf{a-(b+c)=a-b-c}$. (vii) $\mathbf{(ac+bc)\div c=a+b}$.

(iv) $\mathbf{a-(b-c)=a-b+c}$. (viii) $\mathbf{(ac-bc)\div c=a-b}$.

These may be summed up in the following rules :—

Rule I. If a bracket has a + sign in front of it and the bracket is removed, the sign of each term remains the same as it was in the bracket.

Rule II. If a bracket has a − sign in front of it and the bracket is removed, the sign of each term is different from what it was in the bracket.

Rule III. If an expression in a bracket is multiplied or divided by a number and the bracket is removed, each term of the expression must be multiplied or divided by that number.

These results cannot be used for all values of a, b, c. Thus if we put $a=2$, $b=4$, $c=6$, (ii) becomes $2+(4-6)$, *i.e.* $4-6$ added to 2. But $4-6$ has at present no meaning. Again (iii) becomes $2-(4+6)$, *i.e.* $2-10$. This also has no meaning.

When we have taken special cases of these, we have usually chosen whole numbers instead of a, b or c, because it makes the work simpler. But these results are true for any values of a, b, or c.

E.g. $10-(2\frac{3}{4}+1\frac{5}{6})=10-2\frac{3}{4}-1\frac{5}{6}$,

$$(2\tfrac{1}{2}+3\tfrac{1}{5})7=2\tfrac{1}{2}\times 7+3\tfrac{1}{5}\times 7.$$

Example XXIII. Multiply $2x-5y$ by $3x$.

$$\begin{aligned}(2x-5y)3x&=2x\times 3x-5y\times 3x\\&=\mathbf{6x^2-15xy}.\end{aligned}$$

Example XXIV. Simplify $3(2x-5y)-2(x+3y)$.

$$\begin{aligned}3(2x-5y)-2(x+3y)&=(6x-15y)-(2x+6y)\\&=6x-15y-2x-6y\\&=\mathbf{4x-21y}.\end{aligned}$$

To avoid mistakes, it is important to multiply in the first line and to remove the brackets in the second. Do not try to do both in one line.

EXERCISE II. m.*

Simplify the following :

1. $(x+y)z$. **2.** $z(x+y)$. **3.** $(x-y)z$.

4. $z(x-y)$. **5.** $2a(b-c)$. **6.** $(2x-y)2x$.

7. $(2x+3)x^2$. **8.** $(x^2-2x+1)x$. **9.** $(xz+yz)\div z$.

10. $(a^2-ab)\div a$. **11.** $(x^2-2x)\div x$. **12.** $(2x^3-3x)\div x$.

13. $(4x^3+4x^2-8x)\div 4x$. **14.** $(a^2-ab-ac)\div a$.

15. $(x-y)2xy$. **16.** $(a^2-a)\div a$.

17. $a^2-a(a-b)$. **18.** $a(a-b)+ab$.

19. $x(x-1)+2x$. **20.** $x(2x-1)+2(x-1)$.

21. $(x-y)x+(x+y)y$. **22.** $3(2x-5y)-2(x+4y)$.

23. $(10x^3-15x^2y)\div 5x$. **24.** $(8x^2y^2+6xy)\div 2xy$.

25. $4x(x+2y)-3x(x-y)$.

26. $5(2a-3b+4c)-3(a-2b-6c)$.

27. $2a^2(a+3b)+3b(b^2-2a^2)$. **28.** $\left(1-\frac{1}{x}\right)x$.

29. $\left(x-\frac{1}{y}\right)y$. **30.** $\left(\frac{1}{a}+\frac{1}{b}\right)ab$.

It is sometimes useful to have more than one set of brackets in an expression.

Example XXV. A book costs 7 shillings and the discount is 1s. 9d. ; a man orders 6 copies to be sent to him by post (postage 10d.) ; express by brackets the change he will receive out of £2.

Each book when the discount is taken off costs 7s. – 1s. 9d.;

∴ the parcel of 6 books with the postage costs

6(7s. – 1s. 9d.) + 10d. ;

∴ the change he receives is **£2 – (6(7s. – 1s. 9d.) + 10d.).**

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* For additional examples, see E.P. 8, page 251/13.

To prevent confusion, it is better, when one set of brackets contains another between them, to choose brackets of different shapes.

Thus $£2 - \{6(7\text{s.} - 1\text{s. } 9\text{d.}) + 10\text{d.}\}$

or $£2 - (6\ \overline{7\text{s.} - 1\text{s. } 9\text{d.}} + 10\text{d.})$.

In this expression, the inside pair of brackets shows that 7s. – 1s. 9d. is to be multiplied by 6, and the outside pair of brackets shows that the sum of this and 10d. is to be subtracted from £2.

Example XXVI. State in words the process of simplifying $a - \{b(c+d) + e\}$

Multiply the sum of c and d by b.
Add e to this sum.
Subtract the result from a.

EXERCISE II. n.

State in words in a manner similar to that of Example XXVI., the process of simplifying

1. $a + [b - (c + d)]$. **2.** $a[b(c+d) + e]$.

3. $[a(b+c) - d] \div e$. **4.** $a \div [b - c - d]$.

5. Show that the number of pounds in £3. 15s. 7d. is
$$\{(3 \times 20 + 15)12 + 7\} \div 240.$$

6. Express by brackets, as in Ex. 5, the number of tons in (i) 3 tons 7 cwt. 1 qr.; (ii) x tons y cwt. z qr.

7. A sheet of paper is a inches long and $b + c$ inches wide; a square hole, edge x inches, is made in it. Express by brackets the area of the remainder in sq. feet.

Simplify the following:

8. $a + b - \{a - (b + c)\}$ **9.** $2(\overline{x+y} + \overline{x-y})$.

10. $3[4x - 2(x - y)]$. **11.** $3b - [a + 2(b - a)]$.

12. $b - [3c - 2(b + c)]$. **13.** $2x(x - \overline{x-y} - y)$.

14. $[3x(x - 2y) + 9x^2] \div 3x$. **15.** $a - [b - (c - d) + e]$.

16. $4(x^2 - 5xy + y^2) - 3y(2x + y)$.
17. $x - x(2 - \overline{1 + y})$.

18. $[a(a - 4b) + 7a^2c] \div a$.
19. $3(x - 2y) - 2(x - y)$.

20. $1 - (4 - [1 + 1] - 1)$.
21. $[a^2(1 - a) + b(a + a^2)] \div a$.

In Examples 22–26, first express the result by means of brackets without simplifying, and then simplify :

22. Subtract $a - (b - c)$ from $b - (a - c)$.

23. Add $3a + 2(b - c)$ to $4b - 3(a - c)$.

24. By how much does $x^2 + y(2x - y)$ exceed $x(x + y) - 2y^2$?

25. From 1 subtract $a(b - c) - d$.

26. From the sum of $x(x - 2y) - y^2$ and $3x^2 - 2y(x + y)$ subtract $2x(x - y) - y(2x + y)$.

MISCELLANEOUS EXAMPLES II. p.

1. A boy starts the term with $10x$ shillings. He spends x shillings at once and later on $3x$ shillings. How much has he left ?

2. A man is x years old. How old will he be in one year's time ? How old was he 3 years ago ?

3. Add together x^2, $3x^2$ and $4x^2$. Subtract $6x^2$ from the result.

4. A man cycles at $3x$ miles an hour. How far will he go in 5 hours ? How long will he take to go $10x$ miles ?

5. Simplify $5x - 2(2 - x)$. Find the value of this when $x = 1$; also of the answer when $x = 1$.

6. A walks a miles an hour ; B walks b miles an hour faster than A. How fast does B walk ? How far will B go in x hours at this rate ?

7. Subtract $a - 3b$ from $2a - b + c$.

8. A tank contains $10x^2$ gallons of water. How many buckets, each containing $2x$ gallons, can be filled from it ?

9. How many inches are there in a stick a ft. b in. long ? How many inches in a stick twice as long ?

10. The length of a rectangle is $3x$ inches and its width is $2x$ inches ; what is its perimeter ?

11. Add together $7ab$ and ab, and subtract $5ab$ from the result.

12. How many pence are there in (i) y shillings, (ii) z half-crowns ?

13. Which of the following are factors of $12xy$?—$24xy$, $6x$, 12, $3x^2$, $12x^2y^2$, $4xyz$, xy, $60xy^2$, $2xz$. Which are common factors of $12xy$ and $12xz$?

14. Find the value of (i) $3x^2+4x^2$, (ii) $3x^2+4x$, (iii) $3x^2 \times 4x$.

15. A boy is now $2x$ years old ; what will be his age in y years ?

16. Write down the value of (i) $4 . 10+7$, (ii) $6 . 10+2$, (iii) $3 . 10+9$. How would the number $10x+y$ be written, if x and y were whole numbers, not greater than 9 ?

17. Simplify (i) $50x^2yz \div 2x$, (ii) $(3x^3y^3)^2$, (iii) $\frac{2}{5} \times x$.

18. A pound of tea costs x shillings and a pound of coffee y shillings ; what is the cost of 2 lb. of tea and 3 lb. of coffee ? what is the cost of a lb. of tea and b lb. of coffee ?

19. There are a boys in a school ; b leave and c new boys come ; how many boys are there now in the school ?

20. If $a=17$, $b=10$, $c=7$, find the value of (i) $a-b+c$, (ii) $a-(b+c)$.

21. Find the H.C.F. of $8ayz$ and $10bxz$. Reduce $\frac{8ayz}{10bxz}$ to its lowest terms.

22. There are $10a+b$ people at a meeting ; there are $7a$ men and the rest are women ; how many women are there ?

23. How many pence are there in £$a+b$s. ?

24. A plank $30x$ inches long is cut into three pieces ; two of them are x inches and $10x$ inches long ; what is the length of the third piece ?

25. From $5a+b$ take $3a+c$.

26. Simplify (i) $\frac{xy}{axy}$, (ii) $\frac{x}{y} \times by$, (iii) $\frac{b}{c} \div \frac{b}{d}$.

27. On a bookshelf are $2x$ one-volume novels and $3x$ two-volume novels. How many volumes are there altogether ?

28. Simplify (i) $1-\frac{3}{5}$, (ii) $1-\frac{4}{9}$, (iii) $1-\frac{m}{n}$.

29. A bucket holds x lb. of water. A cistern when empty weighs y lb. and holds z buckets of water. What will be its weight when full ?

30. A man buys 24 lb. of butter ; $3x$ lb. of this prove to be bad ; how much is fit to use ? If I buy a third of the good butter, how much do I buy ? If I give as. per lb. for it, how much does it cost ?

31. ABC is a straight line ; D is the middle point of AC. If $AB=8x$, $BC=6x$ inches, find DB.

32. Express by means of brackets the remainder after subtracting three times the sum of m and n from five times $m-n$.

33. Find the value of (i) $x-2y+z$, (ii) $x-2(y+z)$, when $x=13$, $y=3$, $z=1$.

34. Express by means of a bracket the number of shillings in $b+2c$ pence.

35. Fill in the blank space in

$$3\tfrac{1}{2}\times 17+2\tfrac{3}{4}\times 17-4\tfrac{1}{4}\times 17=(\qquad)17.$$

What is the value of $3\frac{1}{2}\times 17+2\frac{3}{4}\times 17-4\frac{1}{4}\times 17$?

36. Simplify

$$\text{(i) } \frac{4a}{11}+\frac{6a}{11},\quad \text{(ii) } \frac{4a}{11}+\frac{3a}{22},\quad \text{(iii) } \frac{5}{11a}-\frac{9}{22a}.$$

37. Find the L.C.M. of $4p^2$ and $2pq$. Simplify $\frac{1}{4p^2}+\frac{1}{2pq}$.

38. A man buys a penny stamps and b halfpenny stamps ; how much change will he receive from 10s. ? Answer in pence.

39. Multiply $12a-b$ by $2ab$. What is the quotient if $24a^2b-2ab^2$ is divided by $12a-b$?

40. How many hours are there between $(n-2)$ o'clock and n o'clock ?

41. Simplify

$$\text{(i) } \frac{b}{a}+\frac{c}{a}+\frac{d}{a},\quad \text{(ii) } \frac{p}{q}\times\frac{q}{r},\quad \text{(iii) } 1-\frac{2m}{4m}.$$

42. Simplify (i) $x(x^2-2x+1)+2(x-1)$.

(ii) $\frac{1}{a}(2a^2+ab)-\frac{1}{b}(ab+2b^2)$.

43. Divide $(2xy)^2-(4yz)^2$ by $4y^2$.

44. Express by means of brackets the following. Multiply $x-y$ by $2x$, add y^2 and subtract the result from $4a^2$.

45. Express by means of brackets the number of shillings left after taking 17 shillings from £$(x+1)$.

46. Express, by means of a bracket, the square of the difference between $2x$ and y. Find the value of it when $x=1$ and $y=\frac{1}{2}$.

47. A is walking $x-1$ miles an hour to Brighton and is 10 miles from Brighton. How far will he be from Brighton two hours later?

48. An engine pumps x gallons of water per minute, and a second engine pumps 50 gallons more than the first each minute. How much will the two together pump in an hour?

49. Write down (i) the square of $5ab^3$, (ii) a square root of $9x^4y^4z^4$.

50. Simplify

$$\text{(i) } \frac{a^2}{bc}\times b, \quad \text{(ii) } \frac{1}{a}-\frac{1}{ab}, \quad \text{(iii) } \frac{x+y}{3}+\frac{x-y}{3}.$$

51. Show that the number of pence in £4. 8s. 5d. is $(4\times 20+8)12+5$. Write down in a similar way the number of pence in £x. ys. zd.

52. Rewrite the product 1359×49, putting $50-1$ for 49. Get rid of the bracket, and so find the value of 1359×49.

53. (i) Find the H.C.F. of $12ax^2$ and $20axy$, (ii) simplify $\frac{12ax^2}{20axy}$. Divide $12ax^2+20axy$ by $4ax$.

54. Simplify

$$\text{(i) } 4ab^2c\times 4a^2bc^2, \quad \text{(ii) } \frac{8acy}{4y}, \quad \text{(iii) } m\div\frac{1}{n}.$$

55. The length of a rectangle is a ft. and the width is $(a-3)$ ft. (i) What is its perimeter? (ii) What is its area?

56. Simplify

$$\text{(i)}\ \left(x \div \frac{1}{y}\right) + \left(y \div \frac{1}{x}\right), \quad \text{(ii)}\ \frac{2b^2}{c^2} - \left(\frac{ab}{c} \div \frac{ac}{b}\right).$$

57. Write down a general statement that includes the following as special cases:

$$a^7 \times a^3 = a^{10}, \quad a^{12} \times a^5 = a^{17}, \quad a^{26} \times a^2 = a^{28}, \quad a^{10} \times a^{40} = a^{50}.$$

58. Simplify (i) $\frac{a}{x} - \frac{a-10}{x}$, (ii) $x - \frac{x}{100}$.

59. How much is $a-b$ greater than $a-2b$? Show that your answer is right for the special case in which $a=10$, $b=3$.

60. Multiply $2a^2-2ab$ by b and divide the result by $2ab$.

61. A rectangular block of wood is a in. long, b in. wide, c in. thick. Write down the sum of the areas of all the faces. What does your answer become, if b and c are each equal to a?

62. Each shorter side of a rectangle is a in. and each longer side $(a+b)$ in.; a square has the same area as this rectangle. What is the length of a side of the square?

63. Show that $(a+b)^2 = a^2+2ab+b^2$ when $a=6$, $b=4$.

64. What fraction with $a+b$ as numerator has the same value as $\frac{1}{c}$?

65. What fraction with $ac+bc$ as denominator has the same value as $\frac{1}{a+b}$?

66. What fraction with z as denominator has the same value as $\frac{x}{y}$? Use your answer to express $\frac{2}{3}$ as a fraction with 100 as denominator.

67. Take the number a; multiply it by 2 and add b. Take this result, multiply by 2 and add b. Take this second result, multiply by 2 and add b. Do not simplify your results, but use brackets.

CHAPTER III.

GENERALISED ARITHMETIC.

PROBLEMS. FORMULAE.

We shall now consider arguments and problems in Algebra which correspond to the types of problems usually worked in Arithmetic.

Before beginning the chapter the pupil is advised to read carefully the following general remarks.

I. Remember that the letters which you use in Algebra always stand for *numbers* and not for *numbers-of-things*. *E.g.* they may stand for 2, 3, 57, $\frac{3}{4}$, 1·06, etc., which are numbers, but not for numbers-of-things, such as two-pence, three-gallons, 57-days, $\frac{3}{4}$-mile, etc.

Any one of these, *e.g.* three gallons, implies *two* things, (*a*) a number, in this case three; (*b*) a unit, in this case a gallon. A letter can stand for the first of these two only.

If some one asked you the length of a field, it would not be enough to say that it was 250, since that might mean 250 feet, or 250 yards, or 250 metres, etc. In the same way, it is not enough to say that the speed of a train is x, but it is quite definite to say that it is x miles an hour or x metres a minute. You can talk about x miles, x gallons or x days. But never say—let x be the cost of a cow, or—let x be the time a man takes over a journey. x is merely a number, and if you are dealing with numbers of things, you must in addition state the unit you are using—let x pounds be the cost of a cow, let x hours be the time a man takes over a journey, etc.

It is therefore important to remember that:

All letters used stand for numbers only.

In every problem dealing with numbers of things, the units should always be stated.

We shall use the word *quantity* to mean 'number of things.' Thus x pounds is a quantity, but x is a number.

II. In Arithmetic we are concerned with problems which deal with definite numbers; the nature of the method employed, however, is not in general affected by the value of these numbers.

Consider the following:

How many pence are there in 7 shillings? 7×12 pence.

If instead of 7 we put any other number, we should still find the number of pence by multiplying this number by 12. Consequently we have the following:

How many pence are there in x shillings? $x \times 12$ pence.

From these remarks we can deduce the following hint:

If you do not see how to work out a question in which letters are used, invent for yourself a similar problem which contains numbers instead of letters and solve it; then use the same method for the given problem.

EXERCISE III. a.

Criticise any of the statements 1-6 that seem unsatisfactory, and suggest improvements.

1. If each book costs x, then two books cost $2x$.

2. If $2x$ is an even number, then $2x+2$ is also an even number.

3. If the length of a room is a feet and the width b feet, the area is ab.

4. The temperature was x and rose 3 degrees, so that it is now $x+3$.

5. Let the price of sugar be x.

6. Let the price of sugar per pound be x.

7. Let the numerator of a fraction be x; then, if its value is $\frac{1}{2}$, its denominator must be $2x$.

8. If the number of inches in a side of a square is x, its area is x^2.

9. Let one book cost x, the other y; then, since one cost 3s. more than the other, $x-y=3\text{s}$.

10. In 1 mile the speed of the train has increased x miles.

UNITARY METHOD, ETC.

Example I. If a train is travelling at the rate of a miles an hour, how long will it take to go x miles?

Invent a similar question, putting numbers instead of the letters a and x: If a train is travelling at 40 miles an hour, how long will it take to go 57 miles?

It goes 40 miles in 1 hour;
∴ it goes 1 mile in $\frac{1}{40}$ hour;
∴ it goes 57 miles in $\frac{1}{40}\times 57$ hours,
i.e. $\frac{57}{40}$ hours.

If instead of 40 we put any other number, that number will always occupy the same position in the answer, *i.e.* it will *always* be in the denominator. So, if instead of 57 we put any other number, that number will *always* be in the numerator. Many people think that the position of the number which is taken instead of 40 depends on whether it is greater or less than 57. This is not so. It will *always* be in the denominator.

We can now work out the original question.

The train goes a miles in 1 hour;
∴ it goes 1 mile in $\frac{1}{a}$ hours;
∴ it goes x miles in $\frac{1}{a}\times x$ hours,
i.e. $\frac{x}{a}$ **hours.**

EXERCISE III. b.*

1. If 1 gallon of water weighs 10 lb., how much will x gallons weigh ?

2. If 10 yards of silk cost 30 shillings, how much will x yards cost ?

3. If 10 yards of silk cost 30 shillings, how much can be bought for y shillings ?

4. If a yards of silk cost 30 shillings, how much will x yards cost ?

5. If a yards of silk cost b shillings, how much can be bought for y shillings ?

6. The railway fare for a journey of x miles is y shillings ; how much will a journey of $3x$ miles cost at the same rate ?

7. A clock loses x seconds an hour ; how many seconds will it lose in 5 days ?

8. If a sheet of cardboard 8 in. long and 5 in. wide weighs 4 oz., what is the weight of a sheet $3x$ in. long and 2 in. wide ?

9. A man walks 3 miles an hour. How many minutes will he take to walk (i) 1 mile, (ii) x miles ?

10. A man cycles 12 miles an hour ; how far will he go in y minutes ?

11. If £1 $= x$ francs, how many pounds are there in 1 franc ?

12. A cubic foot of stone weighs x lb. ; find the number of cubic feet in a block of stone weighing a ton.

13. If two-thirds of the population of a town is x, what is the whole population ?

14. How many coins, each weighing $\frac{2}{5}$ oz., can be made from x oz. of metal ?

15. If 1 oz. $= x$ gm., find the number of pounds in a kilogram.

16. If 8 km. $=$ 5 miles, how many metres are there in $x+y$ miles ?

17. If x men can clear a field of stones in 5 days, how long will $5x$ men take ? How long will y men take ?

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* For additional examples, see E.P. 9, page 251/14.

18. The railway fare for a journey of x miles is y shillings; what is the fare for y miles at the same rate?

19. A town has provisions for a men for b days; how long would these provisions last if there were x men?

20. On a railway the telegraph poles are placed d feet apart, and a train passes n poles per minute. Find the speed of the train in miles per hour. If $d=176$ and $n=20$, find the speed.

21. My income after paying an income tax of a shilling in the pound was £x. What was it before I paid the income tax?

22. If a ton of beetroot produce x lb. of sugar, how much beetroot will be required to produce a ton of sugar?

23. If 30 lb. of tea cost £3. 10s., how much can be bought for £$x+y$s.?

24. If I buy a War Savings Certificate for 15s. 6d. now, I get £1 for it in five years' time. How many such certificates must I buy so as to get £x in five years? How much shall I get in five years' time if I spend £x now in buying certificates?

25. If I buy x lb. of sugar at $4\frac{1}{2}$d. per lb. and y lb. at 5d. per lb., what is the average cost of the sugar per lb.?

AREAS, VOLUMES, ETC.

We shall assume the following formulae:

1. The area of a triangle of base b ft. and height h ft. is $\frac{1}{2}bh$ sq. ft.,

2. The circumference of a circle of radius r ft. is $2\pi r$ ft.,

3. The area of a circle of radius r ft. is πr^2 sq. ft.,

4. The volume of a figure of constant cross-section A sq. ft. (*e.g.* a cylinder or prism) and length l ft. is Al cu. ft.

Instead of ft., sq. ft., cu. ft., in these formulae we may use any corresponding units.

π stands for the number 3·14159...., and is sometimes taken to be 3·14 or $3\frac{1}{7}$.

Example II. If x tiles, each 4 in. square, are needed to cover a path, find the area of the path.

The area of each tile is $\frac{1}{3} \times \frac{1}{3}$ sq. ft.,
i.e. $\frac{1}{9}$ sq. ft.;
$\therefore$ the area of the path is $\frac{1}{9} \times x$ sq. ft.,
or $\mathbf{\frac{x}{9}}$ **sq. ft.**

Example III. The volume of a glass tube is x c.c. and its length is y metres; find its cross section.

The volume is x c.c.
The length is $100y$ cm.;
$\therefore$ the cross-section is $\mathbf{\frac{x}{100y}}$ **sq. cm.**

EXERCISE III. c.*

1. A wall is 8 ft. high and x yd. long; what is its area in square feet?

2. A path is 4 ft. wide and x yd. long; what is its area in square yards?

3. A tile is x inches square; how many of them go to the square foot?

4. What is the cost (in pounds) of a carpet a ft. long and b ft. wide, at 5s. per sq. ft.?

5. The area of a door is a sq. ft. and its height is 7 ft.; what is its width?

6. Pieces of steel 2 cm. wide and 12 cm. long are required for aeroplane fittings. How many of these can be cut from a sheet of steel x cm. square, if there is no waste?

7. The page of a book is 7 in. by $4\frac{1}{2}$ in. How many square feet of paper will be required for a book of x pages?

8. A brick is x in. by y in. by z in. What is its volume?

9. If x cu. ft. of water are pumped into a tank a ft. long and b ft. wide, what will be the depth of the water?

10. A vessel 20 cm. long and 12 cm. wide is filled with mercury to a depth of h millimetres. How many c.c. of mercury are there in it?

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* For additional examples, see E.P. 10, page 251/16.

11. The cross-section of the inside of a glass tube is A sq. cm. and it is a metre long. How many c.c. will it hold?

12. How many bricks 9 in. by 4 in. by 3 in. will be required to make a stack of bricks x yd. long, y ft. high and 18 in. thick?

13. A field is $2x$ yd. long and x yd. wide; find the area of a fence, h ft. high, round the field.

14. A garden path l yd. long and $1\frac{1}{2}$ yd. wide is covered with a layer of gravel 2 in. thick. How many loads of gravel will be required if a load of gravel contains a cubic yard?

15. The thickness of a sheet of paper is $\frac{1}{x}$ of an inch. How many sheets will a pile of paper, y in. high, contain?

16. How much will it cost to fill a swimming bath x ft. long, y ft. wide and h ft. deep, if a cubic foot of water $=6\frac{1}{4}$ gallons and 1000 gallons cost 1s. 6d.?

17. If a man pushes a mowing machine 10 in. wide a distance l yards, how many square yards has he mown?

18. A 14-inch mowing machine must be pushed n times across a lawn in order to mow it; how many times will be required for a 10-inch machine?

19. What is the circumference of a circle, of diameter d in.?

20. What is the area of a circle, of diameter d in.?

21. A cubic foot of copper is drawn out into a wire, the cross-section of which is $\frac{1}{x}$ sq. in. What will be its length?

22. The base of a triangle is $2x$ in. and its height $x-y$ in.; what is its area?

23. A block of stone is a ft. long, b ft. wide and h ft. high. What is the area of its surface?

24. The minute hand of a clock is r ft. long. How far does the end of it move in 20 minutes?

25. A water tank is in the form of a circular cylinder, of r ft. radius and height h ft. How many cubic feet of water will it hold? How many gallons will it hold, if 1 cu. ft. $=6\frac{1}{4}$ gallons?

PERCENTAGE. INTEREST.

If the pupils have not done the corresponding work in Arithmetic, this section can be postponed.

Example IV. How much per cent. is x of $5x$?

$$\text{As in Arithmetic, it is } \frac{x}{5x} \times 100\,\%,$$
$$\textit{i.e.}\ \tfrac{1}{5} \times 100\,\%,$$
$$\textit{i.e.}\ 20\,\%.$$

Example V. A horse is sold for £a which cost £b; find the profit per cent.

Take a similar question in Arithmetic.

A horse is sold for £37 which cost £32; find the profit per cent.

The profit is £(37 − 32).
The cost price is £32;

$$\therefore \text{ the profit is } \frac{£(37-32)}{£32} \times 100\,\%.$$

In the given example,

The profit is £$(a-b)$.
The cost price is £b;

$$\therefore \text{ the profit is } \frac{£(a-b)}{£b} \times 100\,\%,$$
$$\textit{i.e.}\ \frac{a-b}{b} \times 100\,\%, \text{ or } \frac{100(a-b)}{b}\,\%.$$

EXERCISE III. d.

1. What is 10 per cent. (i) of 80, (ii) of x?

2. What is the result of increasing by 20 per cent. (i) 60, (ii) x?

3. How much per cent. is a of b?

4. Express as a percentage, (i) $\frac{1}{2}$, (ii) $\frac{4}{5}$, (iii) $\frac{1}{x}$, (iv) $\frac{x}{y}$.

5. Find the Simple Interest on £200 at x per cent. for 3 years.

6. Find the Simple Interest on £P at 3 per cent. for 1 year.

7. Find the Simple Interest on £P at r per cent. for 3 years.

8. What is the number that exceeds n by 5 per cent. ?

9. If x per cent. of a candle is burnt, what percentage of it is left ?

10. If x per cent. of a candle is burnt every hour, how long will the whole candle last ?

11. If n seeds are sown and 75 per cent. of them come up, how many of them are wasted ?

12. If a man lends £x at 5 per cent., show that he gets xs. a year interest.

13. Tea is bought at a shillings per lb. and sold at a profit of 25 per cent. What are 8 lb. sold for ?

14. A debtor can pay as. + bd. in the pound ; how much per cent. can he pay ?

15. A mixture is made of a lb. of copper and b lb. of tin. How much per cent. of the mixture is copper ?

16. If a man puts £x in the Post Office Savings Bank, he gets $6x$ pence every year as interest. If he puts in £10, how much interest will he get ? What is the rate per cent. per annum?

17. A pig cost £$5x$ and is sold at a profit of 30 per cent. ; find the profit.

18. The price of cycles is increased 5 per cent. If the old price of a cycle was £x, what is the new price ?

19. What will be the Simple Interest on £100 for n years at r per cent. ? What will be the amount after n years ?

20. In a school of 500 boys, x are on the Classical Side. What percentage of the school is not on the Classical Side ?

21. A bookseller deducts $5p$ per cent. from the published price of his books for cash. How many shillings in the pound is this ?

22. Of the people of the British Isles m per cent. are women, and of the women n per cent. are married. What percentage of the population are the married women ?

23. To 100 grams of gold x grams of copper are added. What percentage of the alloy is copper ?

24. Find the value (in shillings) of a per cent. of £10 + b per cent. of £7. 10s.

25. The price of wheat has increased x per cent. in the last two years. It is now 60s. per quarter. What was it two years ago ?

FORMULAE.

As we have seen in Chapter I., it often happens that a number of different arithmetical results all obey the same law. This law can sometimes be stated by using letters. The result of doing this is to make what is called a *Formula.* In other words, a formula is a way of stating the method (or law) by which a certain result is to be obtained.

Example VI. To find the law for Simple Interest.

Here the problem is to find the way in which the interest is connected with the sum of money lent, the time during which it is lent and the rate per cent. at which interest is paid.

In order to obtain a general law, *i.e.* one which applies to all cases, we make use of letters.

Let a sum of £P be invested for T years at R per cent., and let the interest be £I.

What connection is there between P, T, R, I ? If we can find out this, we have obtained the formula for which we are looking.

The interest on £100 for 1 year is £R.
,, ,, for T years is £$R \times T$.
,, £1 ,, £$\frac{R \times T}{100}$.
,, £P ,, £$\frac{P \times R \times T}{100}$.

But the interest is £I ;

$$\therefore\ I = \frac{P \times R \times T}{100}.$$

This is the required formula.

When the formula has been obtained, every question, to which this law applies, can be solved by substituting in the formula special numbers instead of the letters.

For example, the interest on £250 at 3 per cent. for $1\frac{1}{2}$ years is £$\frac{250 \times 3 \times 1\frac{1}{2}}{100}$.

EXERCISE III. e.

1. The area of the floor of a room x ft. long and y ft. wide is A sq. ft. What relation connects A, x, y? [This is the formula for the area of a rectangle.]

2. Find a formula for the volume (V cubic feet) of a rectangular block of given length, breadth and height. [Represent length, breadth and height by means of letters, and then state a relation between them and V.]

3. 8 kilometres may be taken to be equal to 5 miles. Find a formula which will express kilometres in miles. [In other words, if m miles and k kilometres are the same length, find m in terms of k.] Use this formula to find the number of miles in 20 Km.

4. A pint of water weighs a pound and a quarter. If P pints of water weigh W lb., find a relation between P and W.

5. A tradesman takes two pence in the shilling off the price 'marked up.' Find the formula giving the price (p shillings) paid for an article marked x shillings. Use the formula to find the price paid for an article marked 18s.

6. Find the formula for the perimeter of a field whose length and breadth are given.

7. Find a formula for expressing in farthings per pound a price given in shillings per cwt. [x shillings per cwt., how many farthings is this per lb.?] Use your formula to express 21s. per cwt. in farthings per pound.

8. Find the formula giving the number which exceeds x by R per cent. Use it to find the wages of a man who used to earn 24s. per week and has had them increased by $12\frac{1}{2}$ per cent.

9. The formula giving the volume (V cu. in.) of a sphere of radius r in. is $V = \frac{88}{21} \cdot r^3$; find the formula giving the volume of a sphere of diameter d in.

10. If two angles of a triangle are given, find a formula for the third angle.

11. Find a formula for the average of four numbers.

12. Find a formula for the sum of the areas of all the faces of a cube, given the length of an edge.

13. Given the length, breadth and height of a room, find the formula for the area of its walls.

14. Find a formula for the length of the day, given the time of sunrise. [You may assume that 12 o'clock is the middle of the day.] If the sun rises at 5 o'clock, find (from your formula) the length of the day.

15. The barometer at sea level reads 30 in., and for moderate heights you may assume that it decreases 1 in. for every 900 ft. that you rise. Find the formula giving the height (in feet) in terms of the barometer reading (in inches).

16. The circumference of a bicycle wheel is 88 in. Find a formula giving the speed in miles per hour, when it is known that the wheel makes r revolutions in 15 seconds If it makes 56 revolutions in 15 seconds, how fast is it going ?

17. An author gets no profit on the first two thousand copies of his book and 5d. on each copy after that If he sells n thousand copies, find a formula giving his profit in pounds.

18. If BAC is any angle and CA is produced to E, and AD bisects the adjacent angle BAE. find a formula for the angle DAC, in terms of BAC.

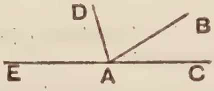


FIG. 4.

19. On March 1st the length of the day is about $10\frac{3}{4}$ hr., and increases about 4 min. each day. Find a formula for its length on the nth day of March.

20. To get local time from Greenwich time you add on 4 min. for every degree of longitude that you are east of Greenwich. What is your local time in l degrees east longitude when it is 6 o'clock at Greenwich ?

MISCELLANEOUS EXAMPLES.

EXERCISE III. f.

1. What is the angle (in degrees) between the hands of a clock at (i) 1 o'clock, (ii) 5 o'clock, (iii) x o'clock (if x is an integer not greater than 6) ?

2. A ton of coal costs x shillings in July but y shillings in December. How much does a man save if he buys 50 tons in July instead of December ?

3. A tank contains x cu. ft. of water. If it is 4 ft. long and 3 ft. wide, what is the depth of the water ?

4. The average age of n boys is 15 years. What is the sum of their ages ?

5. A man's salary is £a for his first year and increases £b every year. What will it be for his fifth year ?

6. The income tax is 2s. in the pound. If my income is £x and I pay tax on all of it, how many pounds do I pay ?

7. If 25 fr. are exchanged for £1, how many shillings shall I get for x francs ?

8. A ship requires 40 tons of coal per day. How long will x tons last ?

9. How many two-shilling pieces have the same value as x half-crowns ?

10. The perimeter of a square is $2a$ inches. What is its area ?

11. The pressure of the air is x gm. per sq. cm. Express it in kilograms per square metre.

12. A penny weighs $\frac{1}{3}$ oz., a halfpenny $\frac{1}{5}$ oz. Show that x pennies and x halfpennies together weigh more than $\frac{x}{2}$ oz.

13. The base of a triangle is $2a$ ft. and its height is equal to its base. Find its area.

14. Eggs are xd. per dozen ; how many do you get for a shilling ? If their price goes up 1d. per dozen, how many do you now get for a shilling ?

15. Express xs. in the pound as a percentage.

16. The circumference of a circle is l ft. What is its diameter in inches? Give the answer in terms of π.

17. From A to B is s miles. If you walk from A towards B at $3\frac{1}{2}$ miles an hour for x hours, how far are you from B?

18. What is the formula giving the simple interest in terms of the principal, rate and time? Find the simple interest when the principal, rate and time are £20, 4 per cent. and 6 months.

19. If 8 Km. = 5 miles, show that to find the number of kilometres in x miles we can take $2x$, and from it subtract a fifth of $2x$.

20. A roll of wall-paper is 12 yd. long and 21 in. wide. Show that to cover x sq. yd. of wall will require $\frac{x}{7}$ rolls of paper.

21. Find the number of days from the 15th of January to the nth of February, counting only one of the 'end days.'

22. If a verst $= \frac{2}{3}$ mile, how many versts are there in x miles?

23. An Ordnance Survey map is x in. long and y in. wide, and is on a scale of 6 in. to the mile. How many square miles does it represent?

24. A sovereign is composed of $\frac{11}{12}$ gold, $\frac{1}{12}$ alloy. If the gold in a sovereign weighs w grains, what is the weight of the sovereign?

25. A rough formula for the volume (in cu. in.) of a sphere of diameter d in. is $\frac{1}{2}d^3 + \frac{1}{20}$ of $\frac{1}{2}d^3$. Simplify this. If the correct formula is $\frac{\pi}{6} \cdot d^3$, where $\pi = 3{\cdot}142$, show that this formula is nearly correct.

26. When it is 12 o'clock at Petrograd it is 10 o'clock at Greenwich. What is the time at Greenwich when it is x o'clock at Petrograd? Why is this formula wrong when x is 2 or less than 2?

27. If 1 Kg. $= 2\frac{1}{5}$ lb., by how much per cent. is half a kilogram greater than a pound? By how much per cent. is x half-kilograms greater than x lb.?

28. A boy buys papers at 8d. a dozen, sells some at 1d. each and returns the rest, getting 7d. a dozen for them. How much does he gain if he buys 10 dozen and sells x dozen of them?

29. For what values of n is n per cent. of a sum of money greater than the sum of money ?

30. If a metric ton $=2205$ lb., (i) how many metric tons are there in x English tons, (ii) how many English tons are there in x metric tons ?

31. A's income is $\frac{1}{2}$ of B's and $\frac{2}{3}$ of C's. If B's income is £I, what are the incomes of A and of C ?

32. If a tons of flour will last b men for c days, how long will the same amount last x men ?

33. Two clocks are set right. One gains x seconds a day, the other loses y seconds. After how many days will they be 2 minutes apart ?

34. A can just give B a start of 2000 points in 9000 at billiards ; when B's score is x, what ought A's to be ? When A's score is y, what ought B's to be ?

35. The average of four numbers is x and the average of three of them is y ; what is the fourth number ?

36. A man buys 10 cows for £$(2a+b)$ each ; he sells 6 of them for £$(a+b)$ each and the rest for £$2a$ each. What does he lose ?

37. By how much per cent. is x guineas greater that £x ? By how much per cent. is £x less than x guineas ?

38. If x is the smallest prime number greater than 50, and y the largest prime number smaller than 50, what is the value of $x-y$?

39. A man is paid x shillings for each day that he works and is fined y pence for each day that he is absent. He works for four days out of the six. What will be his week's wages in shillings ?

40. A sheet of steel 2 ft. long and 6 in. wide weighs 2 lb. What will be the length of a sheet of the same thickness which is 9 in. wide and weighs w lb. ?

41. How many seconds will a train travelling at x feet a second take to pass through a station a yd. long, the length of the train being b yd. ?

42. The average age of a form of 20 boys is 15 years. If another boy joins it, whose age is $15+x$ years, what is now the average age ?

43. It costs £x to feed y horses for a year. How many horses can be fed for half a year for £z ?

44. If a florin weighs $\frac{2}{5}$ oz., how many florins can be made from x oz. of silver ?

45. If a man sold his bicycle for £a he would lose 10 per cent. on the cost price ; if at £b he would gain 10 per cent. Find a formula connecting a and b.

46. £a amounts to £b in 3 years. What would it amount to in 1 year ?

47. A number is increased by r per cent. Show that it has been multiplied by $\frac{100+r}{100}$.

48. The section of a tunnel is of the shape of a semi-circle above a rectangle. The height of the rectangle is h ft. ; the radius of the semi-circle is r ft. What is the perimeter of the section ? Find the perimeter when $h=15$, $r=10$.

CHAPTER IV.

GRAPHICAL WORK.

(POSITIVE NUMBERS.)

OA, OB are two straight lines at right angles.
P is a point in the angle AOB, PM is parallel to OB.

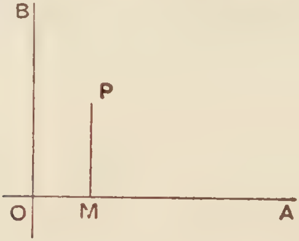


Fig. 5.

Then we know the position of the point P, if we know the length of OM and of PM.

The point O is called the **origin**; the lines OA, OB the **axes**; the lengths OM, PM the **coordinates** of the point P.

The point which has a for its coordinate along OA and b for its coordinate parallel to OB is called 'the point (a, b).'

EXERCISE IV. a.

1. If, in Fig. 6, OA represents 10, what will the following represent? (i) OM_1, (ii) OM_2, (iii) OM_3, (iv) OM_4, (v) OM_5, OM_6?

2. What do the same lines represent, if OA represents 1 inch ?

3. What do the same lines represent, if OA represents 2 lb. ?

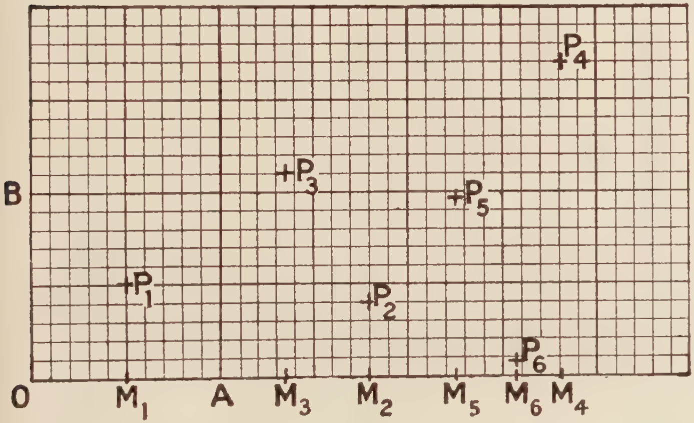


FIG. 6.

4. What do the same lines represent, if OA represents 5 sec. ?

5. Write down the coordinates of (i) P_1, (ii) P_2, (iii) P_3, (iv) P_4, (v) P_5, (vi) P_6, if OA represents 10 and OB represents 10.

6. Write down the coordinates of the same points, if OA represents 1 in. and OB represents 1 in.

7. Write down the coordinates of the same points, if OA represents 2 and OB represents 10.

8. Write down the coordinates of the same points, if OA represents 2 in. and OB 10d.

9. Draw a straight line joining the point O to the point (5, 3), taking OA, OB each to represent 1. Which of the following points are on this straight line: $P_1(2, 1{\cdot}2)$, $P_2(2, 3{\cdot}2)$, $P_3(3{\cdot}5, 2{\cdot}1)$, $P_4(0{\cdot}43, {\cdot}26)$, $P_5(4, 3{\cdot}3)$, $P_6(4{\cdot}7, 2{\cdot}82)$?

INFORMATION GIVEN BY MEANS OF DIAGRAMS.

Example I. The following table gives the average price of a quarter of barley for the years 1900 to 1910:

1900	1901	1902	1903	1904	1905	
24·9	25·2	25·7	22·7	22·3	24·3	shillings
1906	1907	1908	1909	1910		
24·2	25·1	25·8	26·8	23·1	shillings.	

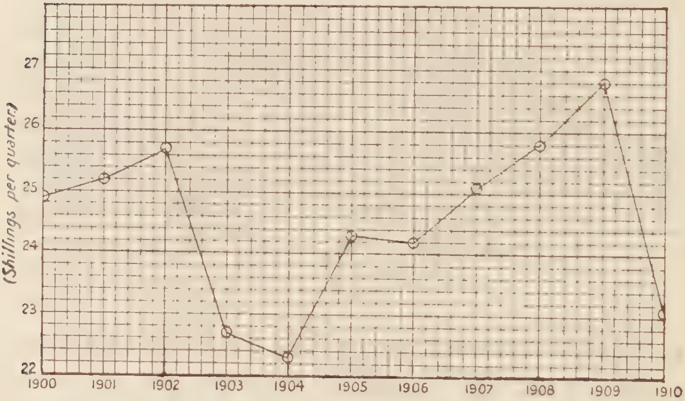


Fig. 7.

From this diagram we can see at a glance that:

(i) The highest price was in 1909, the lowest in 1904.

(ii) The price was rising nearly all the time from 1904 to 1909.

(iii) That the greatest fall in any year was from 1909 to 1910, etc.

EXERCISE IV. b.

1. The following table gives the average price of a 4 lb. loaf in Great Britain for the years 1907 to 1911:

1907	1908	1909	1910	1911	
5·5	5·9	6·2	6·0	5·7	pence.

Represent these on a diagram, taking 1 in. to represent a penny. Join the points by straight lines. How does the price change?

2. The following table gives the average number of pounds of potatoes per head imported into the United Kingdom for each of the years 1903-1912. Plot these values, taking 1 small division to represent 1 lb. on the vertical scale. Write down the year in which

(i) the amount was greatest.
(ii) ,, ,, least.
(iii) the rise was greatest.
(iv) the fall was greatest.

1903	1904	1905	1906	1907	1908	1909	1910	1911	1912
24	26	9	9	21	18	10	8	8	14

3. The following table gives the average number of people to the square mile in Ireland every tenth year from 1811 to 1911. Plot these values, taking 1 large division to represent 10 people, starting from 130. Write down (i) the year when the number was greatest, (ii) the period during which it was increasing, (iii) the period during which it was decreasing, as far as you can tell from these figures.

1811	1821	1831	1841	1851	1861	1871	1881	1891	1901	1911
186	209	239	251	201	178	167	159	144	137	135

4. The following table gives the average price of silver per ounce (to the nearest halfpenny) for the years 1902-1912. Plot these values, and write down

(i) the year in which it was more than 30d.
(ii) ,, years ,, less than 25d.
(iii) the longest period during which it was rising.

1902	1903	1904	1905	1906	1907
24	25	$26\frac{1}{2}$	28	31	30 pence
1908	**1909**	**1910**	**1911**	**1912**	
$24\frac{1}{2}$	$23\frac{1}{2}$	$24\frac{1}{2}$	$24\frac{1}{2}$	28 pence.	

5. Of every 1000 men and boys working in coal mines in England and Wales in 1901 there were 17 under 15 years of age; 106 under 20; 270 under 25; 592 under 35; 804 under 45; 934 under 55; 988 under 65. Plot these values and draw a smooth curve through the points. Show that about a half are under 32 years of age.

6. A boat is rowed across a river, and the depth of the river is measured every 10 feet; the depths in feet are 0, 1·2, 4·8, 12, 16·5, 17, 17, 16·5, 16, 15, 10·8, 7·4, 4·2, 0. Draw a section of the river. Let one large division represent 10 ft. horizontally and vertically.

FUNCTIONS OF A VARIABLE.

Example II. Plot the values of $\frac{x}{2}+3$ as x increases from 0 to 6.

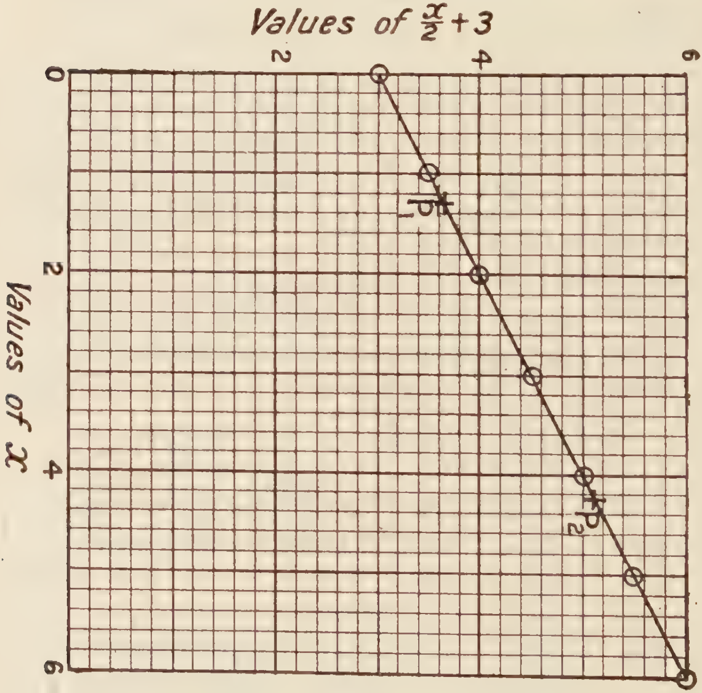


Fig. 8.

In all the examples that have been worked in Exercise IV. *b*, it has been possible to plot only a small number of points.

Thus, in Exercise IV. *b*, Ex. 1, only five values are given, and we can plot only five points. But in this question we can give x any value and from it obtain a point. We have given x the values 0, 1, 2, 3, 4, 5, 6. The values of $\frac{x}{2}+3$ can be worked mentally. The plotted points are found to lie on a straight line. To show that this still happens for less simple values of x, take $x=1{\cdot}3$ and 4·24, and we get the points P_1 and P_2.

The graph shows us that as x increases from 0 to 6, $\frac{x}{2}+3$ increases steadily from 3 to 6.

An expression such as $\frac{x}{2}+3$, the value of which depends on the value of x, is called a **function** of x; x is called the **variable** of the function.

The diagram of Example II. shows us how the value of the function $\frac{x}{2}+3$ changes, as x changes its value between 0 and 6.

Example III. Plot the squares of numbers from 0 to 10. From your curve write down, (i) the square of 5·6, (ii) a square root of 14.

For the scale of x, let 1 large division represent 1.

For the scale of x^2, let 1 large division represent 10.

First take the values 0, 1, 2, 3 .... 10 for x. When x is greater than 7, the plotted points are rather far apart, so it is best to take more values, *e.g.* 7·5, 8·5, 9·5. Draw a smooth curve through the points.

(i) $x=5{\cdot}6$ gives us the point M_1;

$\therefore$ the length P_1M_1 gives us the square of 5·6;

$\therefore$ $5{\cdot}6^2=31{\cdot}4$, about.

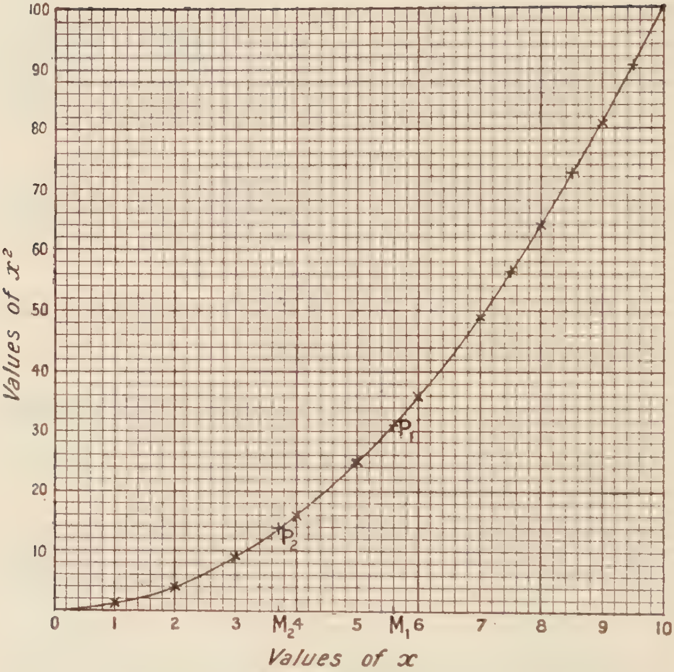


FIG. 9.

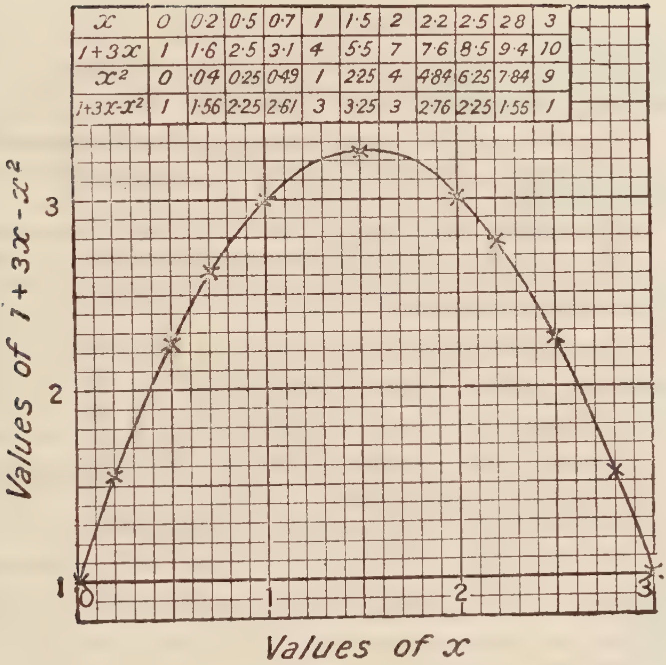
(ii) Consider the point P_2, such that $P_2M_2{}^4 = 14$.

$OM_2 = 3{\cdot}73$; $\therefore$ $3{\cdot}73^2 = 14$, about;

$\therefore$ a square root of 14 is 3·73, about.

Example IV. Plot values of $1 + 3x - x^2$ for values of x between 0 and 3, taking 2 large divisions to represent 1 on each scale.

Write down all your working in the form of a table, the values of x in the top line, of $1 + 3x$ in the second line, of x^2 in the third line; then subtract the third line from the second. This makes it easier to find out mistakes.



x	0	0·2	0·5	0·7	1	1·5	2	2·2	2·5	28	3
$1+3x$	1	1·6	2·5	3·1	4	5·5	7	7·6	8·5	9·4	10
x^2	0	·04	0·25	0·49	1	2·25	4	4·84	6·25	7·84	9
$1+3x-x^2$	1	1·56	2·25	2·61	3	3·25	3	2·76	2·25	1·55	1

FIG. 10.

We can see at once from the curve several things:

(i) As x increases from 0 to 3, $1 + 3x - x^2$ at first increases and then decreases.

(ii) The greatest value of $1 + 3x - x^2$, as far as we have considered it, is about 3·25.

(iii) The value of x which gives $1 + 3x - x^2$ its greatest value is about 1·5.

(iv) As x changes from 0 to 1·5, $1 + 3x - x^2$ increases more and more slowly, and as x gets very near to 1·5, it hardly increases at all.

(v) There are two values of x that make $1 + 3x - x^2$ equal to 3, and the same thing is true if, instead of 3, we put any number between 1 and 3·25.

EXERCISE IV. c.

NOTE.—*It is assumed that this work is done on squared paper ruled in half inches and tenths of an inch.*

1. Plot values of $2x$, taking 1 small division to represent 1 on each scale. The points should lie on a straight line through the origin.

2. Plot values of $x+2$, taking 2 large divisions to represent 1 on each scale. Do not give x whole number values in every case. The points should all lie on a straight line, but it does not go through the origin.

3. You may assume that x inches is the same length as $\frac{5x}{2}$ centimetres. Draw a graph giving values of $\frac{5x}{2}$, taking 2 large divisions to represent 1 on the scale of x and 1 large division to represent 1 on the scale of $\frac{5x}{2}$. You may assume that the graph is a straight line.

Use your graph to find the number of centimetres in 4·3″.

4. You may assume that x miles per hour is the same as $\frac{3x}{2}$ ft. per sec. Draw a graph that will enable you to change miles per hour into feet per second. Write down $4\frac{3}{4}$ mi. per hr. in feet per second.

5. Plot values of $10-x$, taking 1 large division to represent 1 on each scale. Notice that, as x increases, $10-x$ decreases.

6. The charge for x lb. of luggage for a certain railway journey is $\frac{x}{2}-15$ pence. Plot this for values of x between 30 and 100. What is the charge for 56 lb. of luggage? How much free luggage is allowed? Of course $\frac{x}{2}-15$ pence does not give the charge, if there is less than 30 lb. of luggage. Why not?

7. Plot values of x^2 from $x=0$ to $x=2$, taking 2 large divisions to represent 1 on each scale. Take 0·2, 0·4, 0·6, 0·8, 1, 1·2, 1·4, 1·6, 1·8, 2 for the values of x. Draw a smooth curve through the points. Write down (i) the square of 1·36, (ii) the square root of 3.

Notice from your curve (*a*) that, when x is small, x^2 is much smaller, (*b*) that, when x is less than 1, x^2 is less than x, (*c*) that, when x is greater than 1, x^2 is greater than x, (*d*) that, as x increases, x^2 increases more and more rapidly. If you increase x beyond 2, you will soon find that your sheet of paper is not large enough.

8. The distance that a stone will fall in t seconds is $16t^2$ feet (neglecting the resistance of the air). Plot values of $16t^2$ for values of t between 0 and 6. Take 2 large divisions to represent 1 in the scale of t and 2 large divisions to represent 100 in the scale of t^2. Taking Clifton Suspension Bridge as being 275 ft. above the river, find how long a stone will take to fall.

9. If $F°$ Fahrenheit is the same as $C°$ Centigrade, $F=\frac{9}{5}.C+32$. Draw a graph that you can use to change temperatures from the Centigrade to the Fahrenheit scale, for values of C between 0 and 25. Take 2 large divisions to represent 10° on each scale. Change 8·5° Centigrade to Fahrenheit.

10. Sketch roughly the graphs of (i) $2x$, (ii) $2x+1$, (iii) $2x+3$, for values of x between 0 and 4.

11. The expression $\frac{1}{x}$ is called the reciprocal of x. Plot the reciprocal of x for values of x between 0·1 and 10. Take one large division to represent 1 on each scale. Use your curve to find the reciprocal of (i) 0·38, (ii) 4·73. What do you know about the value of $\frac{1}{x}$, (iii) when x is very small, (iv) when x is very large? Notice from your curve, (*a*) that $\frac{1}{x}$ decreases as x increases, (*b*) that $\frac{1}{x}$ is less than 1 when x is greater than 1, (*c*) that $\frac{1}{x}$ is greater than 1 when x is less than 1.

12. The area of a circle of radius R is given by the formula $A=\frac{22}{7}.R^2$. Draw a graph to give you the areas of circles with radii not greater than 5 in. Take 2 large divisions to represent 1 in. on the scale of R and 2 large divisions to represent 10 sq. in. on the scale of A. What is the area of a circle of radius 3·2 in.? What is the radius of a circle of area 40 sq. in.?

13. Plot values of x^3 for values of x from 0 to 5, taking 2 large divisions to represent 1 on the scale of x and 1 large division to represent 10 on the scale of x^3. Use your curve to answer the following questions: (i) How do you know that as x increases from 0 to 5, x^3 is always increasing? (ii) For what values of x is x^3 less than 1? (iii) What is the value of $3{\cdot}85^3$? (iv) What value of x makes $x^3=20$? (v) What is the value of $\sqrt[3]{20}$?

14. Sketch roughly the graphs of (i) x^2, (ii) x^2+3, for values of x between 0 and 3.

15. Plot values of $4x-x^2$ for values of x between 0 and 4, taking 2 large divisions to represent 1 on each scale.

For what values of x (shown on your graph) does $4x-x^2$ increase as x increases? What value of x gives $4x-x^2$ its greatest value? What is its greatest value?

Sketch roughly the graph of $4x-x^2+1$ on the same sheet; also the graph of $4x-x^2+2$.

16. The length of a degree of longitude in latitude $x°$ is very nearly $69-\left(\frac{x}{10}\right)^2$ miles. Draw a curve, giving the length of a degree of longitude in latitudes between 0° and 60°. What is the length of a degree of longitude in the latitude of London ($51\frac{1}{2}°$)?

17. If a man is x ft. above sea level the horizon is about $\sqrt{\frac{3x}{2}}$ miles away. Plot values of $\sqrt{\frac{3x}{2}}$ for values of x from 0 to 500. Find how far the Skerryvore light, which is 139 ft. above sea level, is visible.

CHAPTER V.

NEGATIVE NUMBERS.

THE EXTENDED SCALE OF NUMBERS.

For an account of a method of introducing this chapter, see Introduction, pp. xxii to xxviii.

In Arithmetic, if we count downwards from, say, 4, we count 4, 3, 2, 1 and then stop. We shall now imagine other numbers below 1 and write them as follows: 4, 3, 2, 1, 0, (-1), (-2), (-3), (-4) ..... The number (-1) is called 'minus one,' and so on. Numbers below 0 on this scale are called negative numbers. Negative vulgar and decimal fractions correspond to positive vulgar and decimal fractions; *e.g.* $(-1\frac{1}{2})$ is half way between (-1) and (-2); counting downwards from (-1) in tenths, we get $(-1{\cdot}1)$, $(-1{\cdot}2)$, etc.

Scale of centimetres.

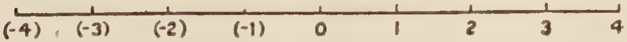


Fig. 11.

We shall sometimes find it convenient to write the positive numbers, 1, 2, 3, etc., in the form $(+1)$, $(+2)$, $(+3)$.

EXERCISE V. a.

1. Write down four consecutive numbers, of which (i) 10 is the greatest, (ii) 2 is the greatest, (iii) (-2) is the greatest.

2. Write down four consecutive numbers, of which (i) 3 is the least, (ii) (-6) is the least, (iii) (-2) is the least.

3. The temperature is 3° C.; if it falls 10°, what is the new temperature?

4. The temperature is (-4)° C.; if it rises (i) 5°, (ii) 4°, what are the new temperatures?

5. The temperature changes from 3° C. to (-5)° C.; how many degrees has it fallen?

6. The temperature changes from 0° C. to (-10)° C.; how many degrees has it fallen?

7. The temperature changes from (-10)° C. to 0° C.; how many degrees has it risen?

8. The surface of Lake Morar is $(+30)$ ft. above sea level; a lump of lead is lowered into it at the rate of 5 ft. per second. What will be the height of the lead above sea level after (i) 1 sec., (ii) 2 sec., (iii) 6 sec., (iv) 10 sec.?

9. A boat is $1\frac{1}{2}$ mile above Caversham Bridge and is drifting down the river at the rate of 1 mile an hour; how far above the bridge will it be (i) 1 hour, (ii) 2 hours later?

10. An aeroplane is moving through the air at 50 miles an hour. How far does it move forward on its course in an hour if there is a wind against it of (i) 20, (ii) 50, (iii) 60 miles an hour?

11. A man buys five articles at £100 each and sells them at £120, £94, £80, £100, £130 respectively; write down his *gain* in each of the five cases. Also write down his *loss* in each of the five cases.

12. At noon, the clocks in a house show the following times: 12.5 p.m., 11.55 a.m., 11.52 a.m., 12.1 p.m. Write down the error of each clock, denoting it by a positive number if the clock is fast, by a negative number if it is slow.

13. A squad of soldiers is judging distances. The true distance of a given point is 560 yd.; some of the estimates are 600 yd., 500 yd., 550 yd., 580 yd. Give the error in each case, representing it by a positive number if the estimate was too great; by a negative number if it was too small.

14. Fill in the following table:

PRICES (PER CWT.).

	1907.	1908.	Increase.
Wheat - -	7·7s.	8·4s.	
Sugar - -	12·0s.	13·0s.	
Tea - -	76s.	75s.	
Cotton - -	66s.	61s.	
Jute - -	22·4s.	16·6s.	

15. Rewrite the following phrases, without using negative numbers:

(i) A temperature of (-5) degrees Centigrade.
(ii) The surface of the Caspian Sea is on the average (-85) ft. above sea-level.
(iii) A clock is (-2) minutes fast.
(iv) A bank balance of £(-80).
(v) In the mile race A had a start of (-20) yd.
(vi) During the term he gained (-4) lb. in weight.
(vii) Newhaven is $8\frac{1}{2}$ miles, Worthing (-10) miles, east of Brighton.

The temperatures 12 *degrees above freezing point,* 6 *degrees below freezing point, freezing point,* 3·5 *degrees below freezing point, may be expressed more shortly as follows:* 12, (-6), 0, $(-3{\cdot}5)$ *degrees above freezing point. Express in a similar way the quantities in Questions* 16 *and* 17.

16. A wins £5, B loses £3, C loses £8, D neither wins nor loses. Express as gains.

17. The time at Berne is 1 hr. ahead of Greenwich time, at Petrograd 2 hr. ahead, at Halifax 4 hr. behind, at New York 5 hr. behind Greenwich time. Express as times ahead of Greenwich time.

18. A is 2 miles east of B; how many miles is B east of A?

19. A is 200 feet above sea-level; B is 250 feet below A; what is the height of B above sea-level?

20. P is 2 miles east of A, Q is (-3) miles east of A; how many miles is P east of Q?

21. The surface of Lake Como is 700 ft. above sea-level, and the bottom is (-600) ft. above sea-level. What is the depth of the Lake?

22. The surface of the Dead Sea is (-1300) feet above sea-level; and the top of Mt. Nebo is 2600 feet above sea-level. How much is the top of Mt. Nebo above the surface of the Dead Sea?

23. The surface of the Dead Sea is (-1300) feet above sea-level; taking it to be 1300 feet deep, what is the height of the bottom above sea-level?

24. A man gains £3 and then loses £5. What is his total gain? How much must he gain after this, if he neither gains nor loses, altogether?

25. On a map of the west coast of Scotland, the heights of three 'bench-marks' are given as 14 ft., 50 ft., 83 ft. above sea-level. There was a time when the sea was 50 ft. higher than it is now: what would have been the heights of these points above sea-level at that time?

26. The water in a harbour is 8 feet above mean sea-level, and is falling at the rate of 4 feet per hour. What will be its height above mean sea-level (i) one hour later, (ii) two hours later, (iii) three hours later?

27. Which is the greater, (-5) or (-3)? By how much?

28. Which is the greater, $(+2)$ or (-7)? By how much?

29. An express train is approaching Chichester Station from the west and is travelling 20 yds. per sec. It is 150 yds. west of the station; how many yards *west* of the station will it be (i) 6 secs. later, (ii) 10 secs. later? Explain the meaning of the last answer.

30. A stone is thrown up with a speed of 100 feet per sec., and at the end of each second its speed is less than at the beginning of that second by 32 feet per sec.; thus, after 1 sec., its speed is 68 feet per sec.; after 2 secs. its speed is 36 feet per sec. What will it be after (i) 3 secs., (ii) 4 secs.? Explain the meaning of the last answer.

ADDITION AND SUBTRACTION.

A discussion of the suggested line of argument in extending addition and subtraction, to negative numbers, which the following examples are intended to illustrate, will be found in the Introduction, pp. xxii to xxviii.

It is suggested that Exercises V. *b* and V. *c* should be taken orally.

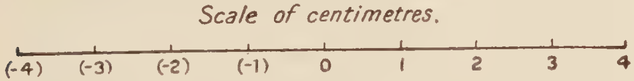


FIG. 12.

ADDITION.

EXERCISE V. b.

Start from the number (-2) *and count* 3 *as follows:* (-1), 0, 1;

$$\therefore (-2)+3=1.$$

In a similar way work out Examples 1-3.

1. Prove that $(-3)+4=1$. **2.** Prove that $(-3)+3=0$.

3. Prove that $(-3)+2=(-1)$.

4. Start at the point marked 1 in Fig. 12 above and move to the right over 4 centimetres. What point do you arrive at? What addition does this illustrate?

5. Start at the point marked (-2) in Fig. 12 and move to the right over 5 centimetres. What point do you arrive at? What addition does this illustrate?

6. Start from the number 1 and count 3, then go on and count another 2. What number do you arrive at? What addition does this illustrate?

7. Start from the number (-5), count 3, then go on and count another 4. What number do you arrive at? What addition does this illustrate?

8. What general formula includes all the following statements as special cases?

$2+3=3+2$; $7+11=11+7$; $1+10=10+1$; $3\frac{1}{2}+1\frac{3}{5}=1\frac{3}{5}+3\frac{1}{2}$.

9. Assuming that the general statement $a+b=b+a$ is true, whether a and b are positive or negative, write down the special form of the statement when (i) $a=3$, $b=7$; (ii) $a=(-3)$, $b=7$; (iii) $a=(-2)$, $b=(-5)$. Do not work out the value of $a+b$ or $b+a$.

10. Express by means of an equation: a gain of £5 + a loss of £3 is equal to a loss of £3 + a gain of £5.

11. Express by means of an equation: a rise in temperature of 3° followed by a fall of 7° gives the same change as a fall of 7° followed by a rise of 3°.

12. Do you agree with the following statements? Give your reasons. (i) "So far we have given no meaning to $+(-3)$." (ii) "So far we have given no meaning to $5+(1-3)$."

13. (i) What is the value of $(-3)+5$? Give your reasons.
(ii) What is the value of $5+(-3)$? Give your reasons.
(iii) Show that the value you have found for $5+(-3)$ is the same as you would have found by starting from 5 and counting 3 downwards.

14. What is the meaning (i) of (-4), (ii) of $(-4)+1$, (iii) of $1+(-4)$? (iv) What is the value of $(-4)+1$? (v) What is the value of $1+(-4)$? (vi) Show that the value you have found for $1+(-4)$ is the same as you would have found by starting at 1 and counting 4 downwards.

15. Start at the point marked (-2) in the diagram on p. 69 and move to the left 4 cm. What point do you arrive at? What addition does this illustrate?

16. Express by an equation: a fall of temperature of 2° C. followed by a fall of 4° C. is equivalent to a fall of 6° C.

17. Simplify (i) $(-2)+(-3)$; (ii) $0+(-2)$; (iii) $(-4)+0$.

SUMMARY OF RESULTS.

(1) To add b to a (where b is positive) start from a on the scale of Fig. 12 and move a distance b cm. to the right.

(2) To add $(-b)$ to a (where b is positive) start from a on the scale and move a distance b cm. to the left.

SUBTRACTION.

The rules for subtraction can be made to depend on those for addition. For, instead of ‘subtract *b* from *a*,’ we may say, ‘What number added to *b* will make *a* ?’

EXERCISE V. c.

1. (i) What number must be added to 3 to give 8 ?
(ii) Subtract 3 from 8.

2. (i) Fill in the blank in $7 + \quad = 4$.
(ii) Simplify $4 - 7$.

3. (i) What number must be added to 3 to give (-4) ?
(ii) Subtract 3 from (-4).

4. (i) What number must be added to (-2) to give 3 ?
(ii) Subtract (-2) from 3.

5. (i) What must be added to (-4) to give 5 ?
(ii) Subtract (-4) from 5.

6. (i) Fill in the blank in $(-5) + \quad = (-1)$.
(ii) Subtract (-5) from (-1).

7. (i) Subtract 3 from 8.
(ii) Start from 8 and count 3 down.

8. (i) Subtract 7 from 4.
(ii) Start from 4 and count 7 down.

9. (i) Subtract 3 from (-4).
(ii) Start from (-4) and count 3 down.

10. (i) Subtract (-2) from 3.
(ii) Start from 3 on the scale on p. 69 and count 2 to the right.

11. (i) Subtract (-4) from 5.
(ii) Start from 5 on the scale and count 4 to the right.

12. (i) Subtract (-5) from (-1).
(ii) Start from (-1) and count 5 to the right.

13. How do you move on the scale if you wish (i) to subtract 3 ; (ii) to subtract (-3) ?

14. How do you move on the scale if you wish (i) to add 4 ; (ii) to add (-4) ; (iii) to subtract 4 ; (iv) to subtract (-4) ?

15. Show that the result of subtracting 5 is the same as that of adding (-5).

16. Show that the result of subtracting (-3) is the same as that of adding 3.

17. Prove that $5-3=2$.

18. Prove that $5-6=(-1)$.

19. Prove that $0-4=(-4)$.

20. Prove that $(-2)-3=(-5)$.

21. Prove that $5-(-2)=7$.

22. Prove that $0-(-2)=2$.

23. Prove that $(-3)-(-2)=(-1)$.

Summary of Results.

(1) To subtract *b* from *a* (where *b* is positive) start from *a* on the scale of Fig. 12 and move *b* cm. to the left.

(2) To subtract $(-b)$ from *a* (where *b* is positive) start from *a* on the scale and move *b* cm. to the right.

ADDITION AND SUBTRACTION.

EXERCISE V. d.

Write down the values of

1. $1+(-3)$.
2. $(-8)+2$.
3. $0+(-4)$.
4. $6+(-2)$.
5. $(-2)+(-9)$.
6. $(-3)+0$.
7. $(-7)+11$.
8. $5+(-4)$.
9. $(-6)+6$.
10. $3+(-9)$.
11. $(-12)+(-1)$.
12. $(-1)+20$.
13. $(-20)+1$.
14. $(-7)+(-3)$.
15. $6+(-8)$.
16. $10+(-10)$.
17. $5-2$.
18. $2-5$.
19. $2-(-1)$.
20. $(-4)-3$.
21. $(-4)-(-1)$.
22. $(-7)-(-10)$.
23. $0-6$.
24. $8-11$.
25. $(-1)-8$.
26. $7-(-5)$.
27. $0-(-2)$.
28. $8-(-1)$.
29. $(-6)-(-6)$.
30. $(-3)-0$.
31. $4-12$.
32. $2-(-10)$.
33. $1+(-45)$.

34. $(-17)+3$. **35.** $12-(-6)$. **36.** $0+(-10)$.

37. $6-(-11)$. **38.** $(-15)-2$. **39.** $(-8)+(-7)$.

40. $(-12)-12$. **41.** $(-12)-(-12)$. **42.** $6+(-19)$.

43. $40-(-20)$. **44.** $1-27$. **45.** $18-(-11)$.

46. $(-10)+10$. **47.** $(-15)+(-11)$. **48.** $(-2)-20$.

49. Add together a gain of £5 and a loss of £8, and express the result as a gain.

50. Add together a fall of 20° C. and a rise of 15° C., and express the result as a rise in temperature.

51. Add together a movement of (-3) inches to the right and a movement of (-7) inches to the right, and express the result as a movement to the left.

52. Add together the following increases in weight: 2 lb., (-5) lb., 1 lb., and express the result as an increase. Express each of these three increases as a decrease, add them and express the result as a decrease.

53. (i) A man weighs a lb.; a year later he weighs b lb. How much has he gained in weight?

(ii) A man weighs 170 lb.; a year later he weighs 174 lb. How much has he gained in weight?

(iii) A man weighs 170 lb.; a year later he weighs 168 lb. How much has he gained in weight?

What subtractions are illustrated in (ii) and (iii)?

54. (i) The temperature is x° C. at 9 a.m. and y° C. at 6 p.m. What is the rise of temperature?

(ii) The temperature is 4° C. at 9 a.m. and (-2)° C. at 6 p.m. What is the rise of temperature?

(iii) The temperature is -12° C. at 9 a.m. and 0° C. at 6 p.m. What is the rise of temperature?

(iv) The temperature is (-4)° C. at 9 a.m. and (-10)° C. at 6 p.m. What is the rise of temperature?

What subtractions are illustrated in (ii), (iii), (iv)?

55. Work through the whole of Ex. 54, finding the fall of temperature in each case.

56. Give a meaning, similar to those in Ex. 54, to

(i) $50°\text{C.}-45°\text{C.}=5°\text{C.}$ (ii) $50°\text{C.}-(-45)°\text{C.}=95°\text{C.}$

57. A man possesses £50; later he is £20 in debt. How much has he lost? What subtraction does this illustrate?

58. Give a meaning to £(-20) − £(-100) = £80.

59. A miner starts from a place 140 ft. below sea-level and ascends to the top of a hill 80 ft. above sea-level. What is the total height that he ascends? What is the value of $80-(-140)$?

60. A miner starts from a place b ft. above sea-level and ascends to a height of a ft. above sea-level. What is the height that he ascends? What does the question become when $b=-50$, $a=100$? What is the answer in this case?

The examples of this exercise illustrate the following results:

$$+(+a) = +a.$$
$$+(-a) = -a.$$
$$-(+a) = -a.$$
$$-(-a) = +a.$$

The last result means that to subtract $(-a)$ is the same as to add a. Similarly for the others.

These results are expressed by the following rules:

I. If a bracket has a + sign in front of it, this + sign does not alter the sign in the bracket, when the bracket is removed.

II. If a bracket has a − sign in front of it, this − sign alters the sign in the bracket, when the bracket is removed.

EXERCISE V. e.

Write in full the meaning of each of the following:

1. $(+5)+(+4)$. **2.** $(+8)-(+5)$. **3.** $(-8)-(+3)$.
4. $(-6)+(-5)$. **5.** $7+(-2)$. **6.** $16-(+3)$.
7. $a+(-b)$. **8.** $x-(+y)-(-z)$.

Give in words the meaning of each of the following statements:

9. $+(-5)=-5$. **10.** $-(-5)=+5$.
11. $+7+(-4)=+3$. **12.** $7+(-4)=3$.
13. $+2+(-6)=-4$. **14.** $2+(-6)=(-4)$.
15. $a-(-b)=a+b$. **16.** $a+(-b)=a-b$.

Represent the following algebraically. Do not simplify:

17. Subtract (-1) from $(+2)$.

18. Add (-1), then subtract (-9), then add $(+2)$.

19. To subtract (-7) is the same as to add $(+7)$.

20. If we add (-11) to $(+1)$ we get (-10).

MULTIPLICATION AND DIVISION.

A discussion on the line of argument which the following examples are intended to illustrate will be found in the Introduction, pp. xxvi-xxvii.

It is suggested that this exercise should be taken orally.

Note.—We may write $(-a)b$ or $(-a)(b)$ for $(-a) \times b$, just as ab is sometimes written for $a \times b$.

EXERCISE V. f.

1. 3×5 is the same as $3+3+3+3+3$. What is $x \times 5$ the same as? What is $(-2) \times 5$ the same as?

2. Simplify (i) 4×3, (ii) $(-4) \times 3$.

3. What general formula includes all the following facts? $2 \times 5 = 5 \times 2$; $4 \times 11 = 11 \times 4$; $12 \times 8 = 8 \times 12$; $3 \times 10 = 10 \times 3$.

4. From the general statement $a \times b = b \times a$, write down special cases when (i) $a = 6$, $b = 9$; (ii) $a = (-4)$, $b = 3$; (iii) $a = 5$, $b = (-2)$.

5. (i) Multiply (-4) by 3; (ii) multiply 3 by (-4).

6. Simplify $5 \times (-7)$.

7. What is the value of (i) $4(5-3)$; (ii) $4 \times 5 + 4 \times (-3)$; (iii) $4 \times 5 - 4 \times 3$?

8. Prove that $3(11-7) = 3 \times 11 - 3 \times 7$.

9. From the general statement $a(b-c) = ab - ac$, write down special cases when (i) $a = 3$, $b = 9$, $c = 5$; (ii) $a = 3$, $b = 5$, $c = 7$; (iii) $a = 5$, $b = 0$, $c = 4$, and find the value of $a(b-c)$ in each case.

10. $(-4)[5+(-2)]$ is the same as $(-4)(5)+(-4)(-2)$. What is the value of $(-4)[5+(-2)]$? What is the value of $(-4)(5)$? What must be the value of $(-4)(-2)$?

11. $(-3)[7+(-5)]$ is the same as $(-3)(7)+(-3)(-5)$. Use this fact to find the value of $(-3)\times(-5)$.

12. Make up an example, like Nos. 10 or 11, to show that $(-6)\times(-7)$ equals 42.

13. Simplify $(-8)\times(-3)$. **14.** Simplify $(2-6)\times(5-8)$.

15. What general formula includes all the following facts? $(3)(-5)=-(3)(5)$; $(2)(-7)=-(2)(7)$; $(11)(-6)=-(11)(6)$.

16. What general formula includes all the following facts? $(-4)(7)=-(4)(7)$; $(-8)(5)=-(8)(5)$; $(-9)(2)=-(9)(2)$.

17. What general formula includes all the following facts?

$$(-5)(-3)=+(5)(3);\quad (-7)(-11)=+(7)(11);$$
$$(-12)(-8)=+(12)(8).$$

18. Simplify $(2)(-3)(-4)$. **19.** Simplify $(-3)(-2)(5)$.

20. Simplify $(-1)(-2)(-3)$.

21. By what number must (-2) be multiplied to give (-6)? Simplify $(-6)\div(-2)$.

22. By what number must 3 be multiplied to give (-15)? Simplify $(-15)\div(3)$.

23. By what number must (-4) be multiplied to give 12? Simplify $12\div(-4)$.

24. Prove that $(24)\div(-8)=(-3)$.

25. Prove that $(-8)\div(-8)=1$.

26. Prove that (i) $0\times3=0$, (ii) $3\times0=0$.

27. What is the value of (i) $0\times6\frac{1}{2}$, (ii) $a\times0$?

28. Can you think of any number such that, when multiplied by 0, the product is 3? Can you think of any number equal to $3\div0$?

The examples of this exercise illustrate the following results:

$$(-a)\times(b)=-(a)(b) \text{ or } -ab,$$
$$(a)\times(-b)=-(a)(b) \text{ or } -ab,$$
$$(-a)\times(-b)=+(a)(b) \text{ or } ab,$$

and $$(-a)\div(b) = -(a)\div(b) \text{ or } -\frac{a}{b},$$

$$(a)\div(-b) = -(a)\div(b) \text{ or } -\frac{a}{b},$$

$$(-a)\div(-b) = +(a)\div(b) \text{ or } \frac{a}{b}.$$

These results are expressed by the following rule:

In multiplication and division of one single-term expression by another, like signs give a positive sign, unlike signs give a negative sign.

Note.—(1) If any number is multiplied by 0, the product is 0.

(2) We shall never try to divide by 0.

(3) Since $$a \times a = a^2$$
and also $$(-a)\times(-a) = a^2,$$
we see that either a or $(-a)$ is a square root of a^2.

We shall take $\sqrt{a^2}$ to mean the *positive* square root of a^2.

EXERCISE V. g.

Write down the values of:

1. $3\times(-10)$. **2.** $(-7)\times 5$. **3.** $(-5)\times(-6)$.

4. $8\times(-1)$. **5.** $(-3)(3)$. **6.** $(-2)(-8)$.

7. $(-1)(-3)$. **8.** $10\div(-5)$. **9.** $(-4)\div(-1)$.

10. $48\div(-12)$. **11.** $(-36)\div(-9)$. **12.** $(-18)\div 6$.

13. $(-10)\div(-10)$. **14.** $(-3)^2$. **15.** $(-1)^2$.

16. $(-2)^3$. **17.** $(-3)(-6)\div(-9)$.

18. $(-8)(3)\div(-6)$. **19.** $(-6)^2\div(-4)$.

20. $36\div(-3)^2$.

21. Simplify (i) $0\times(-3)$; (ii) $(-2)\times 0$.

22. Simplify $0\div(-4)$.

23. The increase in the population of a town is (-20) per week. What will be the increase in 5 weeks at this rate?

24. A man finds that he is losing £100 regularly each year. At the present moment he has £5000. Prove that in x years' time he will have £$5000+(-100)(x)$. What is the result of putting -10 instead of x, and what does this mean?

25. A man has saved £200 in 5 years. How much is this per year? If he had saved £(-200) in 5 years, how much would it have been per year?

26. (i) If a stone is dropped, its speed is increased by 32 feet per second during each second; how much is it increased in 5 seconds?

(ii) If it is thrown upwards it is increased by (-32) feet per second during each second; how much is it increased in 5 seconds?

(iii) If it were thrown up with a speed of 80 feet per second, what will its speed be at the end of 5 seconds?

27. What is the square of 3? What is the square of (-3)? What are the square roots of 9? What is the value of $\sqrt{9}$?

28. Write down the square roots of $25a^2$. Write down the value of $\sqrt{25a^2}$.

GRAPHICAL ILLUSTRATIONS.

Example I. To draw the graph of $4-x$ and extend it by including negative numbers.

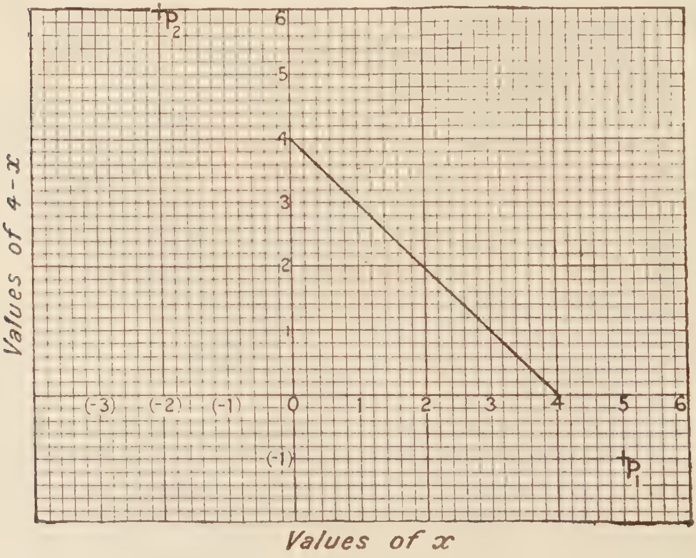


Fig. 13.

Take 1 large division to represent 1 on each scale.

Take the origin near the middle of the paper, and mark the horizontal scale in the usual way as far as 6. Reading the scale from right to left, the graduations are 6, 5, 4, 3, 2, 1, 0. It seems reasonable to extend it to (-1), (-2), (-3). Similarly we extend the vertical scale.

If we plot points for values of x between 0 and 4, we find that they lie on the straight line AB.

In all the work we have done with negative numbers, we have tried to give such meanings to symbols and to make such rules that when we work with letters we do not generally have to consider whether the letters represent positive or negative numbers. So here it seems reasonable to suppose that other values of x, not between 0 and 4, ought to give us points on the same straight line.

If $x=5$, $4-x=4-5=(-1)$, this gives us the point P_1.

If $x=(-2)$, $4-x=4-(-2)=6$, this gives us the point P_2.

These points will be found to be on the straight line AB produced.

Try other values of x.

EXERCISE V. h.

(NOTE.—*Before starting this Exercise it may be advisable to work through a few simple examples of Exercise V. l.*)

1. The following table gives the temperature at 9 a.m. each day for a week and also the average temperature on that day. Draw a diagram, with the horizontal scale giving the dates and the vertical scale giving the amount that the temperature is above the average. Join the points by straight lines.

	Jan. 9th	10th	11th	12th	13th	14th	15th
Actual Temp.	50°	42°	32°	29°	33°	39°	20° F.
Aver. Temp.	38°	38°	38°	38°	38°	38°	38° F.

2. The following table gives clock time when the sundial shows 12 o'clock on the first day of each month. Plot points giving the amount that the clock is ahead of sundial time for each month.

Jan.	Feb.	Mar.	Apr.	May	June
12.3	12.14	12.13	12.4	11.57	11.57
July	**Aug.**	**Sept.**	**Oct.**	**Nov.**	**Dec.**
12.3	12.6	12.0	11.50	11.44	11.59

3. The following table gives the price of pig iron per ton at the beginning of the years 1901-1908. Represent graphically the increase in the price during each year.

1901	1902	1903	1904	1905	1906	1907	1908
63s.	65s.	63s.	58s.	63s.	70s.	74s.	64s.

4. Taking 0 near the middle of the paper and one large division to represent 1, draw a graph of $2+x$ for values of x between 0 and 5. The graph will be a straight line; call it AB. Find the values of (i) $2+(-1)$, (ii) $2+(-3)$. Plot the points corresponding to $x=(-1)$ and $x=(-3)$. Show that these points are on AB produced.

5. Taking 0 rather below the middle of the paper and one large division to represent 1, draw a graph of $3-x$ for values of x between 0 and 3. Call this AB. Produce AB both ways. Assuming that for *all values of* x you get points on this straight line, read off from your graph the values of (i) $3-6$, (ii) $3-(-2)$. Show that these are the values you would get for these expressions by simplifying them.

6. Take 0 near the middle of the paper and plot values of $3-2x$ for values of x between (-2) and 4, using the rules for negative numbers. Show that all these points lie on a straight line.

7. Draw a graph of $2x-1$ for values of x between 1 and 4, taking 0 near the middle of the page and one large division to represent 1. Produce your straight line both ways. Assuming that for *all values of* x you get points on this straight line, show that when $x=(-3)$ you get the same value for $2x-1$ whether you read it off the graph or work it out by the rules of this chapter. Do the same for $x=\frac{1}{2}$.

8. Draw a graph for changing height above sea-level from feet to metres, for heights between 1000 ft. and (-200) ft. Take one large division to represent 100 ft. and two large divisions to represent 100 m. Take 10 ft. $=3$ m.

9. If a temperature is $C°$ in the Centigrade and $R°$ in the Réaumur scale, $R = \frac{4}{5}C$. Draw a graph giving values of R for values of C between 100 and (-20). Take one large division to represent 10° C. or 10° R.

10. Draw a graph of x^2 for values of x between (-3) and 3. Take two large divisions to represent 1 on the scale of x and one large division to represent 1 on the scale of x^2.

11. Draw a graph of x^3 for values of x between (-3) and 3. Take two large divisions to represent 1 on the scale of x and 10 on the scale of x^3.

12. If a temperature is $F°$ in the Fahrenheit and $C°$ in the Centigrade scale, $C = \frac{5}{9}(F-32)$. Plot values of C for values of F between 40° and $(-10)°$, and draw the graph. Let one small division represent 1° on each scale.

13. Draw a graph giving values of $2x - x^2$ for values of x between (-2) and 4.

14. Find by means of a graph the values of x for which $3-5x$ is positive.

15. Find by means of a graph the values of x for which $3x - 2x^2$ is negative.

So far we have represented negative numbers as follows: (-1), (-2), (-3), etc. From this point we shall omit the brackets and represent them as a rule as follows: -1, -2, -3, etc. Similarly we shall not write $(+1)$, $(+2)$, etc., but 1, 2, etc.

It will be necessary in some cases to keep the brackets. *E.g.* we shall do so in such an expression as $3+(-5)$, to avoid having two + or − signs next to one another.

We shall consequently use '$+a$' to represent two different things, (i) the number $+a$; (ii) the operation 'add a.' So '$-a$' will represent either (i) the number $(-a)$, or (ii) the operation 'subtract a.'

E.g., instead of $(-3)-4$, we shall now write $-3-4$, which will mean 'from the number -3 subtract 4.' But $-3-4$ may also mean 'subtract 3 and then subtract 4.'

In Chapter II. (p. 30) we gave the following results :

(i) $a+(b+c)=a+b+c$. (v) $(a+b)c$ or $c(a+b)=ac+bc$.
(ii) $a+(b-c)=a+b-c$. (vi) $(a-b)c$ or $c(a-b)=ac-bc$.
(iii) $a-(b+c)=a-b-c$. (vii) $(ac+bc)\div c=a+b$.
(iv) $a-(b-c)=a-b+c$. (viii) $(ac-bc)\div c=a-b$.

These may be summed up in the following rules :—

Rule I. If a bracket has a + sign in front of it and the bracket is removed, the sign of each term remains the same as it was in the bracket.

Rule II. If a bracket has a − sign in front of it and the bracket is removed, the sign of each term is different from what it was in the bracket.

Rule III. If an expression in a bracket is multiplied or divided by a number and the bracket is removed, each term of the expression must be multiplied or divided by that number.

We could not use these rules in Chapter II. for all values of a, b, c. For example, if $a=2$, $b=4$, $c=6$, the left side of (ii) becomes $2+(4-6)$. This means add $4-6$ to 2, which we were not able to do, as $4-6$ had no meaning at that stage. But, now that we have introduced negative numbers, these results have a meaning for all values of a, b, c, whether positive or negative. Also, if we use the rules of this chapter to find their meanings, these results will be found to be true for all values of a, b, c.

Consequently we can now state these results for all values of a, b, c.

ALGEBRAICAL EXAMPLES.

EXERCISE V. k.*

Add

1. $6x$ and $-x$. **2.** $-2a$ and a. **3.** $-3b$ and $-4b$.
4. $-x$ and x. **5.** $3y$ and $-10y$. **6.** $-3a$ and $-8a$.
7. $-3a^2$ and $4a^2$. **8.** x, $2x$ and $-4x$. **9.** $7ab$ and $-2ab$.
10. $-x^2$ and $-5x^2$. **11.** $3a$, $-5a$, $-2a$. **12.** a^2, $-2a^2$, $-a^2$.

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* For additional examples, see E.P. 11, page 251/18.

Subtract

13. $2a$ from $-3a$.
14. $-4x$ from $2x$.
15. $3c$ from c.
16. $-8x$ from $-6x$.
17. $-5b$ from $5b$.
18. $6x$ from 0.
19. $-2y$ from $10y$.
20. x^2 from $-x^2$.
21. $3ab$ from $2ab$.
22. $-a^2$ from $-b^2$.
23. $-2y$ from x.
24. 1 from $-2a$.

Multiply

25. $2a$ by $-3a$.
26. $-3a$ by $2a$.
27. $-x$ by y.
28. $3x$ by -5.
29. $-3x^2$ by 2.
30. ab by -1.
31. $-2a$ by $-3b$.
32. $6a^2$ by $-4a$.
33. 1 by $-3xyz$.
34. $-2a$ by $-2bc$.
35. $-x^2y$ by $-xy^2$.
36. 0 by $-2a$.

Divide

37. $-6a^2$ by $2a$.
38. $10xy$ by $-2y$.
39. $14x^2$ by $-7x$.
40. $-20xy$ by -5.
41. $4x^2$ by -1.
42. $-9a^2$ by $-9a$.
43. $3xyz$ by $-xyz$.
44. $-12x^4$ by $-x$.
45. $4a^2b^2$ by $-2ab$.
46. $15x^3y^3$ by $-5x^2y$.
47. abc by $-abc$.
48. 0 by $-x$.

Simplify

49. $a+(-2a)$.
50. $-x+(-x)$.
51. $x-3x$.
52. $-x+7x$.
53. $2y-(-3y)$.
54. $2x^2+(-4x^2)$.
55. $-5a^2+a^2$.
56. $xy-7xy$.
57. $-3x^3-x^3$.
58. $-2ab+(-6ab)$.
59. $-y-(-8y)$.
60. $a^2-(-2a)$.
61. $(-6x)\times 5x$.
62. $(-12x)\div 3x$.
63. $8x^3\times(-2x)$.
64. $20a^3\div 10a$.
65. $a^6\div(-a^3)$.
66. $(-2ab)\times(-ab)$.
67. $(-2a^3)\div a$.
68. $x^2y^2\div(-xy)$.
69. $(-x)^2$.
70. $(-3ab)^2$.
71. $-(2xy)^2$.
72. $(-2a)^3$.

Simplify

73. $\dfrac{-x}{y}$.
74. $\dfrac{x}{-y}$.
75. $\dfrac{-x}{-y}$.
76. $\dfrac{-2ab}{b}$.
77. $\dfrac{-x}{-xy}$.
78. $\dfrac{-1}{xy}$.
79. $\dfrac{a^2}{-ab}$.
80. $\dfrac{a^3b}{-ab}$.
81. $\dfrac{-a^2b}{-ab^2}$.
82. $\left(-\dfrac{a}{b}\right)\times b$.
83. $\left(-\dfrac{x}{y}\right)\times(-xy)$.
84. $\dfrac{-1}{a}\times\dfrac{a}{2}$.

Simplify

85. $2x-3x-x$.
86. $3x^2-5x-2x^2+x$.
87. $2x-1-(5x-1)$.
88. $a-2b+c+a+b-2c$.
89. $(a-b-c)\times(-3)$.
90. $(x^2-xy)-(xy-y^2)$.
91. $(-x^2y+xy^2)\div(-xy)$.
92. $(-x+y-z)+(-x-y+z)$.
93. $(-a^2+2a-1)\times(-2a)$.
94. $a(a-b)-b(a+b)$.
95. $x^2+2x-1-(2x^2+x-3)$.
96. $(-2x^3-6x^2+10x)\div(-2x)$.
97. From $y-z$ take $z-y$.
98. Add together $a-b$, $b-c$, $c-a$.
99. Add together $2x-1$, x^2-x and $x-3$.
100. Multiply $x-y$ by y and subtract the product from y^2.
101. Subtract $-x-y-z$ from 0.
102. Divide x^2-x+1 by -1.
103. Multiply x^2-x+1 by -1.
104. Multiply x^2-2x^2 by $2x-3x$.
105. Multiply $1+3a$ by 2 and subtract the product from $a-2$.
106. Simplify $(2a^2-a+3)a-(-a^2+2a-2)$.
107. Divide x^2-xy by x and y^2-xy by y and add the results.
108. Multiply a^2-ab+b^2 by $-b$ and subtract the result from a^3-b^3.

EXERCISE V. 1.

If $a=3$, $b=-3$, $c=0$, $m=2$, $n=-2$, $x=1$, $y=-1$, $z=-2$, find the value of

1. $a+x$.
2. $b+x$.
3. $b-x$.
4. $a-y$.
5. $a+b$.
6. $c+x$.
7. $a-2c$.
8. $2x-b$.
9. $2(x-b)$.
10. $b-2x$.
11. $n-3m$.
12. $b-2y$.
13. $3n-2y$.
14. ab.
15. nx.
16. $2an$.
17. b^2.
18. ac.

19. $2ay$.

20. by.

21. $3n^2$.

22. abc.

23. $a^2 - ab$.

24. $b^2 - b$.

25. $2(y + z)$.

26. $x - 2(y - z)$.

27. $(a - b)y$.

28. $x^2 + y^2$.

29. $\frac{1}{2}(y^2 - z^2)$.

30. $\frac{x}{y}$.

31. $\frac{y}{z}$.

32. $\frac{x - 2y}{z}$.

33. $(x - y)(y - z)$.

34. $(x - y)^2$.

35. $m^2 - 2mn + n^2$.

36. $x + \frac{1}{x} - \left(y + \frac{1}{y}\right)$.

37. If $a = 1$, $b = 2$, $c = -3$, prove that $a^3 + b^3 + c^3 = 3abc$.

So far we have added, subtracted, multiplied, etc., by means of brackets, but we can, if we wish, set out the work in the same way as in Arithmetic. In addition and subtraction arrange the expressions so that only those terms which can be added or subtracted come in the same vertical line.

Example II. Add $2c - a + b$ to $3a - b - c$.

$$\begin{array}{rrr} 3a & -b & -c \\ -a & +b & +2c \\ \hline 2a & & +c \end{array}$$

Answer $2a + c$.

Example III. Subtract $2c - a + b$ from $3a - b - c$.

$$\begin{array}{rrr} 3a & -\ b & -c \\ -a & +\ b & +2c \\ \hline 4a & -2b & -3c \end{array}$$

Answer $4a - 2b - 3c$

The terms can be subtracted as before or you can change in your head the signs of the terms in the lower line and add. Thus add $3a$ and $+a$, add $-b$ and $-b$, etc.

Example IV. Divide $3x^3 - 6ax^2 + 12a^2x$ by $-3x$.

$$\begin{array}{r|l} -3x & 3x^3 - 6ax^2 + 12a^2x \\ \hline & -x^2 + 2ax \ \ - 4a^2 \end{array}$$

Answer $-x^2 + 2ax - 4a^2$.

EXERCISE V. m.

Add

1. $a-2b$ and $2a-b$.

2. $a-b+c$ and $a+b-c$.

3. $3x-2y$ and $-x+y$.

4. $-x$ and $2x-y$.

5. $2x-3$ and $x+1$.

6. $x-5$ and $-2x+1$.

7. $3a-2b$ and $b-2a$.

8. $2x-4$ and $3-x$.

9. x^2+x+1 and x^2-x+1.

10. $2x^2-x$ and $2x-1$.

11. $2x^2+5x-1$ and $-x^2+x+4$.

12. $x+y-z$ and $y+z-x$.

13. $3a-b+c$ and $3c-a+b$.

14. x^2-x+2 and $1-2x-x^2$.

15. $1-x$ and $2+x-3x^2$.

16. $a-b$, $b-c$, $c-a$.

17. $2-x$, $1-x^2$, $3-2x+x^2$.

18. $x-y-z$, $y-z-x$, $z-x-y$.

19. $-(x-2)$ and $1-3x$.

20. $2(x^2-x-1)$ and $-3(x^2+2x)$.

Subtract

21. $a+b$ from $2a-b$.

22. $x-y$ from $2x-3y$.

23. $2x-y$ from $x+y$.

24. $a-2b+c$ from $2a-b+c$.

25. $x-y-z$ from $x+y+z$.

26. $x-5$ from $2x-6$.

27. $1-x$ from $2x-1$.

28. $x-2$ from $3x$.

29. x^2-x+2 from $2x^2+x-1$.

30. $2x^2$ from x^2-1.

31. $2x^2+5x+1$ from x^2-x+3.

32. $1-x+x^2$ from x^2+8x.

33. $a-2b+c$ from $2a-b+2c$.

34. $2x^2-x+1$ from x^2.

35. $3-4x$ from $3x-4$.

36. $-(x+2)$ from $2x-5$.

37. $2(x-1)$ from $-(2x-11)$.

38. $1-3x+2x^2$ from x^2-x+4.

39. $3ab-2b^2$ from a^2-ab.

40. $b+c-a$ from $a+c-b$.

Multiply

41. $a+b$ by x.

42. $a+b$ by $-x$.

43. $a-b$ by $-x$.

44. $a-b$ by $-a$.

45. $x-1$ by x.

46. $x-1$ by $-x$.

47. $1-x$ by $2y$.

48. $1-2a$ by $-b$.

49. $a-3x$ by $-2x$. **50.** $a+b$ by $-ab$.

51. x^2-2x by -2. **52.** $2x-3$ by $-5x$.

53. $2x^2-x+3$ by $-x$. **54.** $1-x+3x^2$ by $-x$.

55. $4a+a^2$ by $-2a$. **56.** $x-x^3$ by $-2x^2$.

57. $x-y-z$ by $-xyz$. **58.** $2a-b-3c$ by $-3a$.

59. $3x^2-2x+5$ by $-6x^2$. **60.** $3a^2+ab+2b^2$ by $-2ab$.

Divide

61. $ab+ac$ by a. **62.** $ab+ac$ by $-a$.

63. $ab-ac$ by $-a$. **64.** x^2-2x by x.

65. x^3+x^2 by $-x$. **66.** $6a^2-3ab$ by $-3a$.

67. $2x^3-6x^2+8x$ by $2x$. **68.** $3x^3-6x^2+18x$ by $-3x$.

69. $24+6x-18x^3$ by -6. **70.** $4-4x$ by -4.

71. $-a^2b+4ab^2$ by $-ab$. **72.** $5x^2y^2-10xy$ by $5xy$.

73. $-12a^3b-48ab^3$ by $-12ab$. **74.** $x^2-xy-xz$ by x.

75. x^3-x by $-x$. **76.** $-7x+14x^2-7x^3$ by $-7x$.

77. $a^3bc-abc^3$ by abc. **78.** $5ax^2-5ax-5a$ by $-5a$.

79. $-x^2yz-xy^2z-xyz^2$ by $-xyz$. **80.** a^4-b^4 by -1.

MISCELLANEOUS EXAMPLES.

EXERCISE V. n.

1. How much must be added to -2 to give 5?
How much must be added to 2 to give -5?

2. Add x, $-4x$, $5x$, $-3x$.

3. A is 250 ft. above sea-level; B is -50 ft. above sea-level. How much is A above B?

4. From x^2 subtract the sum of $2x^2$ and $-5x^2$.

5. How much must be subtracted from 0 to give -8?

6. What are the values of (i) $\frac{3}{-1}$; (ii) $\frac{-2}{-1}$; (iii) $\frac{(-2)^3}{(-1)^3}$?

7. Fill in the blank spaces in the following table:

RAINFALL (IN INCHES).

	1913.	1914.	Increase.
January - -	2·8	1·7	
February -	2·4	2·1	
March - -	1·9	3·6	
April - -	2·3	1·8	

8. Multiply $-6x^3$ by $2x$, and divide the result by $-4x^2$.

9. What must be added to $-a$ to give $-b$?

10. From x subtract $2x-y$.

11. At 8 a.m. a man was 3000 ft. above sea-level ; at 10, 4500 ft. ; at 12, 6500 ft. ; at 2, 6000 ft. ; at 4, 7000 ft. ; at 6, 2000 ft. How much did he go up during each period ? What is the sum of these five answers ?

12. By how much does x exceed $-y$?
By how much does $-y$ exceed x ?

13. To find the common difference of an Arithmetic Progression you subtract any term from the following term. What is the common difference of 7, $5\frac{1}{2}$, 4, $2\frac{1}{2}$, 1, $-\frac{1}{2}$?

14. Multiply 3 by 3 ; multiply -3 by -3. Give two square roots of 9.

15. Add together $-\frac{x}{5}$ and $-\frac{1}{5}$. Add the result to $\frac{x-4}{5}$.

16. The mean temperature for London for each month of the year is given below in degrees Fahrenheit. The average for the year is 49·6° F. Write down, for each month how much its mean temperature is above the average.

Jan.	Feb.	Mar.	April	May	June
38·6	39·5	41·9	47·3	53·1	59·4
July	Aug.	Sept.	Oct.	Nov.	Dec.
62·7	61·6	57·2	50·0	43·5	39·9

17. On a slip (*i.e.* a roadway sloping down to the water) are three rings for mooring boats ; they are 2 ft., 5 ft. and 9 ft. above the water. If the water rises 9 ft., what will be their heights above the water ?

18. Add together the following changes of temperature: a fall of 11° C., a fall of 3° C., a rise of 12° C., a fall of 1° C. If the temperature was 2° C. before these changes took place, what was it afterwards ?

19. One place is x mi. E. of a point A and another y mi. E. of A; x is greater than y. How far are they apart ? What does your answer become if you substitute 2 for x and -3 for y ? Show by drawing a figure that your answer is right.

20. In the Centigrade scale freezing point is 0°, boiling point 100°. If a temperature is $x°$ below boiling point, how much is it above freezing point ? What does your answer become if you substitute 110 for x ? Show that this is correct.

21. Show that $a-(b-c)$ and $a-b+c$ are the same when $a=2$, $b=-1$, $c=5$.

22. Draw a graph of $(-x)^2$ for values of x between 3 and -3. Draw a graph of $-x^2$ on the same sheet with the same scales.

23. (i) By what must I multiply $a-b+c$ to get $b-a-c$? (ii) By what must I divide $a-b+c$ to get $b-a-c$?

24. The sum of two numbers is 0. (i) If one of them is 8, what is the other ? (ii) If one of them is x, what is the other ? (iii) If one of them is $m-n$, what is the other ?

25. Multiply $-\frac{1}{x}+\frac{1}{y}-\frac{1}{z}$ by xyz. Divide $yz+zx+xy$ by xyz, giving the answer as the sum of three fractions.

26. Fill in the blank in $-\frac{1}{x}-\frac{1}{y}=-\frac{\quad}{xy}$.

27. There is a ladder on the wall of a quay. The top is 25 ft., the bottom -5 ft. above the water. The water rises till the top is 15 ft. above it; how much is the bottom above it ? If the top is a ft. above the water, how much is the bottom above the water ? How much is the bottom below the water ?

28. Two numbers have the same square. Prove that either (i) they are equal, or (ii) their sum is equal to 0.

29. Simplify (i) $(1-x)\times(-x)-(x-x^2)$,
(ii) $(x-2x^2)\div(-x)-(x-x^2)$.

Arrange each answer in descending powers of x.

30. Find the number which is halfway between (i) 5 and 3, (ii) 10 and -2, (iii) a and $-b$.

31. What does the statement

$$a^3+b^3 \equiv (a+b)(a^2-ab+b^2)$$

become when you put $-b$ for b ?

32. Given that the square of $a+b+c$ is

$$a^2+b^2+c^2+2bc+2ca+2ab,$$

find the square of $a-b-c$.

33. What is the value of $(-1)^m$ when m is (i) an even integer, (ii) an odd integer ?

34. The distance between the points (h, k) and (a, b) is $\sqrt{(h-a)^2+(k-b)^2}$. Find the distance between $(-1, 2)$ and $(-4, -2)$. Check your answer by a figure.

35. If (h, k) and (a, b) are two points, $\left(\frac{h+a}{2}, \frac{k+b}{2}\right)$ is the middle point of the line joining them. Find the middle point of the line joining the points $(3, -2)$, $(-5, 1)$. Check your answer by a figure.

36. In algebra x is greater than y if $x-y$ is positive. Find which is the greater, (i) 1 or -7, (ii) -5 or 0.

37. If the product xyz is negative, how many of the factors x, y and z are negative ?

38. Find whether $(x+y)^2=(x-y)^2+4xy$ when $x=-1$, $y=5$.

39. (i) Show that $3-2x$ is positive for all negative values of x. (ii) Give a positive value of x for which it is positive.

40. Simplify $\frac{a}{-b}-\frac{-a}{b}+\frac{-a}{-b}$.

41. What is the value of $\frac{a}{b}$ if (i) a is 0, but b is not 0; (ii) a is not 0, but b is 0 ?

CHAPTER VI.

PROBLEMS AND SIMPLE EQUATIONS.

STATEMENTS IN THE FORM OF EQUATIONS.

Example I. Obtain an equation from the following statement :

A number is such that, when it is added to 13, the result is 21.

The number is found from the information that

$$13 + \text{the required number} = 21.$$

The form in which this information is given here consists in a statement that an expression or number is equal to another expression or number.

A *statement* of this kind is called an **equation.**

We shall find later that we have to write our statement several times, in different forms. It therefore saves time, if, instead of repeating some phrase such as 'the required number,' we represent the number by a letter. This 'required number' in an equation is sometimes called '*the unknown.*' Consequently, we find it convenient to say :

Let x represent the required number.

Our statement then becomes

$$\mathbf{13 + x = 21.}$$

You will notice that in this example x represents a number. Read again p. 38. It is there explained that letters may be used to represent numbers, but are not used to represent numbers of things. It is important to remember this in problems dealing with quantities, as in the next example.

Example II. A box and its contents are together worth thirty shillings; the contents are worth four times as much as the box. What is the equation that can be obtained from these facts?

The value of the box + four times the value of the box = thirty shillings.

We can rewrite this more shortly if we suppose that the value of the box is x shillings. The statement now becomes

$$\mathbf{x} \textbf{ shillings} + \mathbf{4x} \textbf{ shillings} = \mathbf{30} \textbf{ shillings.}$$

From this it follows that

$$\mathbf{x + 4x = 30,}$$

which is the required equation.

EXERCISE VI. a.

(NOTE.—*The examples in this exercise should be taken orally.*)

In Examples 1–14, write the given statement in the form of an equation, but do not solve the equation

1. If a certain number is subtracted from 15, the result is 8.

2. If a certain number is doubled, the result is 48.

3. If a certain number is divided by 3, and 5 is then added, the result is 20.

4. If 12 is added to a certain number, the result is the same as if we doubled the number.

5. The sum of two consecutive numbers is 25.

6. The excess of a certain number over 17 is 11.

7. The sum of one-half a certain number and one-third of that number is 50.

8. The average of 25, 40 and another number is 38.

9. If a certain number is increased by one-half of itself, the result is 81.

10. If a certain number is increased by 25 per cent., the result is 200.

11. The denominator of a fraction is 3 greater than the numerator, and the fraction is equal to $\frac{1}{2}$.

12. A number is multiplied by twice itself and the result is 72.

13. The sum of a number and its cube is greater than its square by 21.

14. If a certain number is added to the numerator of the fraction $\frac{1}{2}$, the result is greater by $1\frac{4}{5}$ than if the same number had been added to the denominator.

From each of the statements in Examples 15–30 obtain two equations, (i) an equation connecting quantities, (ii) an equation connecting numbers. *In each case state clearly what the unknown letter represents.* See the two forms at the end of Example II. on p. 92. Do not solve the equations.

15. A box and its contents weigh 48 lb.; the contents weigh three times as much as the box.

16. The lengths of two rods are 55 and 42 inches; equal lengths are cut from each, after which one rod is twice as long as the other.

17. The area of a framed picture is 5 sq. ft., and the area of the frame is one-fifth that of the picture.

18. The perimeter of a rectangle is $5\frac{1}{2}$ ft. and the length exceeds the breadth by 3 inches.

19. A certain number of half-crowns and two more than that number of florins are worth altogether two pounds.

20. A certain number of pounds of tea worth 2s. 3d. per lb. are mixed with 20 lb. of tea worth 1s. 9d. per lb.; the mixture is then worth 1s. 11d. per lb.

21. A room is twice as broad and three times as long as it is high; it contains 4000 cu. ft. of air.

22. A is 36 years old and B is 60; a certain number of years ago B was twice as old as A.

23. One angle of a triangle contains three times as many degrees as either of the other two.

24. A man buys a certain number of oranges at a penny each and one-third of that number for twopence each; his bill is half-a-crown.

25. The price of tea is raised 10 per cent. The new price is 2s. 9d. per lb.

26. The perimeter of a circle of a certain radius is $13\frac{1}{2}$ in.

27. A man walks to a railway station at 4 miles an hour and back again at 3 miles an hour; both journeys together take $3\frac{1}{2}$ hours.

28. One tap pours water into a bath twice as fast as another; together they take six minutes to fill the bath, which holds 90 gallons.

29. A certain sum of money, lent for 6 months at 4 per cent., amounts to £85.

30. Each angle of a regular polygon of a certain number of sides is equal to 162°.

In Examples 31–44, write down statements which can be expressed by the given equations. Examples 31–37 refer to numbers.

31. $x-7=19.$

32. $2x+1=27.$

33. $x-\frac{x}{3}=30.$

34. $\frac{3+x}{5+x}=\frac{5}{6}.$

35. $x+\frac{1}{x}=2\frac{2}{3}.$

36. $(x-1)+x+(x+1)=42.$

37. $x-40=90-x.$

Examples 38–44 refer to quantities. The kind of quantity is shown by the words in brackets.

38. $x(x+2)=48.$ [Area.]

39. $x+\frac{4x}{100}=78.$ [Percentage.]

40. $\frac{22}{7}x^2=300.$ [Area.]

41. $x+x+x=180.$ [Angle.]

42. $x+4x=50.$ [Weight.]

43. $\frac{x}{3\frac{1}{2}}+\frac{x}{4}=3\frac{3}{4}.$ [Distance and Time.]

44. $10\times 8\times x=360.$ [Volume.]

SOLVING EQUATIONS.

When we have obtained statements and equations of the kind just considered, which involve an unknown number, we can generally use them in such a way that we are able to discover what this unknown number is. This process of discovering the unknown number is called *solving the equation*.

We shall assume that

(1) **If two numbers are equal and we add equal numbers to them, the results will be equal.** [Axiom I.]

(2) **If two numbers are equal and we subtract equal numbers from them, the results will be equal.** [Axiom II.]

(3) **If two numbers are equal and we multiply them by equal numbers, the results will be equal.** [Axiom III.]

(4) **If two numbers are equal and we divide them by equal numbers (other than zero), the results will be equal.** [Axiom IV.]

We shall now use these four axioms to solve equations.

Example III. Find a value of x for which $5x+11$ and $2x+23$ are equal to each other.

We are given that $5x+11=2x+23$.

Subtract 11 from each of the equal numbers ;

$$\therefore\ 5x=2x+23-11, \qquad \text{[Axiom II.]}$$

$$\text{or } 5x=2x+12.$$

Subtract $2x$ from each of the equal numbers ;

$$\therefore\ 5x-2x=12, \qquad \text{[Axiom II.]}$$

$$\text{or} \qquad 3x=12.$$

Divide each of the equal numbers by 3 ;

$$\therefore\ x=\frac{12}{3}. \qquad \text{[Axiom IV.]}$$

$\therefore$ **the value of x is 4.**

To make sure that this answer is correct, put 4 instead of x in the first expression, and then find its value.

$$5x+11=5\times4+11=20+11=31.$$

Now put 4 instead of x in the second expression.

$$2x+23=2\times4+23=8+23=31.$$

$\therefore$ when $x=4$, the two expressions are equal.

This method of showing that the answer obtained is correct, is called **checking** or **verifying** the answer.

Example IV. Solve the equation

$$\frac{x}{3}-2=\frac{3x}{4}-\frac{x}{2}$$

and check your answer.

$$\frac{x}{3}-2=\frac{3x}{4}-\frac{x}{2}.$$

Multiply each side by the L.C.M. of the denominators, *i.e.*, in this case, by 12;

$$\therefore \quad \frac{12x}{3}-24=\frac{36x}{4}-\frac{12x}{2}, \qquad \text{[Axiom III.]}$$

$$\text{or} \quad 4x-24=9x-6x,$$

$$\text{or} \quad 4x-24=3x.$$

Add 24 to each side;

$$\therefore \quad 4x=3x+24. \qquad \text{[Axiom I.]}$$

Subtract $3x$ from each side;

$$\therefore \quad 4x-3x=24, \qquad \text{[Axiom II.]}$$

$$\text{or} \quad x=24.$$

To check the answer:

$$\text{Left side}=\frac{x}{3}-2=\frac{24}{3}-2=8-2=6.$$

$$\text{Right side}=\frac{3x}{4}-\frac{x}{2}=\frac{3\times24}{4}-\frac{24}{2}=18-12=6;$$

$\therefore$ when $x=24$, Left side = Right side.

When checking an equation:

(1) Substitute for x in each side *separately*. First find the value of the left side by itself; and, when you have *finished* that, start again and find the value of the right side. Never simplify the two sides together.

(2) Always substitute the value of x in the equation as it is given; do not substitute in the simplified form of the equation. The object of checking is to make certain that there is no mistake in your work. You cannot be certain of this unless you substitute in the first line of all.

When solving an equation:

(i) Always start by copying down the equation you have to solve exactly as it stands in the book; do not make any simplification in your head.

(ii) Do not write your work in the following way:

$$2x = 8$$
$$= x = 4$$

An equation is a statement, and though you may equate numbers or quantities, the sign "=" is not used between two statements to show that one follows from the other.

$$2x = 8;$$
$$\therefore\ x = 4$$

is, however, quite correct.

EXERCISE VI. b.

In Examples 1–20, solve the equations, explaining each step of the argument, as has been done in Examples III. and IV. pp. 95, 96.

1. $7x = 28.$ **2.** $5x = x + 8.$ **3.** $3x - 5 = 7.$

4. $-3x = 12.$ **5.** $35 = 5x.$ **6.** $x - 6 = 0.$

7. $x - 7x = 18.$ **8.** $\frac{x}{3} = 10.$ **9.** $4x = 1\frac{1}{2}.$

10. $\frac{2x}{3} = 14.$ **11.** $\frac{12}{x} = \frac{3}{4}.$ **12.** $x - \frac{x}{2} = 36.$

13. $10x - 1 = 3x + 13.$ **14.** $8x + 20 = 5x - 4.$

15. $10 - x = x - 7.$ **16.** $0 = 3x - 12.$

17. $x + \frac{x}{5} = 18.$ **18.** $5x = 3x.$

19. $4x + 5 - 11x = 12.$ **20.** $\frac{x}{2} + \frac{x}{3} = 20.$

Solve the equations :

21. $3x=x+16.$ **22.** $25-x=4x.$

23. $2x-7=0.$ **24.** $3x-5=x+12.$

25. $15x+10=5x+90.$ **26.** $2(x-1)=14.$

27. $12=7x.$ **28.** $9(x-2)=7.$

29. $3x+1=0.$ **30.** $4x+7x=0.$

31. $4x-25+3x-10=0.$ **32.** $3(2x-3)=x+1.$

33. $4-4(1-x)=36.$ **34.** $10(x+4)-7(x-3)=100.$

35. $2(5x-1)-3(2+x)=6.$

36. $3(x-1)-2(x+5)=10(2+x)+12.$

37. $5x-(2x-9)=15.$ **38.** $6(2x-7)=5(5x-11).$

39. $2(x-2)+3(x-3)+4(x-4)=0.$

40. $78=8x-3(7-x).$ **41.** $5x=7\frac{1}{2}.$

42. $\frac{3x}{4}=2.$ **43.** $\frac{x}{3}=\frac{7}{9}.$ **44.** $\frac{5}{2x}=\frac{3}{4}.$

45. $2x=3{\cdot}8.$ **46.** $\frac{1}{2}x=0{\cdot}7.$ **47.** $\frac{x}{8}\div\frac{3}{2}=\frac{5}{6}.$

48. $1{\cdot}2x=6.$ **49.** $0{\cdot}1x=2{\cdot}7.$ **50.** $3{\cdot}5x=5{\cdot}6.$

51. $x-\frac{3x}{4}=7.$ **52.** $2x+\frac{2x}{3}=40.$ **53.** $\frac{x}{2}-\frac{x}{5}=6.$

54. $\frac{1}{6}x+\frac{1}{2}=0.$ **55.** $\frac{4x}{7}-2=18.$ **56.** $\frac{x}{4}+1=\frac{2x}{3}-\frac{3}{2}.$

57. $x+\frac{1}{6}x=5\frac{1}{4}.$ **58.** $\frac{2x}{3}+\frac{3x}{5}=5-\frac{2x}{5}.$

59. $\frac{x}{2}+\frac{x}{3}=0.$ **60.** $1{\cdot}7x+1{\cdot}5=0{\cdot}6x+7.$

Solve the equations :

61. $\frac{1}{2}(x-1)=6.$ **62.** $\frac{2}{3}(x+5)=16.$

63. $\frac{1}{4}(2x+1)=2\frac{1}{4}.$ **64.** $\frac{2}{3}(x-3)=\frac{1}{2}(x-2).$

65. $3-\frac{1}{5}(x-2)=0.$ **66.** $1-\frac{1}{4}(2x+1)=0.$

67. $x-\frac{2}{5}(2x-3)=3.$ **68.** $3-\frac{1}{7}(3x-1)=1.$

69. $x=4-\frac{2}{3}(x+1).$ **70.** $\frac{1}{3}(x+1)-\frac{1}{4}(x-5)=0.$

71. $\dfrac{x-1}{2}=6.$ **72.** $\dfrac{2x+10}{3}=16.$ **73.** $\dfrac{3x-1}{5}=2\frac{1}{5}.$

74. $\dfrac{x-1}{x-2}=\dfrac{4}{3}.$ **75.** $3-\dfrac{x-2}{5}=0.$ **76.** $7-\dfrac{2x+5}{3}=0.$

77. $2x-\dfrac{x-6}{4}=9.$ **78.** $2+\dfrac{x-2}{3}=5-\dfrac{5x+1}{6}.$

79. $\dfrac{3-4x}{4}-\dfrac{3(x-1)}{2}=0.$ **80.** $3-\dfrac{x+5}{8}=2-\dfrac{x+1}{6}.$

Solve the equations:

81. $80-32t=-80.$ **82.** $n-(2n-1)=5-2(n-1).$

83. $5W=3\,(100-W).$ **84.** $x=2(90-x).$

85. $r+\dfrac{22r}{7}=10.$ **86.** $x+\dfrac{5x}{100}=63.$

87. $\dfrac{3}{5}=\dfrac{5}{x}.$ **88.** $x-\dfrac{30x}{100}=35.$

89. $\frac{17}{2}\{2+(17-1)\,d\}=221.$ **90.** $42=20\,.\,3-\frac{1}{2}\,.\,f\,.\,3^2.$

91. $\frac{2}{3}(b+5)=\frac{3}{4}(b+4).$

92. $x(x+3)-5(x-7)=x^2+1.$ **93.** $n(n-1)-(n^2-1)=3.$

94. $\dfrac{5}{x}+\dfrac{3}{7}=\dfrac{41}{14}.$ **95.** $\dfrac{2}{y}+\dfrac{3}{5y}=\dfrac{1}{5}.$

96. If, in the equation $A=lb$, $A=14$ and $b=4$, find l.

97. If, in the equation $A=\frac{1}{2}bh$, $A=15$ and $b=10$, find h.

98. If, in the equation $A=\frac{1}{2}(a+b)h$, $A=30$, $a=8$ and $b=6$, find h.

99. If, in the equation $s=\frac{1}{2}ft^2$, $s=24$ and $t=4$, find f.

100. If, in the equation $A=\frac{1}{2}(a+b)h$, $A=19\frac{1}{2}$, $a=5$, $h=3$, find b.

101. If, in the equation $f=\dfrac{M-m}{M+m}\cdot g$, $f=4$, $g=32$ and $M=9$, find m.

102. If, in the equation $S=\dfrac{44r}{7}(h+r)$, $r=3$, and $S=100$, find h.

If we are given that $a+b=c$ and we subtract b from each of these equal numbers, we get $a=c-b$.

If we take other examples like this we shall find that the changes we make may all be summed up in the following rule :

We may move any term from either side of an equation to the other if we change the sign in front of it. This process is called **transposing** the term.

If we are given that $\frac{a}{b}=\frac{c}{d}$ and we multiply each of the equal fractions by bd, we get $ad=bc$. This process is called **cross-multiplying**.

It often enables us to get rid of fractions from an equation. *E.g.* if we are solving the equation $\frac{x-3}{x-5}=\frac{6}{7}$, by cross-multiplying we get $7(x-3)=6(x-5)$.

PROBLEMS (NUMBERS).

EXERCISE VI. c.

(NOTE.—*Check your answer in each question.*)

1. For what value of x is $3x$ the same as 27 ?

2. For what value of x are the expressions $2x+1$ and 37 equal ?

3. For what value of x are $\frac{1}{2}x+1$ and $\frac{1}{3}x+4$ equal ?

4. For what value of x is $3x+7$ twice as much as $x+5$?

5. If $8x-3$ exceeds $5x+1$ by 11, find the value of x.

6. Find a number which is less than 20 by one-third of itself.

7. The result of adding 42 to a certain number is the same as multiplying the number by 4. What is the number ?

8. The sum of one-half and one-third of a number is 30. What is the number ?

9. The sum of two consecutive numbers is 39. What are the numbers ?

10. What number is half way between 55 and 17 ?

PROBLEMS (QUANTITIES).

It has already been stated (see pp. 38-39) that in Algebra *a letter is used to represent a number, and should never be used to represent a number of things.*

Whenever a letter is introduced in solving a problem, it is important to take care that it is used properly; it is also important to state what it represents as carefully as possible.

E.g. Never say—Let the man's age be x.

But you may say—Let the man be x years old now.

Or, again, *Never say*—Let the price of the tea be x.

But you may say—Let 1 lb. of the tea cost x shillings.

[Here x does not represent a sum of money, but only the *number* of shillings in the cost of 1 lb. of tea.]

You may find the following instructions helpful in solving problems:

(1) Read the question very carefully, till you are quite familiar with it.

(2) Take a letter to represent the unknown number (sometimes there is more than one, and more than one letter may be necessary, but such cases will not arise at present). State what the letter represents as carefully as possible.

(3) Rewrite the question, using this letter to make the statement of the problem more complete (see Example VI. p. 103). You may have to do this more than once.

(4) Try, by reading this statement, to write down an equation.

(5) Solve the equation; check your answer, *not in the equation*, but in the problem itself.

(6) Whenever possible, draw a diagram and represent the data on it (see p. 102).

Example V. A man buys fifteen copies of a book for £3. 3s., some of them bound in leather at 5s. each and the rest in cloth at 3s. each. How many did he get in the more expensive binding ?

Let x represent the number of leather-bound copies that he bought.

[Now rewrite the question and use x to make the statement as complete as possible.]

A man buys fifteen copies of a book for £3. 3s. ; he buys x copies at 5s. each and $15-x$ at 3s. each.

[Or, writing it out again.]

A man buys fifteen copies of a book for 63 shillings ; he buys x copies for $5x$ shillings and $15-x$ copies for $3(15-x)$ shillings.

[Now try to get an equation from this statement.]

$$\therefore\ 5x \text{ shillings} + 3(15-x) \text{ shillings} = 63 \text{ shillings};$$

$$\therefore\ 5x + 3(15-x) = 63;$$

$$\therefore\ 5x + (45-3x) = 63;$$

$$\therefore\ 5x + 45 - 3x = 63;$$

$$\therefore\ 2x = 18;$$

$$\therefore\ x = 9;$$

∴ he buys **9 copies** in the more expensive binding.

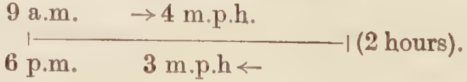
To check your answer, do not substitute in the equation, because your equation may be wrong. Substitute in the original question.

9 copies at 5s. each cost 45s.

∴ he buys 15 − 9 or 6 copies at 3s. each ; these cost 18s. ;

∴ the total cost is 45s. + 18s., *i.e.* 63s.

Example VI. A man leaves his house at 9 a.m. and walks to a town at 4 miles an hour. He spends two hours there and then walks back at 3 miles an hour, arriving home at 6 p.m. How far is the town from his house ?



Let the town be x miles from his house.

[Now rewrite the question, giving more information.]

A man leaves his house at 9 a.m. and walks x miles to a town at 4 miles an hour. He spends two hours there and then walks back x miles at 3 miles an hour, arriving home at 6 p.m.

[Or, writing it once more.]

A man leaves his house at 9 a.m. and takes $\frac{x}{4}$ hours to walk x miles to a town. He spends two hours there and then takes $\frac{x}{3}$ hours to walk x miles home. He arrives home at 6 p.m., which is 9 hours after he started.

[Now read this over and discover a relation between the various times mentioned.]

$$\frac{x}{4} \text{ hours} + 2 \text{ hours} + \frac{x}{3} \text{ hours} = 9 \text{ hours};$$

$$\therefore \frac{x}{4} + 2 + \frac{x}{3} = 9;$$

$$\therefore 3x + 24 + 4x = 108;$$

$$\therefore 7x = 84;$$

$$\therefore x = 12;$$

$\therefore$ the house is **12 miles** from the town.

Check the answer from the original question, not from the equation.

To walk 12 miles at 4 miles an hour takes 3 hours.

He spends 2 hours in the town.

To walk 12 miles at 3 miles an hour takes 4 hours.

$$\therefore \text{total time} = 3 + 2 + 4 \text{ hours}$$
$$= \mathbf{9\ hours.}$$

Time between 9 a.m. and 6 p.m. is **9 hours.**

EXERCISE VI. d.

(NOTE.—*All these Examples should be worked by Algebra.*)

1. If $A\hat{O}B = 2B\hat{O}C$ (see figure), find the size of each in degrees.

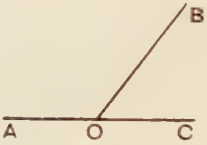


FIG. 14.

2. If $A\hat{O}B$ is greater than $B\hat{O}C$ by a right angle, find the size of each in degrees.

3. A field is twice as long as it is wide, and I walk 240 yd. in going round it. Find its length.

4. The length of a flagstaff is one-quarter of the height of the tower on which it is erected. The top of the flagstaff is 150 ft. above the ground. How high is the tower ?

5. The price of wheat is increased by one-third, and is now 63s. per quarter. What was the price before the increase ?

6. Find the angles of a triangle, if each angle at the base is twice the third angle.

7. A weathercock turns from x degrees East of North to $x+10$ degrees East of South : if the wind has shifted through 120 degrees, find its first direction.

8. Where must a pole, 40 ft. long, be cut so that one part may be two-thirds the length of the other ?

9. Divide 10s. between two people so that one has a shilling more than the other.

10. The sum of the angles of a figure of n sides is $2n-4$ right angles. If the sum of the angles of a figure is 10 right angles, how many sides has it ?

11. The total number of children in a school is 266. If there are 38 more girls than boys, find the number of each.

12. A man, whose age is 40, has a son who is 9 years old. When will the father be just twice as old as his son ?

13. There are 24 gallons of water in one tank and 5 gallons in another. Equal quantities of water are pumped into each. How much has been added when one contains twice as much as the other ?

14. It costs eight times as much to buy a house as to furnish it. The total cost of both is £3600. How much did it cost to furnish it ?

15. One angle of a triangle is 90° ; one of the remaining angles is twice the other. Find the other angles.

16. The sun rises at x o'clock and sets at x o'clock. Find x, if the middle of the day is at 12 o'clock.

17. (i) Find the number which, when increased by one-fifth, is 42. (ii) Find the number which, when increased by 20 per cent., is 42.

18. The price of flour has increased 50 per cent., and is now $7\frac{1}{2}$d. per gallon. How much has it increased ?

19. Of the 2000 people in a town 200 are soldiers. How many extra soldiers must be quartered there so that one-fifth of all the people in the town may be soldiers ?

20. The sum of the angles of a figure of n sides is $2n-4$ right angles. If they are all equal, what is the size of each angle ? If each angle of a figure is $1\frac{1}{2}$ right angles, how many sides has it ?

21. If the day is three hours longer than the night, what is the length of each ?

22. If a direction may be described either as x degrees E. of N. or as $(x-5)$ degrees N. of E., what is the direction ?

23. From A to B is 15 miles. If two men start at the same time, one cycling from A to B at 12 miles an hour and the other walking from B to A at 4 miles an hour, where do they meet ?

24. On a railway journey 60 lb. of luggage is free, and the charge for every pound above that is $\frac{1}{2}$d. How much luggage do I carry if I pay at the rate of 1d. for 5 lb. on the whole of it?

25. A man is allowed a discount of 3d. in the shilling on the published price of a book. If he pays 12s. for the book, what was the published price ?

26. A first-class ticket from London to Earlswood costs twice as much as a third-class. If 8 first-class and 20 third-class tickets cost £3, what is the cost of each kind ?

27. A first-class return ticket costs $1\frac{3}{4}$ times as much as a third-class for the same distance. If the former costs 1s. $10\frac{1}{2}$d. more than the latter, what is the cost of each ?

28. A man finds that he would pay the same income-tax, whether he paid 6d. in the pound on all his income or 2s. in the pound on all his income above £120. What was his income ?

29. The price of coal diminishes 30 per cent., and is then 21s. per ton. What was the first price ?

30. *ABCD* is a rectangular field, and *E* is the middle point of *CD*. It is 100 yards longer from *A* to *E* one way round than the other. Find the length of each side of the field if $AB = \frac{1}{2}BC$.

31. A man cycles to a town at 8 miles an hour and returns at 12 miles an hour. If the whole journey takes 5 hours, how far away was the town ?

32. The number of people at a concert is 220. Some pay one shilling and some two shillings for a ticket, and the amount of money taken was £13. How many people took two-shilling tickets ?

33. A man buys 800 bulbs for a guinea, some of them at a shilling for 50 and the rest at 3s. per 100. How many were there of each kind ?

34. The same number is added to the numerator and denominator of the fraction $\frac{5}{12}$, and it is then equal to $\frac{1}{2}$. Find the number.

35. A room is 24 ft. long and 10 ft. high. If the area of the walls is 800 sq. ft., find its width.

36. A man rows upstream at 4 miles an hour and back to the same place at 6 miles an hour, and takes $2\frac{1}{2}$ hours altogether. How far did he go upstream ?

37. At a concert 100 people paid half-a-crown each and the rest paid sixpence each ; if the average paid per head was a shilling, how many people were there altogether ?

38. A man pays an income tax of one shilling in the pound on all his income above £160. If the tax is 3 per cent. of all his income, what was his income ?

39. Show that any two numbers in the ratio of 2 to 3 may be represented by $2x$ and $3x$.

Divide 300 into two parts in the ratio of 2 to 3.

40. Find two numbers in the ratio of 5 to 3 whose difference is 17.

41. In a mill the men get 5 shillings and the women 4 shillings a day. There are 200 hands, and the wages are £46 a day. How many men were employed ?

42. *AC* is 20 ft., *BC* is 12 ft. Steps from *A* to *B* are each 8 in. high ; some are 12 in. and some 15 in. wide. How many are there of each ?

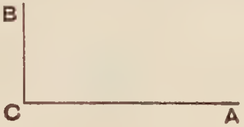


FIG. 15.

43. The sides of a triangle are proportional to 3, 4 and 5. If the sum of the sides is 18 in., what is the length of the shortest side ?

44. A gardener reckons that on his ground for every gallon of potatoes he plants he will get a crop of seven gallons. How many gallons must he plant so as to have 150 gallons to eat and enough left over to plant the same quantity next year ?

45. A man wants to find two consecutive whole numbers, the squares of which differ by 37 ; is this possible ? If so, find the numbers.

46. A slow train, travelling at 25 miles an hour, leaves a station 30 min. before a fast train travelling at 40 miles an hour. When and where will they meet ?

47. A man has £100 and invests it at 3 per cent. simple interest ; another invests £80 at 5 per cent. If neither spends his interest, when will they have the same amount ?

48. What must be a man's income if he has £450 a year after paying an income-tax of 2s. in the pound ?

49. How much tea at 1s. 6d. per lb. must be mixed with 12 lb. of tea at 10d. per lb., so that the mixture may be worth 1s. 3d. per lb. ?

50. A man reckons that the cost of growing potatoes is 2s. per bushel. He keeps 12 bushels for his own use and just gets back his expenses by selling the rest at 2s. 4d. per bushel. How many bushels did he grow ?

51. One man starts with an income of £150, which is increased by £20 at the end of each year; another starts with an income of £225, which is increased by £7. 10s. each year. When will their incomes be the same ?

52. A mass of ore weighing 15 lb. contains 10 per cent. of silver and the rest is lead. How much lead must be removed so that it may contain 80 per cent. of silver ?

53. From Croydon to Brighton is 40 miles. A starts from Croydon to walk to Brighton at $3\frac{1}{2}$ miles an hour; B starts half an hour later from Brighton to cycle to Croydon at 10 miles an hour. Where do they meet ?

54. What is the number of two digits, with x for the ten digit and 5 for the unit digit ? If the number is five times the sum of the digits, what is the ten digit ?

55. A man pays 2s. in the pound income-tax on the whole of his income above £160. Next year he pays 2s. 6d. on all above £120. If he pays £8 more the second year than the first, find his income.

56. A publisher offers to pay an author either 7d. on every copy of his book that is sold, after the first 500 copies, or 5d. on every copy sold. If the author chooses the former and gains by it, what can you tell about the number of copies sold ?

GRAPHICAL WORK.

All the equations that we have considered in this chapter have been *equations of the first degree*, *i.e.* they can be reduced to the form $ax+b=0$, where a and b may be numbers or may be expressions containing letters, so long as they do not contain x. Thus, either $2x-15=0$ or $(p^2-p)x+q^2-q=0$ is an equation of the first degree in x.

If we represent $ax+b$ graphically we shall find that the graph is always a straight line, and is never parallel to the axis of x. It therefore always cuts the axis of x at one point, and the value of x which corresponds to this point is the value of x which satisfies the equation $ax+b=0$.

For example, we will plot $2x-3$.

When $x=0$, $2x-3=-3$, when $x=5$, $2x-3=7$. If we join these two points by a straight line and take any other values of x, the points we shall get will lie on this straight line. Also, by considering the point P, we see that $2x-3=0$, when $x=1\cdot5$.

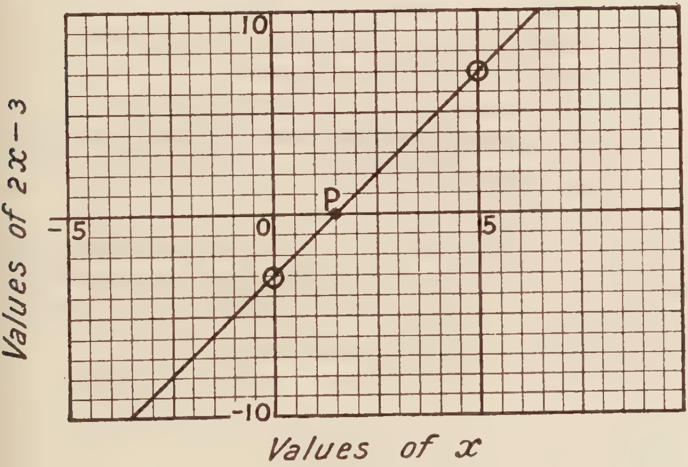


FIG. 16.

EXERCISE VI. e.

1. Plot $5-3x$ for any six values of x, and show that the points you obtain all lie on a straight line. (i) What value of x corresponds to the point at which the straight line cuts the axis of x? (ii) What value of x makes $5-3x$ equal to 0? (iii) What value of x satisfies the equation $5-3x=0$?

2. Plot two points on the graph of $\frac{x}{10}-2$. For values of x take 1 small division to represent 1; for values of $\frac{x}{10}-2$, take 1 large division to represent 1. Join the two points by a straight line. Use this straight line to solve the equation

$$\frac{x}{10}-2=0.$$

3. By plotting two points and drawing the straight line through them, obtain the graph of $2-5x$. Take any three other values of x and show that the points you obtain lie on this straight line. (i) For what values of x is $2-5x$ negative? (ii) For what values is it positive? (iii) Use the graph to solve the equation $2-5x=0$.

4. By plotting two points and drawing a straight line through them obtain a solution of $2{\cdot}7x-3{\cdot}7=0$.

5. By plotting two points obtain a graph of $10-3{\cdot}4x$. For what values of x is it greater than 5?

6. A cyclist is carried up a hill by his own momentum, without working, and his velocity after t seconds is $15-1{\cdot}8t$ feet per second. (i) Draw a graph giving his velocity at any time, and read off his velocity after $2\frac{1}{2}$ seconds. (ii) How long will it be before he stops?

7. A man pays income-tax of 1s. 3d. in the pound on all his income above £160. Draw a graph giving the tax in pounds for any income between £300 and £400. What is the tax on £365?

What formula gives the tax on an income of £x?

8. Coffee is worth 1s. 8d. per lb. and chicory 4d. per lb. Draw a graph to show the value of the mixture when to 1 lb. of coffee x lb. of chicory are added, for values of x from 0 to 3. Is your graph a straight line? What are the proportions if the mixture is worth 1s. 1d. per lb.?

9. One man has £450 and invests it at 8 per cent. simple interest. Another has £500 and invests it at 3 per cent. Plot the amount after t years in each case. When will they have the same amount, if neither spends his interest?

10. A slow train, travelling at 30 miles an hour, leaves a station an hour before another train travelling at 52 miles an hour. Plot the distance travelled by each x hours after the first started. Where will they meet?

CHAPTER VII.

PROBLEMS AND SIMULTANEOUS EQUATIONS.

SOLVING SIMULTANEOUS EQUATIONS.

Example I. Solve the equations :

$$5x+6y=16, \quad \text{..................................}(1)$$
$$3x-4y=2. \quad \text{..................................}(2)$$

Multiply each side of equation (1) by 2 and each side of equation (2) by 3 ;

$$\therefore\ 10x+12y=32,$$

and $$9x-12y=6.$$

Adding, we have $$19x=38;$$
$$\therefore\ x=2.$$

Substitute for x in equation (1) ;

$$\therefore\ 10+6y=16;$$
$$\therefore\ 6y=6;$$
$$\therefore\ y=1.$$

$\therefore$ the solution is $x=2$, $y=1$.

Check the answer by substituting in equation (2).

The first side $=3x-4y=6-4=2$

$=$ the second side.

Notes.—1. The process of obtaining from two equations in x, y a single equation which does not contain y is called "**eliminating y**."

2. When you have found one unknown, it does not matter in which equation you substitute to find the other. Take the easiest equation you have. But to check the answer you should not substitute in the same equation.

Example II. Solve the equations :

$$2x - \frac{y}{5} = 4\tfrac{1}{6}, \quad\text{..}(1)$$

$$\frac{3x}{4} + \frac{y}{2} = 3. \quad\text{..}(2)$$

Multiply each side of equation (1) by 5 and each side of equation (2) by 4 ;

$$\therefore\ 10x - y = 20\tfrac{5}{6}, \quad\text{..}(3)$$

and $$3x + 2y = 12. \quad\text{..}(4)$$

To eliminate y, multiply each side of equation (3) by 2 ;

$$\therefore\ 20x - 2y = 41\tfrac{2}{3} \quad\text{..}(5)$$

Add equations (4) and (5) ;

$$\therefore\ 23x = 53\tfrac{2}{3} ;$$
$$\therefore\ x = 2\tfrac{1}{3}.$$

Substitute for x in equation (3) ;

$$\therefore\ 23\tfrac{1}{3} - y = 20\tfrac{5}{6} ;$$
$$\therefore\ y = 23\tfrac{1}{3} - 20\tfrac{5}{6}$$
$$= 2\tfrac{1}{2}.$$

$\therefore$ the solution is $x = 2\tfrac{1}{3}$, $y = 2\tfrac{1}{2}$.

Check these values by substituting in (2).

Notes.—1. If x or y have fractional coefficients, the first step is to clear these terms of fractions ; you need not remove fractions from the terms which do not include x or y.

2. In substituting, choose the easiest equation ; this is usually one from which fractions have been removed. But to check the answer you should not choose the same equation, and you should take it in the form in which it was originally given. You may have made a mistake in clearing the x and y terms of fractions.

EXERCISE VII. a.*

Rewrite the equations in Examples 1–3, so that the x terms are the same.

1. $x + 5y = 13.$ $\quad 3x + 7y = 23.$

2. $4x + y = 13.$ $\quad 6x - 5y = 13.$

3. $10x + y = 3.$ $\quad 15x - y = 2.$

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* For additional examples, see E.P. 12, Nos. 1-42, page 251/20.

Rewrite the equations in Examples 4–6, so that the y terms are the same, except that the signs may be different.

4. $6x+y=20.$
$5x+2y=19.$

5. $2x-7y=-5.$
$7x+2y=9.$

6. $x+8y=5.$
$5x-6y=2.$

7. Eliminate x from the equations:

$$5x+y=7\,; \quad x+2y=6.$$

8. Eliminate y from the equations:

$$7x-3y=2\,; \quad 5x-2y=3.$$

9. Eliminate y from the equations:

$$y=2x\,; \quad 13x-4y=17.$$

10. Eliminate x from the equations:

$$x=3(y-2)\,; \quad x=2(y+1).$$

Solve the equations:

11. $x+y=17.$
$x-y=3.$

12. $2x+y=11.$
$x+y=7.$

13. $2x+y=9.$
$x-3y=1.$

14. $2x-3y=5.$
$x+2y=6.$

15. $x-3y=-1.$
$2x-4y=2.$

16. $5x-2y=1.$
$2x+y=13.$

17. $y=2x+1.$
$y=3x-2.$

18. $x-2y=0.$
$4x+3y=11.$

19. $x=2y.$
$4x-y=22.$

20. $x+3y=2.$
$4x+5y=6.$

21. $6x-2y=5.$
$4x+7y=-5.$

22. $5x-7y=20.$
$9x-11y=44.$

23. $3x+7y=4\frac{3}{4}.$
$2x+3y=2\frac{3}{4}.$

24. $3x-4y=2.$
$2x=3y.$

25. $3x+7y-7=0.$
$5x+3y+36=0.$

26. $12x=4+8y.$
$10y=3-x.$

27. $50x-20y=10.$
$91x-26y=39.$

28. $5(x-1)+2(y+1)=42.$
$x=2y-3.$

29. $0\cdot3x+0\cdot2y=4.$
$2\cdot5x-y=12.$

30. $8x+y=4\cdot3.$
$3x-4y=0\cdot3.$

31. $1\cdot5x-2y=4\cdot5.$
$0\cdot6x-2\cdot5y=-8\cdot4.$

32. $0\cdot1x+1\cdot3y=8\cdot2.$
$0\cdot7x+5y=32\cdot8.$

33. $x+5\cdot1y=3\cdot85.$
$x+9\cdot3y=4\cdot95.$

34. $0\cdot3x+1\cdot2y=20.$
$2x-0\cdot4y=3\cdot6.$

35. $x - 3 = y - 5 = 10.$
36. $2x + y = 3x - y = 15.$
37. $3x + 5 = 8y + 4 = 7x - 7y - 1.$
38. $0 = 3x - 7y - 5 = y - x + 3.$
39. $x + 5y - 1 = 3x + 2y + 1 = x + y + 7.$
40. $2(x + y + 2) = 3(2x - 3y + 3) = 6(x - y).$

Rewrite the equations in Examples 41–46, so that the coefficients of x and y are whole numbers.

41. $\frac{x}{2} + y = 3.$
$x + \frac{y}{3} = 4.$

42. $\frac{3x}{4} - \frac{y}{2} = 1.$
$x - \frac{y}{6} = \frac{1}{4}.$

43. $\frac{x}{2} + \frac{3y}{5} = 7.$
$\frac{3x}{2} - \frac{y}{10} = 1\tfrac{1}{3}.$

44. $\frac{3x}{7} - \frac{y}{2} = 2.$
$\frac{5x}{9} + y = 4\tfrac{1}{4}.$

45. $2x - y = 3\tfrac{3}{17}.$
$\frac{x}{4} - 2y = 0.$

46. $\frac{x}{3} + \frac{y}{4} = 3\tfrac{1}{5}.$
$\frac{x}{4} + \frac{y}{3} = 2\tfrac{4}{5}.$

Solve the equations:

47. $\frac{x}{2} - 2y = 5.$
$\frac{x}{2} + y = 8.$

48. $x - \frac{5y}{2} = -2.$
$x - 2y = -1\tfrac{1}{2}.$

49. $\tfrac{1}{2}x - \tfrac{1}{3}y = 3.$
$\tfrac{1}{4}x - 5y = 1\tfrac{1}{2}.$

50. $\frac{3x}{7} + 2y = 4.$
$\frac{5x}{7} + 3y = 7.$

51. $\frac{3x}{4} + \frac{5y}{6} = 3\tfrac{1}{6}.$
$\frac{2x}{5} - \frac{y}{4} = 1\tfrac{11}{20}.$

52. $x = \tfrac{1}{3}(y - 4).$
$y = \tfrac{1}{2}(x - 6).$

53. $x + y = 3\tfrac{1}{14}.$
$x - y = 2\tfrac{1}{12}.$

54. $\frac{5x}{2} - \frac{4y}{3} = 7.$
$\frac{x}{4} + \frac{y}{3} = 3\tfrac{1}{2}.$

55. $\tfrac{1}{2}(x - 2) = \tfrac{1}{3}(y - 3).$
$x + y = 10.$

56. $3x + y = \tfrac{1}{44}.$
$3x - 2y = 1\tfrac{1}{55}.$

57. $x + y = 17.$
$\frac{x}{y} = \frac{2}{3}.$

58. $\frac{x}{y} = \frac{3}{4}.$
$\frac{x - 5}{y - 5} = \frac{7}{11}.$

59. $\frac{x + 1}{y - 2} = \frac{2}{3}.$
$\frac{x - 1}{y + 2} = \frac{4}{7}.$

60. $\frac{2x + 5}{3y + 7} = \frac{2x}{3y}.$
$x - y = 2.$

CONSTANTS IN FORMULAE.

Example III. If a weight W grams is fastened to the lower end of an elastic string, its length is $a+bW$ cm. Find a and b if the length is 55·6 cm. when the weight is 200 grams and 58 cm. when it is 500 grams. Also find the length when there is no weight.

Substituting these values of W, we get

$$a+b\times 200=55{\cdot}6,$$
$$a+b\times 500=58.$$

Subtracting, $$300b=2{\cdot}4\,;$$
$$\therefore\ b=0{\cdot}008.$$

Substituting in the first equation,

$$a+0{\cdot}008\times 200=55{\cdot}6\,;$$
$$\therefore\ a=55{\cdot}6-1{\cdot}6=54.$$

∴ the length with weight W is $54+0{\cdot}008W$ cm.

Put $W=0$;

∴ the length when there is no weight is **54 cm.**

EXERCISE VII. b.

1. On a railway journey the charge for n lb. of luggage is $an-b$ pence, where a and b are constants, so long as the charge is not negative. If the charge for 70 lb. is 4d. and for 112 lb. 1s. 4d., find a and b. How much luggage is allowed free ?

Draw a graph giving the charge for any quantity of luggage up to 200 lb.

2. A temperature of t degrees Fahrenheit is the same as $at+b$ degrees Centigrade, where a and b are constants. Find a and b, given that 32° F. is the same as 0° C. and 212° F. the same as 100° C.

3. If a man's income is £x, x being between 160 and 400, he pays $(ax+b)$ pounds income tax. Find a and b, given that for an income of £300 he pays £22. 10s. and for £360 he pays £30.

Draw a graph giving the tax on any income between £160 and £400.

4. By means of an arrangement of pulleys I can raise a weight of W lb. by a pull of P lb. Find whether P and W can be connected by the equation $P=aW+b$, when a and b are constants, if (i) $P=6$, when $W=20$, (ii) $P=10$, $W=30$, (iii) $P=16$, $W=45$.

5. Cheap market tickets (return) are issued from Horsham as follows: Dorking ($13\frac{1}{2}$ mi.) 1s. 6d.; East Grimstead (15 mi.) 1s. 8d.; Hassocks ($22\frac{1}{2}$ mi.) 2s. 6d.; Pulborough (12 mi.) 1s. 4d. Show that the cost of the ticket is ax pence, where a is a constant and x mi. is the distance of the place. Find a, and find the cost of a ticket to Chichester ($31\frac{1}{2}$ mi.).

6. The cost of producing a book is (i) a sum of money necessary to set up the type, etc., before a single copy is printed, (ii) a certain sum in addition to this for each copy printed. Give a formula for the total cost (in pounds) of producing an edition of n copies of a book. If, in a particular case, 1000 copies cost £98. 15s. and 2000 would have cost £117. 10s., find the initial expense before any copies were printed.

7. If a spiral spring has a weight W lb. hung from its lower end, its length is $a+bW$ inches. Find a and b, given that when no weight is hung the length is 15 in. and when 2·5 lb. is hung the length is 16·5 in.

Draw a graph giving the length for any weight up to 6 lb.

8. An author gets nothing for the first a copies of a book that are sold, but b shillings for every copy after that. How much (in shillings) will he get if n copies are sold?

In a special case he gets £6. 5s. for 600 copies and would have had £31. 5s. for 1000 copies. What would he have had for n copies?

Draw a graph showing the amount he would have received for any number of copies up to 800.

9. A series of numbers is written down such that the nth number is $a+nb$, where a and b are constants. If the 5th number is 4 and the 17th 10, find the first four.

PROBLEMS.

EXERCISE VII. c.*

Solve the following by means of simultaneous equations.

1. The sum of two numbers is 25 and their difference is 3; find them.

2. The perimeter of a sheet of paper is 30 in. and its length is 3 in. more than its breadth. Find its area.

* For additional examples, see E.P. 12, Nos. 43-58, page 251/22.

3. The number of boys in a school is 450 and there are 30 more on the Classical than on the Modern side. Find the number on each side.

4. Three pennies weigh as much as five halfpennies. If 12 pennies and 4 halfpennies together weigh 4·8 oz., find the weight of a penny and of a halfpenny.

5. Find two numbers in the ratio of 7 to 11, whose difference is 36.

6. A gramophone with 20 records costs £8. 10s.; the same gramophone with 50 records costs £12. 5s. Find the price of the gramophone alone.

7. Divide 315 into two parts in the ratio of 7 to 8.

8. A man cycles at 10 miles an hour and walks at 4 miles an hour. He cycles from A, has a breakdown and walks on to B. If it is 30 miles from A to B and he takes $3\frac{3}{4}$ hours, how far did he walk?

9. The number of people at a concert is 360. Some take shilling and the rest half-crown tickets. If tickets were sold to the value of £24, how many of each kind were taken?

10. A man's income is £1200. On some of it he pays a tax of 9d. in the pound and on the rest a tax of 1s. 3d. in the pound. Altogether he pays £65. On how much did he pay the larger tax?

11. P lb. and Q lb. are two weights, such that their sum is 20 lb. and P is three-fifths of Q. Find them.

12. One number is greater than another by 20. If the first is divided by the second, the quotient is 2 and the remainder 8. Find the numbers.

13. A number of two digits is equal to 21 times the difference between the digits. If the ten digit is greater than the unit digit, show that it must be twice the unit digit.

14. What is the number of two digits which has x for its ten digit and y for its unit digit? The number is 18 greater than it would be if the digits were reversed, and the sum of the digits is 8. Find the number.

15. If a quadrilateral is inscribed in a circle the sum of each pair of opposite angles is 180°. The number of degrees in the angles of such a quadrilateral are as in the figure. Find x and y.

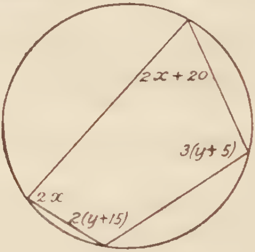

FIG. 17.

16. In the given figure, in which CT touches the circle, it can be proved that

$$\frac{BC}{CA}=\frac{TB}{TC}=\frac{TC}{TA}.$$

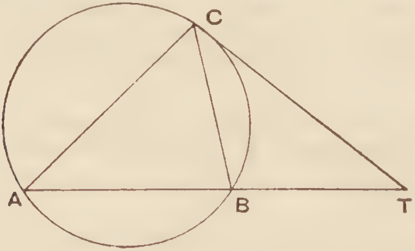

FIG. 18.

(i) If $BC=3''$, $TB=6''$, $TC=8''$, calculate AC and AB.

(ii) If $BC=5''$, $CA=6''$, $AB=8''$, calculate TB and TC.

17. In a row of houses, the rents of the two end houses are £5 higher than those of the others: and the total rent amounts to £190. If the rents of the end houses are increased to double that of the other houses, the total rent would amount to £220. Find the number of houses in the row.

18. On a certain journey, one passenger pays 1 fr. 50 for 50 Kg. of luggage and another pays 2 fr. for 60 Kg. of luggage. If a certain amount is allowed free and the rest is charged at so much per Kg., find the amount allowed free

19. A pamphlet contains 10 full pages of print: if small type is used, a page takes 550 words; but if large type is used, a page takes only 350 words. There are 4600 words in all; how many pages are printed in large type and how many in small type?

20. A shilling weighs $\frac{1}{5}$ oz. and a penny $\frac{1}{3}$ oz. If a man receives in change 11 coins (shillings or pence) and they weigh 2·6 oz., what sum did he receive?

21. The perimeter of a rectangle is 3 feet; if the length is decreased by 3 inches and the breadth increased by 2 inches, the area is unaltered. Find the area.

22. ABC is a triangle and D a point on BC such that $CA = AB = BD$. If $B\hat{A}D = x°$, $D\hat{A}C = y°$, find an equation between x, y. If, further, $B\hat{A}D$ is three-quarters of $B\hat{A}C$, calculate the angles BAD, ABC.

23. ABC is a triangle and BC is produced to D; the lines bisecting the angles ABC, ACD meet at E. If $A\hat{B}C = x°$, $A\hat{C}B = y°$, and if $C\hat{E}B = 3C\hat{B}E$, find an equation between x and y, and prove that $B\hat{A}C = 3A\hat{B}C$. Further, if

$$A\hat{C}D + A\hat{B}C = 120°,$$

find x and y.

24. A contractor promises to complete a piece of work in a certain time and is to forfeit a certain sum from the stipulated price for every day that he is late in finishing the work. If he is ten days late he will receive £1400; if he is thirty days late he will receive £1200. How much will he receive if he finishes it at the right time?

25. A coal merchant has contracted to deliver 200 tons of coal at 28s. a ton; but when the time comes he finds that his stock, which has cost him 21s. a ton, is not large enough, and he has to buy more at 32s. a ton, and consequently loses £18 on the transaction. How much coal did he have to buy?

26. A man, cycling x miles an hour for y hours, travels 24 miles; if he cycled $(x+2)$ miles an hour for $(y+2)$ hours he would travel 56 miles; how far will he go at $(x+3)$ miles an hour for $(y+3)$ hours?

27. $ABCD$ is a rectangle; a quadrant of a circle is drawn with B as centre and BA as radius, meeting BC at P; another quadrant with C as centre and CP as radius meets CD at Q, P being between B and C, and Q being between C and D. If the curved path APQ from A to Q is 20 feet less than the path $ABCQ$ and 10 feet less than the path ADQ, find the sides of the rectangle. Prove also that $AB > \frac{1}{2}AD$.

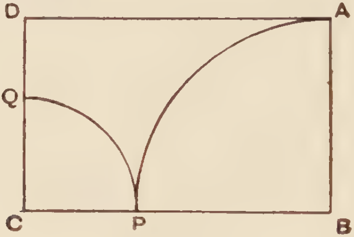

FIG. 19.

CHAPTER VIII.

FACTORS AND MULTIPLIERS OF MORE THAN ONE TERM.

SINGLE TERM FACTORS.

IN this chapter, when we speak of factors we mean *rational* factors only, unless it is stated that other factors are included. Thus we say that x^2+4 has no factors, although it is equal to $(x+2\sqrt{x}+2)(x-2\sqrt{x}+2)$, because these factors contain the irrational terms $2\sqrt{x}$, $-2\sqrt{x}$.

EXERCISE VIII. a.*

1. (i) What is the meaning of $10(x+3y)$?

(ii) What is the value of $10(x+3y)$, expressed without brackets ?

(iii) What are the factors of $10x+30y$?

2. If $7x+7y$ is divided by 7, what is the quotient ? If $7x+7y$ is divided by $x+y$, what is the quotient ?

3. If $ax+ay$ is divided by a, what is the quotient ?

4. What are the factors of $ax-3ay$?

5. If $cx+c$ is divided by c, what is the quotient ? If $cx+c$ is divided by $x+1$, what is the quotient ?

6. What are the factors of $ay-a$?

7. Put into factors x^2-x.

8. Divide $2x^3-6x^2y$ by $2x^2$.

9. Put into factors $3a^4-15a^2b^2$.

10. (i) Put into factors $xy+xz$. (ii) Divide $xy+xz$ by $y+z$.

11. (i) Put into factors $3b^2c-6bc^2$. (ii) Divide $3b^2c-6bc^2$ by $3bc$.
(iii) Divide $3b^2c-6bc^2$ by $b-2c$.

12. Is x^2 a factor of x^4+2xy^2 ?

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* For additional examples, see E.P. 13, page 251/23.

Factorise the following expressions :

13. $2a^3 - 6a^2b$. **14.** $10 - 15x^2$.

15. $xy^2 - x^2y$. **16.** $x^4 + x^2 + x$.

17. $c^3 + c^2d + cd^2$. **18.** $3a^2b^2 - 6abc + 15a^2b$.

19. $x^2(x+y) + x^4$. **20.** $a^2(a-b) + 3a^2b$.

21. $5a^2 - a^{10}$. **22.** $7x^4y^3 - 35x^3y^5 - 14x^5y^4$.

23. What is the H.C.F. of $2xy$ and $3x^2$?

24. What is the H.C.F. of $2(a+b)$ and $3(a+b)$?

25. What is the L.C.M. of $2(a+b)$ and $3(a+b)$?

26. (i) What is the H.C.F. of $a(x+y)$ and $a(x-y)$?

(ii) What are the quotients when each is divided by the H.C.F.?

27. (i) What is the L.C.M. of $a(x+y)$ and $a(x-y)$?

(ii) What are the quotients when the L.C.M. is divided by each expression separately ?

28. (i) What is the L.C.M. of $x^2 + x$ and $xy + y$?

(ii) Divide the L.C.M. by each in turn.

29. What is the L.C.M. of $6x^3 - 2x^2y$ and $18x^2y - 6xy^2$?

30. What is the H.C.F. of $a^3 + a^2b$, $a^2b + ab^2$, $a^2c + abc$?

31. What is the L.C.M. of $3a^2 - 3ab$, $2ab - 2b^2$, $5ab$?

32. What is the H.C.F. of $x^3 + x$ and $x^2 + 1$?

33. Simplify (i) $\frac{30}{40}$; (ii) $\frac{3x}{4x}$; (iii) $\frac{3(x+y)}{4(x+y)}$; (iv) $\frac{5a-5b}{6a-6b}$.

34. Simplify (i) $\frac{2x}{x(x+y)}$; (ii) $\frac{x}{x(x+y)}$.

35. Fill in the blank spaces in

(i) $\frac{3}{10} = \frac{\quad}{50}$; (ii) $\frac{a}{b} = \frac{\quad}{bc}$; (iii) $\frac{a}{b} = \frac{\quad}{b(x+y)}$.

36. Express $\frac{1}{x+y}$ as an equivalent fraction with $x(x+y)$ as denominator.

37. Express $\frac{1}{x}$ as an equivalent fraction with $x(x+y)$ as denominator.

38. Express $\frac{a}{a-b}$ and $\frac{a}{b}$ as fractions which have equal denominators.

39. Simplify $\frac{1}{x}+\frac{1}{x+y}$.

40. Simplify $\frac{1}{a+b}+\frac{1}{a-b}$.

41. Find the H.C.F. and L.C.M. of $x(x-y)$ and $(x-y)^2$.

42. (i) Fill in the blank spaces in $x=\frac{\quad}{xy}$ and $a=\frac{\quad}{a+b}$.

(ii) Simplify $x+\frac{xy}{x-y}$.

Simplify the following expressions :

43. $\frac{1}{x}-\frac{y}{x^2+xy}$.

44. $\frac{1}{2a-2b}-\frac{1}{3a-3b}$.

45. $a-\frac{ab}{a+b}$.

46. $\frac{1}{x^2}-\frac{1}{x^2+x}$.

47. $\frac{1}{2}-\frac{x}{3x-3y}$.

48. $\frac{1}{a}+\frac{1}{b}-\frac{1}{a+b}$.

49. $\frac{1}{y}+\frac{x}{y(x+y)}-\frac{2}{x+y}$.

50. $2+\frac{x}{x+y}-\frac{2y}{x}$.

51. $\frac{a}{a^2+ab}-\frac{b}{(a+b)^2}$.

52. $a+b-\frac{a^2}{a-b}$.

Solve the following equations :

53. $\frac{1}{x+1}+\frac{2}{5}=\frac{1}{2}$.

54. $\frac{3}{2x+8}-\frac{1}{x+4}=\frac{1}{8}$.

55. $\frac{2x-1}{4}-\frac{3}{x}=\frac{x}{2}$.

56. $1-\frac{x-2}{x+2}=\frac{3}{x}$.

57. $\frac{x}{x-1}-1=1-\frac{x-2}{x}$.

58. $\frac{1}{x}+\frac{x}{x+1}=1-\frac{1}{x}$.

59. $\frac{2}{x}+\frac{3}{x+5}=\frac{20}{x(x+5)}$.

60. $\frac{x-1}{x-3}+\frac{2}{x}=1$.

MULTIPLICATION AND DIVISION.

Example I. Multiply $2x-5$ by $3x-4$.

$$\begin{array}{r}
2x-5 \\
3x-4 \\
\hline
6x^2-15x \\
-8x+20 \\
\hline
6x^2-23x+20
\end{array}$$

The product is $\mathbf{6x^2-23x+20}$.

Example II. Divide $4x^2 - 6xy + 7y^2$ by $2x - y$.

$$\begin{array}{r|l|l} 2x - y & 4x^2 - 6xy + 7y^2 & 2x - 2y. \\ & \underline{4x^2 - 2xy} & \\ & \quad\; - 4xy + 7y^2 & \\ & \quad\; \underline{- 4xy + 2y^2} & \\ & \qquad\qquad\; 5y^2. & \end{array}$$

Quotient $\mathbf{2x - 2y}$; remainder $\mathbf{5y^2}$.

Note.—It is advisable that, of the two given expressions which have to be multiplied or divided, either both should be arranged in ascending powers, or both in descending powers of some common letter (see p. 12).

In Examples I and II. each expression is arranged in descending powers of x.

EXERCISE VIII. b.

1. Multiply $x + 3$ by $x + 4$.

2. Multiply $3x + 2$ by $2x + 5$.

3. Multiply $x + 2$ by $x - 7$.

4. Multiply $2x + 3$ by $x - 2$.

5. Multiply $3x - 1$ by $2x - 1$.

6. Multiply $2x - 3y$ by $x + 5y$.

7. Multiply $a + 2b$ by $a - 2b$.

8. Find the square of $5x - 4$.

9. Find the square of $2a - 5b$.

10. Multiply $x^2 - 3x + 5$ by $x + 2$.

11. Multiply $3x^2 + x - 7$ by $x - 3$.

12. Multiply $a^2 - ab + b^2$ by $a + b$.

13. Multiply $x^2 - 2x + 1$ by $x + 2$.

14. Multiply $x^2 + 2x + 4$ by $5x - 3$.

15. Multiply $x + a$ by $x + b$.

16. Multiply $x + a$ by $x - b$, grouping the x terms together.

17. Multiply $x - a$ by $x - b$, grouping the x terms together.

18. Multiply $y - z$ by $y - 1$, grouping the y terms together.

19. Divide $x^2 + 5x + 4$ by $x + 4$.

20. Divide $6x^2 + 5x + 1$ by $2x + 1$.

21. Divide $x^2 + x - 20$ by $x + 5$.

22. Divide $2x^2 + 5x - 18$ by $x - 2$.

23. Divide $x^2 + 7x + 10$ by $x + 2$. What would be the quotient if $x^2 + 7x + 10$ were divided by $x + 5$?

24. Divide $2x^2 + 9x + 7$ by $2x + 7$. What would be the product if the answer were multiplied by $2x + 7$?

25. Divide x^2-3x-4 by $x-4$. What are the factors of x^2-3x-4?

26. Divide x^2+4 by $x+2$. Is $x+2$ a factor of x^2+4?

27. One of the factors of $a^2+10ab+24b^2$ is $a+4b$; what is the other?

28. One of the factors of a^2-b^2 is $a-b$; what is the other?

29. Divide $4x^2-44x+121$ by $2x-11$. What are the factors of $4x^2-44x+121$? Write down a square root of $4x^2-44x+121$. What is the square of $2x-11$?

30. Divide $x^3+9x^2+22x+16$ by $x+2$.

31. Divide x^3-4x^2-5 by $x-3$.

32. Divide $18x^3-5x-2$ by $3x-2$.

33. Divide $x^2+ax+bx+ab$ by $x+a$. What are the factors of $x^2+ax+bx+ab$?

34. Divide $x^2+(a+b)x+ab$ by $x+a$.

35. Divide $x^2+xy-xz-yz$ by $x+y$.

36. Divide $x^2-xy-xz+yz$ by $x-y$. What are the factors of $x^2-xy-xz+yz$?

RELATION BETWEEN $a-b$ AND $b-a$.

It is evident that

$$7+5=5+7,$$
$$11+4=4+11,$$
$$10+15=15+10,$$

etc.

All such statements are included in the general statement

$$A+B=B+A.$$

Also

$$7-5=-(5-7),$$
$$11-4=-(4-11),$$
$$10-15=-(15-10),$$

etc.

All such statements are included in the general statement:

$$A-B=-(B-A).$$

Thus, $A+B$ and $B+A$ are the same quantities, but $A-B$ and $B-A$ are related in the same way as 3 and -3, or 5 and

-5, *i.e.* they have the same arithmetical value but are of different sign.

Consequently, $\dfrac{X}{A+B}=\dfrac{X}{B+A}$.

And, just as $\dfrac{2}{-3}=\dfrac{-2}{3}=-\dfrac{2}{3}$,

so $\dfrac{X}{A-B}=\dfrac{-X}{B-A}=-\dfrac{X}{B-A}$.

Example III. Find the L.C.M. of $y+x$, $2(x-y)$, $3(y-x)$, y^2-x^2.

First arrange all the expressions in descending powers of x (or all in ascending powers of x).

They then become

$$x+y,\quad 2(x-y),\quad -3(x-y),\quad -(x^2-y^2),$$

or
$$x+y,\quad 2(x-y),\quad -3(x-y),\quad -(x+y)(x-y);$$

$\therefore$ **the L.C.M. is $6(x^2-y^2)$.**

Note.—We could, if we wished, say that the L.C.M. is $-6(x^2-y^2)$.

Example IV. Simplify $\dfrac{1}{x-y}+\dfrac{x}{y^2-x^2}$.

Just as $\dfrac{x}{-z}=-\dfrac{x}{z}$, so $\dfrac{x}{y^2-x^2}=-\dfrac{x}{x^2-y^2}$, since y^2-x^2 and x^2-y^2 are related in the same way as z and $-z$, *i.e.* have the same arithmetical value but are of different sign.

$$\therefore\ \frac{1}{x-y}+\frac{x}{y^2-x^2}=\frac{1}{x-y}-\frac{x}{x^2-y^2}$$
$$=\frac{1}{x-y}-\frac{x}{(x+y)(x-y)}$$
$$=\frac{(x+y)-x}{(x+y)(x-y)}$$
$$=\frac{\mathbf{y}}{(\mathbf{x}+\mathbf{y})(\mathbf{x}-\mathbf{y})}.$$

EXERCISE VIII. c.

1. If $x+y=10$ and $x-y=1$, write down the values of

(i) $y+x$, (ii) $y-x$, (iii) $\dfrac{1}{y+x}$, (iv) $\dfrac{1}{y-x}$.

2. If $x-y=3$, write down, where possible, the numerical values of

$$\text{(i) } y-x, \quad \text{(ii) } x+y, \quad \text{(iii) } x^2-y^2, \quad \text{(iv) } \frac{1}{2y-2x}.$$

3. If $x-y=a$, write down, where possible, in terms of a, the values of

$$\text{(i) } y+x, \quad \text{(ii) } y-x, \quad \text{(iii) } \frac{1}{y^2-x^2}, \quad \text{(iv) } \frac{1}{y-x}.$$

4. If $x-y=5$, find the values of

$$\text{(i) } \frac{2}{x-y}-\frac{1}{x-y}, \quad \text{(ii) } \frac{2}{x-y}+\frac{1}{y-x}.$$

5. If $p-q=a$ and $x-y=b$, write down, where possible, in terms of a and b, the values of

$$\text{(i) } \frac{p-q}{x-y}, \quad \text{(ii) } \frac{p-q}{y-x}, \quad \text{(iii) } \frac{q-p}{x-y}, \quad \text{(iv) } \frac{q-p}{y-x},$$

$$\text{(v) } (p-q)(y+x), \quad \text{(vi) } (p-q)(x-y), \quad \text{(vii) } (q-p)(y-x),$$

$$\text{(viii) } (x-y)(q+p), \quad \text{(ix) } (q-p)^2, \quad \text{(x) } (x+q)-(p+y).$$

6. Divide $a-b$ by $b-a$. **7.** Divide $(a-b)^2$ by $b-a$.

8. Simplify $(2x-2y)\div(3y-3x)$.

9. Find the H.C.F. of $2(x-y)$ and $3(y-x)$.

10. Find the L.C.M. of a^2-ab and b^2-ab.

11. Find the L.C.M. of $a+b$, $a-b$, $b+a$, $b-a$.

12. Simplify $(x-y)(y-x)$. **13.** Simplify $-\dfrac{(a+b)(a-b)}{(b+a)(b-a)}$.

14. Simplify $(a-b)^3\div(b-a)^2$.

Simplify the following expressions :

15. $\dfrac{2}{x-1}+\dfrac{1}{1-x}$.

16. $\dfrac{x}{x-y}+\dfrac{y}{y-x}$.

17. $\dfrac{1}{x^2-1}+\dfrac{1}{1-x^2}$.

18. $1+\dfrac{a}{a-b}-\dfrac{b}{b-a}$.

19. $\dfrac{a}{ab-b^2}-\dfrac{b}{ab-a^2}$.

20. $\dfrac{1}{ab+a^2}+\dfrac{1}{ab+b^2}$.

21. $\dfrac{1}{x^2-x}+\dfrac{1}{1-x}$.

22. $\dfrac{2x}{x-1}-\dfrac{x}{1-x}-3$.

23. $\dfrac{2y}{x-y}+\dfrac{2y^2}{(y-x)(y+x)}$.

24. $\dfrac{x^2-xy}{y-x}-\dfrac{y^2-2xy}{2x-y}$.

FACTORS BY GROUPING TERMS.

If you are putting $ax+bx$ into factors, you say "x is a common factor, it goes into ax x times, it goes into bx b times."

$$x \,\underline{\big|\, ax+bx\,};$$
$$\quad a+b$$

$$\therefore\ ax+bx=x(a+b).$$

Exactly in the same way, if you are putting

$$a(x+y)+b(x+y)$$

into factors, you say $x+y$ is a common factor.

$$(x+y)\,\underline{\big|\, a(x+y)+b(x+y)\,};$$
$$\quad a+b$$

$$\therefore\ a(x+y)+b(x+y)=(x+y)(a+b).$$

In all factorisation, if there is a common factor, always take it out *first of all*, and write it down first : because, if you are using the division method, this is the order in which you ought to be thinking of the work.

Example V. Simplify $(a-x)(b-y)$.

$$\begin{aligned}(a-x)(b-y)&=(a-x)b-(a-x)y\\&=(ab-xb)-(ay-xy)\\&=ab-xb-ay+xy.\end{aligned}$$

Example VI. Factorise $a^2-ab-ac+bc$.

$$\begin{aligned}a^2-ab-ac+bc&=(a^2-ab)-(ac-bc)\\&=a(a-b)-c(a-b).\end{aligned}$$

But $(a-b)$ is a common factor ; $\therefore$ we can divide throughout by $(a-b)$, the quotient is $a-c$.

$$\therefore\ \text{the expression}=(a-b)(a-c).$$

Example VII. Factorise $ab-ax-xy+by$.

$$\begin{aligned}ab-ax-xy+by&=(ab-ax)-(xy-by)\\&=a(b-x)-y(x-b)\\&=a(b-x)+y(-x+b)\\&=(b-x)(a+y).\end{aligned}$$

Note.—1. *When you have factorised an expression, multiply the factors together, either in your head or on paper, and so make sure that they give the expression with which you started.*

2. *If there is a common factor of the whole expression, always put that factor down first.*

EXERCISE VIII. d.

Work out the expressions in Examples 1–10, in the same way as in Example V. p. 128.

1. $(a+b)(c+d)$. **2.** $(a+b)(c-d)$. **3.** $(a-b)(a+b)$.

4. $(x+y)(x+z)$. **5.** $(x+y)(x-y)$. **6.** $(x+2y)(a-3b)$.

7. $(3x+2y)(2a-b)$. **8.** $(x-y)(x-1)$.

9. $2(x-a)(y-b)$. **10.** $(1+x)(1+x^2)$.

11. Is $x+1$ a factor of $2x+2$? **12.** Is $x+1$ a factor of $2(1+x)$?

13. Is $1-x$ a factor of $x-1$? **14.** Is $a-3$ a factor of $a+3$?

15. Is $x-y$ a factor of $3(y-x)$?

16. Is $x+y$ a factor of (i) $4(x-y)$, (ii) $ax+ay$, (iii) $-x+y$?

17. Is $x+y$ a factor of $a(x+y)+b(y+x)$?

18. Is $x+y$ a factor of $a(x+y)+b(x-y)$?

19. What are the factors of $x(p+q)+y(p+q)$?

20. What are the factors of $x(p-q)+y(p-q)$?

Have the following expressions factors ? If so, state what they are, and verify by multiplication.

21. $a(x+y)+b(x-y)$. **22.** $c(x+y)+d(y+x)$.

23. $x(x+y)+y(x+z)$. **24.** $a(b+c)+b(a+c)$.

25. $x(a-b)+y(b-a)$. **26.** $x(y+1)+y(x+1)$.

27. $x(2a+2)+y(a+1)$. **28.** $a(2x+y)+2b(x+y)$.

29. $x(a+b)+a+b$. **30.** $a(b-c)-b+c$.

Factorise the following expressions, *when possible.*

31. $x^2+xy+xz+yz$. **32.** $x^2-xy+xz-yz$.

33. $x^2-xy-xz+yz$. **34.** $x^2-xy-xz-yz$.

35. $a^2+ab+2a+2b$. **36.** $a(x+y+z)+b(x+y+z)$.

37. $x^4+a^2x^2+b^2x^2+a^2b^2$. **38.** $x^6+x^4+x^2+1$.

39. $a^2-ab+a-b$. **40.** $x^2-xy-x-y$.

41. $a^2+ab-a-b$. **42.** $x^2(x+a)-bc(x+a)$.

43. $a(x+y)+c(y-x)$. **44.** $2a+6b+3by+ay$.

45. $x^2+(c+d)x+cd$. **46.** $1+a(x+x^2)+x$.

47. $(a+2b)(x+y)+(a+2c)(x+y)$. **48.** $a(bx+cx)+bd+cd$.

49. $ax-bx+bq-aq$. **50.** $z^4+z^3+z^2+z$.

The expression ax^2+bx+c, where a, b, c represent any numbers whatever, is called **a quadratic function of x**.

As has been stated in Chapter II. p. 12.

a is called the coefficient of the term in x^2, or **the coefficient of x**^{2}.

b is called the coefficient of the term in x, or **the coefficient of x.**

c is called **the constant term**.

FACTORS OF x^2+px+q.

x^2+px+q is a quadratic function of x in which the coefficient of x^2 is 1.

If you multiply $x+3$ by $x+11$ you get $x^2+3x+11x+33$ or $x^2+14x+33$.

The coefficient of x is the result of adding 3 and 11.

The constant term is the result of multiplying 3 by 11.

Therefore, in order to put $x^2+14x+33$ into factors, you will look for two numbers whose sum is 14 and whose product is 33.

These are 3 and 11.

$$\therefore\ x^2+14x+33=(x+3)(x+11).$$

So, to find the factors of x^2+px+q, you look for two numbers whose sum is p and whose product is q.

Example VIII. Factorise $x^2+9x+20$.

What are the two numbers whose sum is 9 and whose product is 20?

First take the numbers whose *product* is 20, for (if we keep to whole numbers) there are only a few possibilities.

$$20=20\times 1=10\times 2=5\times 4;$$

$$\therefore \text{ the numbers are 4 and 5};$$

$$\therefore\ x^2+9x+20=(\mathbf{x}+\mathbf{4})(\mathbf{x}+\mathbf{5}).$$

Always verify your answer by multiplication, either in your head or on paper.

Example IX. Factorise $x^2-3x-70$.

What are the two numbers whose sum is -3 and whose product is -70?

$$70=1\times 70=2\times 35=5\times 14=7\times 10;$$

$$\therefore \text{ the numbers are } +7 \text{ and } -10;$$

$$\therefore\ x^2-3x-70=(\mathbf{x}+\mathbf{7})(\mathbf{x}-\mathbf{10}).$$

Example X. Factorise $70+3x-x^2$.

Do not turn the expression round.
Do not forget to verify your answer by multiplication.

$$70+3x-x^2=(\mathbf{7+x})(\mathbf{10-x}).$$

EXERCISE VIII. e.

(NOTE.—*Examples* 1–12 *should be worked mentally.*)

Work out the following expressions :

1. $(x+2)(x+3)$.
2. $(x+5)(x+7)$.
3. $(x-6)(x+9)$.
4. $(x-3)(x-8)$.
5. $(a+1)(a-11)$.
6. $(1+x)(2+x)$.
7. $(2-x)(7+x)$.
8. $(z-20)(z+2)$.
9. $(3+x)(8-x)$.
10. $(1+3x)(1+2x)$.
11. $(9-a)(4-a)$.
12. $(x+2y)(x-3y)$.

Write down the coefficient of x in each of the following expressions :

13. $(x+9)(x+30)$.
14. $(x+50)(x-80)$.
15. $(x+8)(x-8)$.
16. $(x-12)(x-4)$.
17. $(x+7)(x+7)$.
18. $(5+x)(x-8)$.

Write down the constant term in Examples 19–24.

19. $(x-15)(x+10)$.
20. $(a-6)(a-8)$.
21. $(5-y)(y+3)$.
22. $(q+7)(q-4)$.
23. $(1-3x)(1+5x)$.
24. $(2-z)(z-2)$.

25. Find the factors of $x^2+(3+5)x+3\times 5$.

26. Find the factors of $x^2-(c+d)x+cd$.

Find the factors of the following :

27. x^2+6x+8.
28. $x^2+9x+20$.
29. $x^2-8x+15$.
30. $x^2-9x+20$.
31. x^2+3x+2.
32. x^2+7x+6.
33. $x^2-11x+10$.
34. $x^2-2x-35$.
35. $x^2-3x-70$.
36. $x^2+7x+12$.
37. $x^2+12x+36$.
38. x^2-8x+7.
39. x^2-4x+4.
40. $x^2+105x+500$.
41. x^2+2x+1.
42. $x^2+22x+40$.
43. $y^2-7y-44$.
44. $a^2+5a-14$.
45. a^2-25.
46. $z^2-9z-36$.
47. $l^2-50l+600$.
48. $1+3x-10x^2$.
49. $10+7y+y^2$.
50. $24-5x-x^2$.
51. $12-x-x^2$.
52. $35+2y-y^2$.
53. $1-x-42x^2$.
54. $3x^2+9x+6$.
55. $10x^2+110x+280$.
56. $x^2y^2+8xy-9$.
57. $28-3ab-a^2b^2$.
58. $2x^2-6x-36$.
59. $5x^2-10x+5$.

60. What is the L.C.M. of x^2+2x+1 and x^2+x ?

61. Simplify $\frac{x^2+7x+10}{x^2+5x}$.

62. What is the quotient if $x^2+7x+12$ is divided by $x+3$?

63. If $x+2$ is a factor of $x^2+ax+14$, what is the value of a ?

64. If $x-3$ is a factor of $x^2-bx-15$, what is the value of b ?

65. If $x+4$ is a factor of x^2+8x+c, what is the value of c ?

66. Find the H.C.F. of x^2+3x and x^2+5x+6.

67. Find the L.C.M. of $2x+2$ and $3x^2+9x+6$.

68. Simplify $\frac{x-3}{x^2-4x+3}$.

69. If $\frac{2}{x-3}=\frac{a}{x^2-5x+6}$, what is a ?

70. If $x^2-35x+150$ is divided by $x-5$, what is the quotient ?

71. Simplify $\frac{x^2+2x+1}{x^2+5x+4}$.   **72.** Simplify $\frac{1}{x-2}-\frac{1}{x-1}$.

73. One of the factors of $x^3+6x^2+11x+6$ is $x+1$; find the others.

74. Simplify $\frac{x^2-4x-5}{x^2-2x-15}\div\frac{x^2-x-2}{x^2-x-12}$.

75. Simplify $\frac{2x}{x^2+7x+12}-\frac{1}{2x+6}$.

76. If $\frac{x}{x+1}=\frac{x^2+x}{a}$, what is a ?

77. Simplify $\frac{2}{x^2-2x-15}-\frac{1}{x^2-5x}$.

78. What is the coefficient of x in $ax^2+bx+x+c$?

79. What is the coefficient of x in $(ax+b)\left(cx+\frac{1}{a}\right)$?

80. What is the coefficient of x in $(x+a)^2-(x+1)^2$?

Simplify the following :

81. $1-\frac{x+1}{(x+2)(x+3)}$.   **82.** $\frac{x+1}{x+2}-\frac{x+2}{x+3}$.

83. $\frac{2}{1+\frac{x-1}{x+1}}$.   **84.** $\left(1-\frac{x}{x+4}\right)\left(\frac{x}{x-3}-1\right)$.

85. $\dfrac{1}{x(x+1)}+\dfrac{1}{(x+1)(x+2)}+\dfrac{1}{x(x+2)}$. **86.** $-\dfrac{1}{x^2-6x+8}+\dfrac{2}{x^2-4x}$.

87. $\dfrac{x+1}{x^2-x-12}-\dfrac{x+3}{x^2+x-6}$. **88.** $\dfrac{1+\dfrac{a+b}{a-b}}{1+\dfrac{a-b}{a+b}}$.

89. $\dfrac{1}{(x+1)(x+2)}-\dfrac{2}{(x+1)(x+2)(x+3)}$. **90.** $\dfrac{3y}{x-3y}-\dfrac{y}{x-y}$.

Solve the following equations :

91. $\dfrac{x}{x+2}=\dfrac{2x-1}{2x+1}$. **92.** $\dfrac{2x+1}{5x+3}-\dfrac{4x-3}{10(x-1)}=0$.

93. $\dfrac{3x+4}{3x}=1+\dfrac{3}{x+7}$. **94.** $\dfrac{6(x+2)}{3x-1}=\dfrac{2x}{x-5}$.

95. $\dfrac{1}{x+1}+\dfrac{1}{x-2}=\dfrac{2}{x}$. **96.** $\dfrac{3}{x+1}+\dfrac{8}{x-5}=\dfrac{11}{x}$.

97. $\dfrac{4}{2x-1}-\dfrac{1}{x+1}=\dfrac{5-x}{(2x-1)(x+1)}$.

98. $\dfrac{3x}{4(3x+5)}-\dfrac{x+2}{4x+1}=\dfrac{1}{2(3x+5)}$.

99. $\dfrac{3x+2}{x^2+5x+6}-\dfrac{2}{x+2}=\dfrac{3}{x+3}$. **100.** $\dfrac{x+7}{x^2-x-6}=\dfrac{x-5}{x^2-4x+3}$.

FACTORS OF ax^2+bx+c.

Example XI. Factorise $6x^2+11xy-10y^2$.

The product of the first terms of the two brackets is $6x^2$.

The product of the last terms of the two brackets is $-10y^2$.

∴ the first terms are either $6x$ and x or $3x$ and $2x$, and the last terms are either $10y$ and $-y$, $-10y$ and y, $5y$ and $-2y$, or $-5y$ and $2y$.

This gives various possible arrangements from which we must select that one which gives the correct middle term.

For example, $(\underset{\uparrow}{6x}+\overset{\downarrow}{5y})(\overset{\downarrow}{x}-\underset{\uparrow}{2y})=6x^2-7xy-10y^2$,

$$(\underset{\uparrow}{3x}-\overset{\downarrow}{2y})(\overset{\downarrow}{2x}+\underset{\uparrow}{5y})=6x^2+11xy-10y^2.$$

∴ the factors are $(3x-2y)(2x+5y)$.

EXERCISE VIII. f.

(NOTE.—*Examples* 1–12 *should be worked mentally.*)

Work out the following expressions :

1. $(2x+1)(x+4)$.

2. $(3x+2)(4x+5)$.

3. $(3x-2)(4x-5)$.

4. $(3x+2)(4x-5)$.

5. $(2x-7)(3x-1)$.

6. $(x+2y)(2x+y)$.

7. $(4x+y)(x-3y)$.

8. $(4+5x)(3+2x)$.

9. $(1-x)(7-8x)$.

10. $(3a+2b)(4b-a)$.

11. $(5x+11)(2x-7)$.

12. $(3-5x)(7-3x)$.

Write down the coefficient of x in each of the following :

13. $(2x-9)(x-3)$.

14. $(2x+4)(3x-1)$.

15. $(2x+25)(x+10)$.

16. $(7x-9)(2x-11)$.

17. $(3+x)(2x-1)$.

18. $(4-3x)(2x-5)$.

Write down the constant term in each of the following :

19. $(2x+11)(x+12)$.

20. $(3x-1)(x-17)$.

21. $(17x+2)(x-20)$.

22. $(3x-11)(x-1)$.

23. $(3-5x)(12-11x)$.

24. $(4x-5)(10-13x)$.

Find the factors of the following expressions :

25. $3x^2+7x+2$.

26. $3x^2-7x+2$.

27. $2x^2+5x+2$.

28. $2x^2-5x+2$.

29. $5x^2+8x+3$.

30. $2x^2+11x+14$.

31. $3x^2+x-2$.

32. $3x^2-x-2$.

33. $5x^2-2x-7$.

34. $6x^2+7x-5$.

35. $15x^2-11x+2$.

36. $6x^2+5x-6$.

37. $3x^2+14x-5$.

38. $8a^2-14a+3$.

39. $6a^2-4a-2$.

40. $2x^2+11xy-21y^2$.

41. $3-5x+2x^2$.

42. $2-x-15x^2$.

43. $14a^2+29a-15$.

44. $2+7x-15x^2$.

45. $2x^2-xy-21y^2$.

46. $15+14a-8a^2$.

47. $20a^2+44a-15$.

48. $4x^4-13x^2y^2+9y^4$.

49. $6x^4+10x^2yz-4y^2z^2$.

50. Find the L.C.M. of $2x^2+3xy-2y^2$ and $10x^2-7xy+y^2$.

51. Write down the product of $2a+21b$ and $3a-10b$.

52. Simplify $\dfrac{x}{2x-3}-\dfrac{x+2}{2x-1}$.

53. Find the H.C.F. of $5x+10$ and $5(2x^2-5x-18)$.

54. If $\dfrac{2x-3}{3x-11}=\dfrac{y}{3x^2-8x-11}$, what is y?

55. If $6x^2-29x-65$ is divided by $3x+5$, what is the quotient?

56. Find the factors of $(2x^2-5x+2)(7+5x-2x^2)$.

57. $x-1$ is one factor of $2x^3+7x^2+x-10$; find the others.

58. Solve the equation $\dfrac{x}{2x-1}+\dfrac{2}{3x+2}=\dfrac{1}{2}$.

59. Simplify $\dfrac{2}{3x^2+x-2}-\dfrac{1}{4x^2+3x-1}$.

60. Simplify $\dfrac{6x^2-x-2}{3x^2+x-2}\left(2-\dfrac{x-1}{2x+1}\right)$.

FACTORS OF THE DIFFERENCE OF TWO SQUARES.

By multiplication, we find that

$$(A+B)(A-B)=A^2-B^2;$$

$\therefore$ the factors of A^2-B^2 are $A+B$ and $A-B$.

In this, A and B can stand for any numbers or any expressions.

Thus $$x^2-9=x^2-3^2=(x+3)(x-3),$$

$$16x^2-49=(4x)^2-7^2=(4x+7)(4x-7),$$

and $$(x+a)^2-b^2=\{(x+a)+b\}\{(x+a)-b\}$$
$$=\{x+a+b\}\{x+a-b\}.$$

Note.—(i) There are no (rational) factors of A^2+B^2.

(ii) Although x^2-2 has no rational factors, it may be written $x^2-(\sqrt{2})^2$ and $=(x+\sqrt{2})(x-\sqrt{2})$.

(iii) Although $(x-3)^2-5$ has no rational factors, it may be written $(x-3)^2-(\sqrt{5})^2$ and $=\{(x-3)+\sqrt{5}\}\{(x-3)-\sqrt{5}\}$
$$=\{x-3+\sqrt{5}\}\{x-3-\sqrt{5}\}.$$

EXERCISE VIII. g.

1. What is the product of $x+y$ and $x-y$?

2. What is the quotient if x^2-y^2 is divided by $x+y$?

3. What is the quotient if x^2-16 is divided by $x-4$?

4. What are the factors of x^2-a^2? Use your result to find the factors of 23^2-20^2.

Find the factors of the following :

5. x^2-4.
6. a^2-49.
7. a^2-1.
8. $100-b^2$.
9. $4a^2-1$.
10. $4a^2-9$.
11. $9x^2-y^2$.
12. $4a^2-9b^2$.
13. $16y^2-25$.
14. x^2y^2-16.
15. a^2-25b^2.
16. $x^2y^2-z^2$.
17. a^4-9.
18. $25x^6-y^6$.
19. $3a^2-12$.
20. $5-20x^2$.
21. $(x+7)^2-6^2$.
22. $(x-1)^2-81$.
23. x^4-1.
24. $(x+y)^2-1$.
25. $(2x+3)^2-16$.
26. $(a+b)^2-c^2$.
27. $16-(a+b)^2$.
28. $b^2-(a-3x)^2$.
29. $4(a+3b)^2-9$.
30. $25(2a-b)^2-49$.
31. $16-9(x+y)^2$.

32. Write down the product of $3a^4+7$ and $3a^4-7$.

33. Find the H.C.F. of x^2-x and x^2-1.

34. Simplify $\dfrac{(x+3)^2}{x^2-9}$.

35. Find the L.C.M. of $xy-y$ and x^2-1.

36. Show that 41 is a factor of 122^2-81^2.

37. Find the L.C.M. of $4x^2$ and $4x^2-4$.

38. If $\dfrac{2}{x+2}=\dfrac{a}{x^2-4}$, what is the value of a ?

39. Show that 83^2-27^2 is divisible by 11.

40. What is the coefficient of x in $(3x+7)(3x-7)$?

41. Is x^4+25 divisible by x^2+5 ?

42. Simplify $\dfrac{a}{x-a}-\dfrac{a}{x+a}$.

43. Show that the difference between the square of $2n$ and the square of $2n+2$ is divisible by $2n+1$.

44. Solve the equation

$$\frac{x+1}{x-2}-\frac{x-1}{x+2}=\frac{6}{x+1}.$$

45. Solve the equation

$$(x+5)(x-5)+(x+6)(x-6)=x(2x+1).$$

46. Simplify $\dfrac{a^2-\dfrac{1}{a^2}}{a-\dfrac{1}{a}}\times a$.

47. Find irrational factors of x^2-11. Show that either $x=\sqrt{11}$ or $x=-\sqrt{11}$ makes x^2-11 equal to 0.

48. Write down the product of $4a^2+1$ and $4a^2-1$.

49. Write down the product of 22 and 24 as the difference of two squares.

50. What is the L.C.M. of x^2+2x-3 and x^2-9 ?

51. Simplify $\frac{a}{a+b}+\frac{a}{a-b}-\frac{a^2+b^2}{a^2-b^2}$.

52. Simplify $\left(x-\frac{xy}{x+y}\right)\left(x+\frac{xy}{x-y}\right)$.

53. Simplify $\frac{1}{2x+4}+\frac{2}{3x+6}-\frac{x-3}{x^2-4}$.

54. Solve the equation

$$\frac{x}{2x+1}+\frac{x}{2x-1}=1+\frac{2x-3}{4x^2-1}.$$

55. Find irrational factors of x^2-3.

56. Find irrational factors of $(x-1)^2-2$.

57. Factorise $(x-29)^2-9^2$. Show that $x=38$ makes $(x-29)^2-9^2$ equal to 0.

SQUARES AND SQUARE ROOTS.

$$\begin{aligned}(A+B)^2&=(A+B)(A+B)\\&=A^2+AB+AB+B^2\\&=A^2+2AB+B^2.\end{aligned}$$

This gives a rule for writing down the square of any expression which consists of two terms.

Thus $$\begin{aligned}(3x+5y)^2&=(3x)^2+2(3x)(5y)+(5y)^2\\&=9x^2+30xy+25y^2,\end{aligned}$$

and $$\begin{aligned}(3x-5y)^2&=(3x)^2+2(3x)(-5y)+(-5y)^2\\&=9x^2-30xy+25y^2.\end{aligned}$$

It is important to be able to recognise a perfect square when you have to deal with it.

$$\begin{aligned}25x^2-40xy+16y^2&=(5x)^2-2(5x)(4y)+(4y)^2\\&=(5x-4y)^2 \quad\text{or}\quad =(4y-5x)^2.\end{aligned}$$

Notice that an expression has always two square roots.

The square root of 49 is $+7$ or -7.

The square root of $25x^2-40xy+16y^2$ is $+(5x-4y)$ or $-(5x-4y)$.

Example XII. Add a number to $49x^2-126x$ to make it a perfect square.

The first term is $(7x)^2$.

The second term is $2.7x.9$.

$\therefore$ the last term must be 9^2.

EXERCISE VIII. h.

Write down the squares of

1. $x+3$.
2. $x-1$.
3. $x+4y$.
4. $3x-y$.
5. $2a-3b$.
6. $5x^2+7y$.
7. $2xy+z$.
8. x^2+y^2.
9. $a+\frac{1}{x}$.
10. $2b+\frac{3}{c}$.
11. $5bc-\frac{2}{d}$.
12. $3x-\frac{4}{x}$.

Write down a square root of each of the following:

13. x^2-2x+1.
14. $x^2+6xy+9y^2$.
15. $4x^2+20xy+25y^2$.
16. $a^2b^2+10abc+25c^2$.
17. $9a^2-42ab+49b^2$.
18. $25a^2+4b^2-20ab$.
19. $a^4+2a^2b^2+b^4$.
20. $100+60x+9x^2$.
21. $x^2+x+\frac{1}{4}$.

Add a number to the following so as to make the result a perfect square:

22. x^2+6x.
23. x^2-10x.
24. a^2+5a.
25. $4x^2+28x$.
26. x^2-7x.
27. x^2-ax.

Fill in the missing terms in the following:

28. $(x+5)^2=x^2+....+25$.
29. $(2x+7)^2=....+28x+49$.
30. $(x-3a)^2=x^2....+9a^2$.
31. $(2x-a)^2=4x^2....+a^2$.
32. $(x....)^2=x^2-12x+36$.
33. $(2x....)^2=4x^2-28x+49$.
34. $(a....)^2=a^2+8ab+....$.
35. $(....-3a)^2=4....+9a^2$.

In the following, state whether the expression is a perfect square; if it is, give a square root of it.

36. $x^2+4xy+4y^2$.
37. $x^2+6xy+36y^2$.
38. $4x^2-40xy+25y^2$.
39. $x^2+x+\frac{1}{4}$.
40. $a^2+2+\frac{1}{a^2}$.
41. $a^4+6a^2b^2+9b^4$.

42. $x^2-2xy-y^2$. **43.** $1-2x+x^2$.

44. $x^2+5x+6\frac{1}{4}$. **45.** $25x^2y^2+9y^2-30xy^2$.

46. What is the coefficient of x in $(3x-7)^2$?

47. What is the constant term in $5(2x+3)^2$?

48. Give two terms, either of which, when added to x^2+49, makes a perfect square.

49. What single terms, added to $x^4+x^2y^2+y^4$, will make the result a perfect square?

50. Is $\frac{9x^2}{16}+\frac{3}{5}xy+\frac{4}{25}y^2$ a perfect square?

51. Simplify $(x-1)^2-(1-x)^2$.

52. Simplify $(5x-15)^2-25(x-3)^2$.

53. Find the L.C.M. of $(2x-2)^2$ and $(3x-3)^2$.

54. Simplify $\frac{(7x+14)^2}{(x+2)^2}$.

55. One square root of $4x^2+4x+1$ is $2x+1$; what is the other?

MISCELLANEOUS FACTORS.

EXERCISE VIII. k.*

Factorise the following expressions:

1. $3x^2-3x$. **2.** x^2+9x+8. **3.** p^2q-pq^2.
4. $a-4a^2$. **5.** a^4-b^2. **6.** $a^2-15a+26$.
7. $5n^2-25$. **8.** x^2+x-90. **9.** $x(b+c)+y(b+c)$.
10. $1+n+n^2+n^3$. **11.** y^2-64z^2. **12.** $t^2+13t+40$.
13. a^3b-ab^3. **14.** $3z^2+5z-2$. **15.** $x(m-n)+z(n-m)$.
16. x^3-x^2-x+1. **17.** $a^4-4a^2b^2$. **18.** $a(x+1)+b(x+1)+c(x+1)$.
19. $7x^2-28z^2$. **20.** x^3+3x^2+2x. **21.** $4x^2-xy-5y^2$.
22. $a^2-ab+bd-ad$. **23.** x^4-4x^2+3. **24.** x^3-x^2+x.
25. $x^2-x(y+z)+yz$. **26.** $a^2b^2-2ab-35$. **27.** $u^2+4uv+4v^2$.
28. x^4-36x^2. **29.** $a^3b+a^2b^2+ab^3$. **30.** $10a^2+50ax+60x^2$.
31. $ab+xy-ay-bx$. **32.** $1-5x+4x^2$. **33.** $ax-a-x+1$.
34. $4x^2+20xy+25y^2$. **35.** $1-\frac{9}{x^2}$. **36.** $x^2-pq-(p-q)x$.
37. $3x^2+9x-120$. **38.** $cx-dx+dq-cq$. **39.** $72-17x+x^2$.
40. $17x-72-x^2$. **41.** $a^3-a^2b+ab^2-b^3$. **42.** $(a-b)^2-100$.

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* For additional examples, see E.P. 14, page 251/24.

43. $x(2y-3z)+3z-2y$.

44. $(x+z)^2+5(x+z)$.

45. $ah+bh+ch+ak+bk+ck$.

46. $x^2-(a+3)x+3a$.

47. $4a^2-(b+c)^2$.

48. a^5-a^4-a+1.

49. $mp^2-mq^2+lq^2-lp^2$.

50. $5x^2-10xy+5y^2$.

51. $z^2-16(x-y)^2$.

52. $6(a^2-bc)+4ac-9ab$.

53. $31x-10x^2-15$.

54. $49a^2-(3a-4b)^2$.

55. $x(y-1)-y+1$.

56. $a^2b^2+c^2d^2-a^2c^2-b^2d^2$.

57. $4-(1-x)^2$.

58. $m(n-1)+(n-1)^2$.

59. $1+\frac{2}{a}+\frac{1}{a^2}$.

60. n^5-n.

MISCELLANEOUS EXAMPLES.

EXERCISE VIII 1.*

1. What is the quotient if $xy(x-1)$ is divided by (i) x, (ii) $x(x-1)$?

2. What are the factors of x^2-x-42?

3. What is the product of $x+8$ and $x-8$?

4. Write down the product of $x-1$ and $x-13$.

5. Find the H.C.F. and L.C.M of x^2-x-20 and x^2-25.

6. Simplify $\frac{1}{x}-\frac{1}{x+5}$.

7. Express $\frac{x}{x-6}$ as a fraction with $x^2-9x+18$ as denominator.

8. Multiply x^2+7x+6 by $x-3$. What are the factors of the product?

9. What is the meaning of $4(x-1)^2$? What is the quotient if it is divided by $x-1$?

10. Express $\frac{2}{x-1}$ and $\frac{3}{x-2}$ as fractions with the same denominators.

11. What are the factors of 23^2-36?

12. Simplify $\frac{x^2-4x}{2(x^2-2x-8)}$.

13. Divide a^3-ab^2 by (i) $a(a-b)$, (ii) $a-b$.

14. $x-3$ is one factor of $x^3-7x^2-9x+63$. What are the others?

15. Solve the equation $(2x+1)(6x-7)=(3x+4)(4x-6)$.

*** For further practice in Fractions, see E.P. 15, page 251/25.**

16. What is the quotient if $x^3 - x$ is divided by $x^2 - x$?

17. Simplify $\dfrac{1}{x^2 - x} - \dfrac{1}{x^3 - x}$.

18. (i) Factorise $2x^2 - 15x + 18$;
(ii) If $x = 6$, show that $2x^2 - 15x + 18$ is equal to 0.

19. Write down the product of $2x - 7$ and $x - 2$.

20. Solve the equation $\dfrac{12x}{x+2} = 6 + 2\left(\dfrac{3x+2}{x+1}\right)$.

21. What is the quotient if $100^2 - 93^2$ is divided by 7?

22. Simplify $x - \dfrac{x^2 - 3}{x + 3}$.

23. What number must be added to $x^2 + 14x$ to make the result a perfect square, and of what is it then the square?

24. Simplify $\dfrac{x}{2(x+3)} - \dfrac{x}{2(x-3)}$.

25. Solve the equation $\dfrac{6}{3x-5} - \dfrac{1}{x-5} = \dfrac{2}{2x-5}$.

26. Simplify $\left(1 - \dfrac{1}{x}\right)\left(1 + \dfrac{1}{x-1}\right)$.

27. Prove that $\left(1 - \dfrac{1}{x}\right)^2 = \dfrac{1 - 2x + x^2}{x^2}$.

28. Prove that $(x^2 + 5x + 6)(x^2 - 5x + 6) = (x^2 - 4)(x^2 - 9)$.

29. Simplify $(x + 2) \div \left(1 + \dfrac{2}{x}\right)$. **30.** Simplify $\left(1 - \dfrac{1}{x}\right) \div \left(x - \dfrac{1}{x}\right)$.

31. Solve the equation $\dfrac{2x-3}{3x-4} = \dfrac{4x-5}{6x-7}$.

32. What is the square root of $(x^2 - 2x + 1)(x^2 + 4x + 4)$?

33. Prove that $(x^2 + x - 2)(x^2 - 4x + 3)(x^2 - x - 6)$ is a perfect square, and find its square root.

34. Divide $x^3 - 4x$ by $x^2 + 2x$.

35. Prove that $(x - 1)(x + 1)(x^2 + 1)(x^4 + 1) = x^8 - 1$.

36. Factorise $(3x + 1)(x + 2)^2 - (x + 2)(x - 1)^2$.

37. Factorise $(x + 1)(x + 2)(x + 3) - 2x - 2$.

38. Factorise $a(b + 6c) - 3a^2 - 2bc$.

39. Factorise $(x - y)^2 + (xy + 1)^2$. **40.** Factorise $ab^2c - 9ac^3$.

41. Solve the equation $\dfrac{x-4}{x-2} + \dfrac{x+3}{x-5} = 2$.

42. What is the least expression by which $(x^2-4x+3)(x^2+x-2)$ must be multiplied in order to make it a perfect square ?

43. Find the factors of $(a-2b)^2-7(a-2b)+12$.

44. Simplify $\dfrac{1}{1-\dfrac{1}{x}}+\dfrac{1}{\dfrac{1}{x}-1}$.

45. Solve the equation $\dfrac{(x-1)(x-2)(x-6)}{(x-3)^3}=1$.

46. Multiply $3x+7y-800$ by $3x+7y+800$.

47. Factorise $(x-3)^3-a^2b^2(x-3)$.

48. Divide $(2x^2-7x+4)^2-(x^2-2x+5)^2$ by x^2-5x-1.

49. Prove that $(x^2+y^2)(p^2+q^2)=(px-qy)^2+(qx+py)^2$.

50. Factorise $(2x+3y)^2-2x-3y-12$.

51. Factorise $ab-7a+5b-35$.

52. Solve the equation $\dfrac{4(x+5)}{x+6}+\dfrac{3(x+6)}{x+5}=7$.

53. If $x+4$ is a factor of $x^2+ax-28$, what is the value of a and what is the other factor ?

54. Write down the square of $3b-2c$.

55. What is the coefficient of x^3 in the product

$$(1+3x-x^2-2x^3)(2+x-x^2-x^3)\ ?$$

56. Solve the equation $\dfrac{6x+13}{15}-\dfrac{3x+5}{5x-25}=\dfrac{2x}{5}$.

57. Factorise $(x+y)^2(x-1)+(y+z)^2(1-x)$.

58. Simplify $\dfrac{x^2-x-2}{x^2-5x+6}-\dfrac{x^2+4x-5}{x^2+2x-15}$.

59. Evaluate $\dfrac{2240\times 48^2}{64}-\dfrac{2240\times 18^2}{64}$.

60. Simplify $(x-y)(x^2+xy+y^2)-(x+y)(x^2-xy+y^2)$.

61. What single term must be added to x^2-11ax in order to make the result a perfect square ?

62. Solve the equation $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-5}{x-6}-\dfrac{x-6}{x-7}$.

63. Factorise $a^2+4+\dfrac{4}{a^2}$.

64. Factorise $(3a+2b)^2-2(3a+2b)(2a+b)+(2a+b)^2$.

65. Divide $x+\frac{1}{x}-2$ by $\frac{1}{2x}-\frac{1}{2}$.

66. Solve the equation $\frac{x-4}{x+4}+\frac{x-6}{x+6}=2$.

67. Prove that $(a+b)^2+(b-c)^2-2(a+b)(b-c)=(a+c)^2$.

68. Factorise $a(a-1)-b(b+1)$.

69. Simplify $\frac{a^2+2ab+b^2}{a+b}+\frac{a^2-2ab+b^2}{a-b}$.

70. Factorise $(p-q)^2+(p+x)^2-(x-y)^2-(q+y)^2$.

CHAPTER IX.

LITERAL EQUATIONS. FORMULAE (CHANGE OF SUBJECT). IDENTITIES.

LITERAL EQUATIONS.

IN all the equations we have solved, the unknown number has been represented by a letter, but the known numbers have not been represented in this way. By comparing the two examples given below it will be seen that it is possible to represent the known numbers by letters. Such equations are called **literal equations.**

Example I.

(i) Find the number which is as much above 5 as it is below 17.

Let x be the unknown number ;

$$\therefore\ x-5=17-x\ ;$$
$$\therefore\ 2x=5+17\ ;$$
$$\therefore\ x=\frac{5+17}{2}.$$

(ii) Find the number which is as much above a as it is below b.

Here a and b are known numbers.
Let x be the unknown number.

$$\therefore\ x-a=b-x\ ;$$
$$\therefore\ 2x=a+b\ ;$$
$$\therefore\ x=\frac{a+b}{2}.$$

It is generally understood, unless it is otherwise stated or otherwise evident, that letters at the end of the alphabet—

x, y, z, u, v, w—stand for unknown numbers; other letters for known numbers.

Example II. Solve the equation

$$ax - ab = ab - bx.$$

Here a and b are known numbers, x an unknown number.

Since $\qquad ax - ab = ab - bx,$

$\therefore\ ax + bx = 2ab$, transposing the terms $-ab$ and $-bx$;

$$\therefore\ (a+b)x = 2ab.$$

Divide each of the equal quantities by $a + b$;

$$\therefore\ \mathbf{x} = \frac{\mathbf{2ab}}{\mathbf{a+b}}.$$

EXERCISE IX. a.

Solve the following equations:

1. $4x - 3a = 5a + 2x.$ **2.** $2(x+a) = 5(x-a).$

3. $x + a = b.$ **4.** $3(x-p) = x + 3q.$

5. $a - x = x - b.$ **6.** $ax = b.$ **7.** $abx = a + b.$

8. $a(x+b) = 2ab.$ **9.** $(m-n)x = mn.$ **10.** $ax + b = bx + a$

11. $x + ax = b.$ **12.** $ax + bx = c.$ **13.** $ax - a^2 = b^2.$

14. $h(x-h) = k(x-k).$ **15.** $a(x-b) = b(x-a).$

16. $\frac{x}{2} + \frac{a}{3} = 2a - \frac{x}{3}.$ **17.** $\frac{1}{2}(x+a) + \frac{1}{3}(x-a) = a.$

18. $\frac{a}{2}(x-b) = \frac{b}{3}(x-a).$ **19.** $\frac{1}{a-b} - x = \frac{1}{a+b}.$

20. $\frac{x}{a} = b.$ **21.** $\frac{1}{x} = \frac{q}{p}.$ **22.** $\frac{a}{x} = 2a.$

23. $\frac{x}{p} = \frac{p}{q}.$ **24.** $\frac{x+a}{x-a} = 2.$ **25.** $\frac{a-x}{b-x} = \frac{p}{q}.$

26. $\frac{1}{a} - \frac{1}{x} = \frac{1}{b}.$ **27.** $\frac{x}{a} + \frac{x}{b} = 1.$ **28.** $\frac{a}{x} - 1 = 1 - \frac{b}{x}.$

29. $ax + \frac{1}{a} = bx + \frac{1}{b}.$ **30.** $\frac{x-a}{b} + \frac{x-b}{a} = 2.$

31. $\frac{x-a}{b-a} = \frac{x-b}{a-b}.$ **32.** $\frac{p-q}{qx+r} = \frac{p+q}{px-r}.$

33. $\frac{x-a}{x-b} = \frac{x-c}{x-d}.$ **34.** $(x-a)^2 - (x-b)^2 = 0.$

35. $a(x-b)+b(x-c)+c(x-a)=0.$

36. $(x-a)(x-b)=(x+a)(x+b).$

37. $(x+a)^2+(x+b)^2=2x^2.$ **38.** $(x-a)(x-c)=(x-b)^2.$

39. $\frac{x}{bc}+\frac{x}{ca}+\frac{x}{ab}=a+b+c.$ **40.** $\frac{x-a}{a}+\frac{x-b}{b}+\frac{x-c}{c}=1.$

EXERCISE IX. b.

1. (i) A stick is 18 inches long. It is cut into two parts, one of which is three times the other. Find the length of each part.

(ii) A stick is l inches long. It is cut into two parts, one of which is three times the other. Find the length of each part.

2. (i) Find the angles of a triangle, if each angle at the base is double the angle at the vertex.

(ii) Find the angles of a triangle, if each angle at the base is n times the angle at the vertex.

3. (i) The sum of two consecutive numbers is 37; find them.

(ii) The sum of two consecutive numbers is $2n+3$; find them.

4. (i) On one side of a pair of scales is a weight of 7 lb.; on the other side 4 lb. How much must be taken from one side and put on the other so that they may balance?

(ii) On one side of a pair of scales is a weight of m lb.; on the other side a smaller weight of n lb. How much must be taken from one side and put on the other so that they may balance?

5. One angle of a triangle is d degrees and the other two are equal to one another; find their size.

6. Six ordinary chairs and two arm-chairs cost £a. If each ordinary chair costs p shillings less than an arm-chair, find the cost of each arm-chair.

7. The same number is added to the numerator and denominator of the fraction $\frac{a}{b}$, and its value is then $\frac{1}{2}$; find the number.

8. A room is a ft. long and b ft. wide; the area of the walls is $2ab$ sq. ft. Find its height.

9. The average age of a form of 20 boys is n years; the average age of the boys and the master is $n+1$ years; find the age of the master.

10. A chain hangs over a peg; a ft. of it are on the longer side and b ft. on the shorter. What is the distance of the middle point of the chain from the peg?

11. A man pays a tax of a shillings in the pound on the amount of his income above £160. If he pays an average of b shillings on all his income, find his income.

12. The price of coal diminishes 30 per cent., and is then c shillings per ton. What was the first price?

13. Divide 10s. among two people so that one may have a shillings more than the other.

14. Divide 300 into two parts in the ratio of a to b.

15. If r per cent. of a man's income is £p, find his income.

16. (i) Pewter consists of 4 parts tin and 1 part lead; how much tin is there in p lb. of pewter?

(ii) An alloy consists of a parts tin and b parts lead; how much tin is there in p lb. of the alloy?

17. (i) ABC is a triangle, $BC=7''$, $CA=4''$, $AB=6''$. If AD bisects the angle BAC, $\frac{BD}{DC}=\frac{BA}{AC}$. Find the lengths of BD and DC.

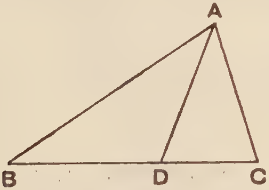

FIG. 20.

(ii) Work out the same question if $BC=a$, $CA=b$, $AB=c$.

CHANGE OF SUBJECT IN A FORMULA.

It is often convenient to write a formula in more than one form.

If the area of a rectangle x inches long and y inches wide is A square inches, we know that

$$A=xy.$$

This is the most suitable form if x and y are known numbers and A is unknown, *i.e.* if we are given the length and breadth and wish to find the area.

Writing the formula in the form $xy=A$ and dividing each side by y, we get

$$x=\frac{A}{y}.$$

This is the most suitable form if x is the unknown number, *i.e.* if we are given the area and breadth and wish to find the length.

This process is called **Changing the Subject** of a formula. In the first case A was the subject; in the second case x has become the subject.

Example III. A room is x ft. long, y ft. wide and h ft. high. If the area of the four walls is A sq. ft., we know that

$$A = 2h(x+y).$$

A is at present the subject of this formula; change the formula so that x becomes the subject.

$$A = 2h(x+y);$$

$$\therefore\ 2h(x+y) = A.$$

Divide each side by $2h$;

$$\therefore\ x+y = \frac{A}{2h}.$$

Subtract y from each side;

$$\therefore\ \mathbf{x} = \frac{\mathbf{A}}{\mathbf{2h}} - \mathbf{y}.$$

EXERCISE IX. c.

1. If x degrees is the vertical angle and y degrees a base angle of an isosceles triangle, $x + 2y = 180$. Make (i) x the subject, (ii) y the subject.

2. If the area of a triangle whose base is b in. and height h in. is A sq. in., $A = \frac{1}{2}bh$. Make (i) b the subject, (ii) h the subject.

3. If C in. is the length of the circumference of a circle of radius r in., $C = 2\pi r$, where π is approximately $3\frac{1}{7}$. Make r the subject. Do not substitute $3\frac{1}{7}$ for π.

4. If the simple interest on £P for T years at R per cent. is £I, $I = \frac{PRT}{100}$. Make (i) P, (ii) T the subject.

5. If y is the result of increasing x by r per cent., $y = x\left(1 + \frac{r}{100}\right)$. Make r the subject. Find r, if $x = 12$ and $y = 17$.

6. If A square inches is the area of a circle of radius r inches, $A = \pi r^2$. Make r the subject.

7. If V cubic feet is the volume of a pyramid, the height of which is h ft. and the area of the base A sq. ft., $V = \frac{1}{3}Ah$. Make h the subject.

8. If F degrees Fahrenheit is the same temperature as C degrees Centigrade, $F=32+\frac{9C}{5}$. Make C the subject. Find C when $F=8$.

9. The pressure p, volume v and temperature t of a quantity of gas are connected by the formula $\frac{pv}{273+t}=10$. Make t the subject. Find t if $p=5$, $v=650$.

10. Given that $v=u+ft$, find (i) t in terms of u, v, f; (ii) f in terms of u, v, t.

11. If the area of the surface of a cube, each edge of which is x inches long, is A square inches, then $A=6x^2$. Make x the subject. Find the edge of a cube, if the area of its surface is 150 sq. in.

12. If the volume of a circular cone of height h inches, and whose base is of radius r inches, is V cubic inches, then $V=\frac{1}{3}\pi r^2 h$. (i) Make h the subject; (ii) make r the subject.

13. If the area of the *total* surface of a circular cylinder of height h inches, and whose base is of radius r inches, is A square inches, then $A=2\pi r(r+h)$. Make h the subject.

14. If the area of a trapezium, whose parallel sides are of length x inches and y inches and are h inches distant apart, is A square inches, then $A=\frac{1}{2}h(x+y)$. (i) Make h the subject; (ii) make x the subject.

15. If $x^2y=\frac{1}{2}z$, find x in terms of y and z.

16. If $a^2+b^2=c^2$, find b in terms of a and c.

17. If $\frac{1}{u}+\frac{1}{v}=\frac{2}{f}$, find u in terms of v and f.

18. If $t=2\pi\sqrt{\frac{l}{g}}$, find l in terms of the other numbers.

19. If $s=\frac{n}{2}\{2a+(n-1)d\}$, find d in terms of the other numbers.

20. If $s=ut+\frac{1}{2}ft^2$, find f in terms of the other numbers.

IDENTITIES.

Compare the statements

$$x+x=3,$$

and

$$x+x=2x.$$

A statement such as the first is true for certain special values of x, in this case for one value, viz. $1\frac{1}{2}$; such a statement is called an **equation**.

The second statement is true for all values of x; such a statement is called an **identity**.

In an identity, the two expressions which are said to be equal are in reality the same expression in two different forms. To show that such a statement is an identity, we must show that one expression can be changed into the other; or that both can be reduced to the same expression.

An identity is often distinguished from an equation by using the symbol $\equiv$ instead of $=$. Thus, we write

$$x+x=3,$$

but

$$x+x\equiv 2x.$$

Example IV. Prove the identity

$$(x+a)(x-a)-b^2\equiv(x+b)(x-b)-a^2.$$

$$\begin{aligned}\text{The first expression}&=(x^2-a^2)-b^2\\&=x^2-a^2-b^2\\&=x^2-b^2-a^2\\&=(x+b)(x-b)-a^2\\&=\text{second expression.}\end{aligned}$$

Or,

$$\begin{aligned}\text{Left side}&=x^2-a^2-b^2.\\\text{Right side}&=x^2-b^2-a^2;\\\therefore\ \text{Left side}&=\text{Right side.}\end{aligned}$$

EXERCISE IX. d.

1. Prove that $x-1=-(1-x)$ is an identity.

2. Prove that $x+1=-(1+x)$ is not an identity. For how many values of x is it true?

3. Find whether $x^2-a^2=(x-a)^2$ is an equation or an identity.

4. Find whether $(x-2)^2=(2-x)^2$ is an equation or an identity.

Prove the following identities:

5. $a-b\equiv-(b-a)$.

6. $(a+d)+a+(a-d)\equiv 3a$.

7. $(y-z)+(z-x)+(x-y)\equiv 0$.

8. $(x-a)^2\equiv(a-x)^2$.

9. $(a+b)(a^2-ab+b^2)\equiv a^3+b^3$.

10. $x^2+y^2\equiv(x+y)^2-2xy$.

11. $(a-b)(c-d)\equiv(b-a)(d-c)$.

12. $(a-b)\{(a+b)^2-ab\}\equiv a^3-b^3$.

13. $(x-y)^2\equiv(x+y)^2-4xy$.

14. $a(b-c)+b(c-a)+c(a-b)\equiv 0.$

15. $x^4+x^2y^2+y^4\equiv(x^2+y^2)^2-x^2y^2.$

16. $(x-a)(b-c)+(x-b)(c-a)+(x-c)(a-b)\equiv 0.$

17. $(y+z)(y-z)+(z+x)(z-x)+(x+y)(x-y)\equiv 0.$

18. $\left(x-\dfrac{1}{x}\right)^2+2\equiv x^2+\dfrac{1}{x^2}.$

If the angles of a triangle are A, B, C degrees, prove the following statements :

19. $\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}=90.$ **20.** $\tfrac{1}{2}(B+C)=90-\dfrac{A}{2}.$

21. $45+\dfrac{A}{4}$, $45+\dfrac{B}{4}$, $45+\dfrac{C}{4}$ degrees are possible angles of a triangle.

If a, b, c are the sides of a triangle and $a+b+c=2s$, prove the following statements :

22. $s-a=\tfrac{1}{2}(b+c-a).$ **23.** $(s-a)+(s-b)+(s-c)=s.$

24. $\tfrac{1}{2}(b+c)+\tfrac{1}{2}(c+a)+\tfrac{1}{2}(a+b)=2s.$

25. $a(2s-a)-b(2s-b)=(a-b)c.$

26. If $x+\dfrac{1}{x}=a+b$ and $x-\dfrac{1}{x}=a-b$, show that $ab=1$.

27. Find numerical values of a and b so that $ax+b\equiv x+2$.

28. Find numerical values of a and b so that

$$a(x-1)+b(x+2)=3x$$

may be an identity.

CHAPTER X.

PROBLEMS AND QUADRATIC EQUATIONS, INCLUDING EASY SIMULTANEOUS QUADRATIC EQUATIONS.

If A and B are two quantities and all that we are told about them is that $AB=2$, we cannot have any idea of the value of A; nor can we of the value of B.

This is still true if, instead of 2, we put any other number except 0. But if

$$AB=0$$

we must have *either* $A=0$ *or* $B=0$. In other words, if you multiply two numbers the result cannot be zero, unless one or other of the numbers is zero.

SOLUTION BY FACTORS.

Example I. Solve $x(x-2)=0$.

If $x=0$, the equation is satisfied. For the first side $=0\times$ some number, and so is equal to 0.

If $x-2=0$, the equation is satisfied. For the first side $=$ some number $\times 0$, and so is equal to 0.

This gives $x=2$.

$$\therefore \text{ the equation is satisfied if } \mathbf{x=0} \text{ or if } \mathbf{x=2}.$$

Example II. Solve $x^2=9$.

Since

$$x^2=9;$$
$$\therefore x^2-9=0;$$
$$\therefore (x-3)(x+3)=0;$$
$$\therefore \text{ either } x-3 \text{ must be equal to } 0$$
$$\text{or } x+3 \text{ must be equal to } 0;$$
$$\therefore \text{ either } \mathbf{x=3} \text{ or } \mathbf{x=-3}.$$

Notice that it follows from this that there are two square roots of 9, viz.: $+3$ and -3.

Similarly, the equation $x^2=b$ is the same as

$$x^2-b=0, \text{ or } x^2-(\sqrt{b})^2=0, \text{ or } (x-\sqrt{b})(x+\sqrt{b})=0;$$

$$\therefore \text{ either } x-\sqrt{b}=0 \text{ or } x+\sqrt{b}=0,$$

$$i.e.\ x=\sqrt{b} \text{ or } -\sqrt{b}.$$

This is sometimes written '$x=\pm\sqrt{b}$.'

Example III. Solve $(x+3)(x+4)=2$.

Since

$$(x+3)(x+4)=2,$$

$$\therefore x^2+7x+12=2;$$

$$\therefore x^2+7x+10=0;$$

$$\therefore (x+5)(x+2)=0;$$

$$\therefore \text{ either } x+5=0 \text{ or } x+2=0;$$

$$\therefore x=-5 \text{ or } x=-2.$$

Check. If $x=-5$, $(x+3)(x+4)=-2\times-1$
$=2.$

If $x=-2$, $(x+3)(x+4)=1\times 2$
$=2.$

Example IV. Solve the equation $x^2-(a+b)x+ab=0$.

This is the same as $(x-a)(x-b)=0$. Therefore the roots are a and b.

Since a and b may have any values, it follows that in any equation, in which the coefficient of x^2 is 1,

(i) The coefficient of x = the sum of the roots with the sign changed;

(ii) The constant term = the product of the roots.

If we take the equation

$$Ax^2+Bx+C=0,$$

it may be put in the form

$$x^2+\frac{B}{A}x+\frac{C}{A}=0,$$

dividing each term by A.

Consequently, (i) the sum of the roots $=-\dfrac{B}{A}$;

(ii) the product of the roots $=\dfrac{C}{A}$.

This gives us a method of checking our answers. For example, if we find that the roots of $2x^2+3x-14=0$ are 2 and $-3\frac{1}{2}$, the sum of these roots is $-1\frac{1}{2}$ and their product -7. This is correct, as is seen by writing the equation in the form $x^2-(-\frac{3}{2})x-7=0$.

EXERCISE X. a.*

1. (i) If $xy=0$ and $x=1$, what must be the value of y?

(ii) If $xy=0$ and $x=0$, what do you know about the value of y?

(iii) If $xy=2$, can you give any information about the value of x?

(iv) If $(a-b)x=0$, what information can you give about the value of x?

(v) If $(x-1)(y-2)=0$ and $x=10$, what do you know about the value of y?

(vi) If $(x+1)(y+2)=0$ and $x=-1$, what do you know about the value of y?

Solve the following equations:

2. $2x^2=0$.

3. $x(x-3)=0$.

4. $2x(x+1)=0$.

5. $(x-2)(x-3)=0$.

6. $(x+2)(2x-1)=0$.

7. $x^2-5x=0$.

8. $x^2=3x$.

9. $2x^2=18$.

10. $4x^2=25$.

11. $2x^2+3x=0$.

12. $x^2-3x+2=0$.

13. $x^2+3x+2=0$.

14. $x^2-x-20=0$.

15. $5x^2=7x$.

16. $3-x^2=x^2-5$.

17. $(x+7)^2=0$.

18. $x^2-8x+12=0$.

19. $x^2+2x=15$.

20. $2x^2-15x+25=0$.

21. $2x^2+5x=7$.

22. $(3x)^2=100$.

23. $(x-2)^2=4$.

24. $4x^2+4x+1=0$.

25. $x^2-8x+15=0$.

26. $2x^2-25x+12=0$.

27. $x^2+11x+24=0$.

28. $x^2+11x+30=0$.

29. $3x^2-2x-1=0$.

30. $x^2+x=72$.

31. $5x^2+2=11x$.

32. $6x^2+5x-6=0$.

33. $x^2-13x+22=0$.

34. $10x^2-30x+20=0$.

35. $3x^2+15x+12=0$.

36. $10x^2-11x+3=0$.

37. $3x^2-8x=3$.

38. $4x+21=x^2$.

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* For additional examples, see E.P. 16, page 251/26.

39. $6x = (x+4)(x-4)$.

40. $\dfrac{n(n+1)}{2} = 28$.

41. $64t - 16t^2 = 48$.

42. $x^2 + (6-x)^2 = 20$.

43. $(x+3)^2 + (x-3)^2 = 20x$.

44. $\dfrac{n}{2}\{6 + 2(n-1)\} = 120$.

45. $\dfrac{24}{x} + \dfrac{24}{x-5} = 11$.

46. $\dfrac{210}{x+2} + \dfrac{210}{x+4} = 72$.

47. $\left(\dfrac{60}{x} + 5\right)(x-6) = 60$.

48. $\dfrac{x}{x-2} + \dfrac{x-2}{x} = 3\tfrac{1}{3}$.

Make equations with the following roots, expressing each equation in the form $Ax^2 + Bx + C = 0$.

49. $0,\ 3$.

50. $0,\ -1$.

51. $2,\ 11$.

52. $2,\ -2$.

53. $-7,\ 1$.

54. $-6,\ -5$.

55. $\tfrac{1}{2},\ 3$.

56. $\tfrac{1}{2},\ -\tfrac{3}{4}$.

57. $-1\tfrac{1}{4},\ -2\tfrac{1}{3}$.

58. Show that $x = 1$ satisfies the equation $x^2 + 17x = 18$. Find the other root.

59. If $x^2 \quad - 15 = 0$ is a quadratic equation with the x term missing and one of its roots is 3, find the other root. Also find the missing term.

60. If $2x^2 + x \quad = 0$ is a quadratic equation with the constant term missing, and one of the roots is $1\tfrac{1}{2}$, find the other root. What is the constant term?

61. One of the roots of the equation $x^2 - 7x + 4 = 0$ is 6·37, to two decimal places. Find the other root, to two decimal places.

Solve the equations:

62. $x^2 - a^2 = 0$.

63. $4x^2 = 9c^2$.

64. $m^2x^2 = n^2$.

65. $x^2 - ax = 0$.

66. $ax^2 + bx = 0$.

67. $x^2 - mx - nx = 0$.

68. $x^2 - 3ax + 2a^2 = 0$.

69. $x^2 + px - 6p^2 = 0$.

70. $3x^2 - 11bx + 10b^2 = 0$.

SOLUTION BY COMPLETING THE SQUARE.

The process of adding a term to x^2+ax to make it a complete square will have been explained in connection with Exercise VIII. h, p. 138. A few examples should be worked before beginning this section.

Example V. Solve $2x^2+3x-14=0$.

Since
$$2x^2+3x-14=0,$$
$$\therefore\ x^2+\tfrac{3}{2}x-7=0,$$
dividing each side by 2.
$$\therefore\ x^2+\tfrac{3}{2}x+(\tfrac{3}{4})^2-7-\tfrac{9}{16}=0,$$
first adding $(\frac{3}{4})^2$ to complete the square, and then subtracting it.
$$\therefore\ (x+\tfrac{3}{4})^2-\tfrac{121}{16}=0;$$
$$\therefore\ (\overline{x+\tfrac{3}{4}}-\tfrac{11}{4})(\overline{x+\tfrac{3}{4}}+\tfrac{11}{4})=0,$$
putting the first side into factors.
$$\therefore\ x+\tfrac{3}{4}-\tfrac{11}{4}=0 \quad\text{or}\quad x+\tfrac{3}{4}+\tfrac{11}{4}=0;$$
$$\therefore\ \mathbf{x}=2 \quad\text{or}\quad \mathbf{x}=-3\tfrac{1}{2}.$$

Check. Since the given equation may be put in the form
$$x^2+\tfrac{3}{2}x-7=0 \quad\text{or}\quad x^2-(-\tfrac{3}{2})x-7=0,$$
the sum of the roots should be $-\frac{3}{2}$ and their product 7. This is seen to be the case.

Example VI. Solve $x^2-5x+2=0$, giving the roots of the equation correct to two significant figures.

Since
$$x^2-5x+2=0,$$
$$\therefore\ x^2-5x+\left(\frac{5}{2}\right)^2+2-\frac{25}{4}=0,$$
first adding $(\frac{5}{2})^2$ to complete the square, and then subtracting it.
$$\therefore\ \left(x-\frac{5}{2}\right)^2-\frac{17}{4}=0;$$
$$\therefore\ \left(\overline{x-\frac{5}{2}}-\frac{\sqrt{17}}{2}\right)\left(\overline{x-\frac{5}{2}}+\frac{\sqrt{17}}{2}\right)=0;$$
$$\therefore\ \text{either } x-\frac{5}{2}-\frac{\sqrt{17}}{2}=0 \text{ or } x-\frac{5}{2}+\frac{\sqrt{17}}{2}=0;$$
$$\therefore\ \text{either } x=\frac{5}{2}+\frac{\sqrt{17}}{2} \text{ or } x=\frac{5}{2}-\frac{\sqrt{17}}{2};$$

$$\begin{array}{r|r|l}
 & 17 & 4{\cdot}12 \\
 & 16 & \\
\hline
81 & 100 & \\
 & 81 & \\
\hline
822 & 1900 &
\end{array}$$

$$\therefore\ x=2{\cdot}5+2{\cdot}06 \text{ or } 2{\cdot}5-2{\cdot}06;$$
$$\therefore\ \mathbf{x}=\mathbf{4{\cdot}56} \text{ or } \mathbf{0{\cdot}44}.$$

Check. The sum of the roots should be 5; this is so. Their product is about $4\frac{1}{2}$ times 0·44, which is nearly 2.

EXERCISE X. b.[*]

Solve the following equations, giving the roots correct to two significant figures :

1. $x^2=2.$
2. $3x^2=1.$
3. $2x^2=25.$
4. $\frac{22x^2}{7}=10.$
5. $x^2+9=36.$
6. $2(x^2+1)=15.$

Solve the equations in Exercise X. a. 25–42 by 'completing the square.'

Check the results by the method of pp. 153, 154.

Solve the following equations, working out each root to two significant figures, if it is not rational :

7. $x^2+4x-2=0.$
8. $x^2-4x-2=0.$
9. $x^2+2x-1=0.$
10. $x^2+5x+2=0.$
11. $x^2+5x-3=0.$
12. $x^2-x-4=0.$

Solve the following equations, working out each root to two places of decimals, if it is not rational :

13. $2x^2+7x=5.$
14. $3x^2=8x-2.$
15. $x^2+20=11x.$
16. $4x^2+3x-4=0.$
17. $x^2+x=120.$
18. $10x^2=x+1.$
19. $2x^2+12x-7=0.$
20. $x(2x+9)=10.$
21. $30x^2-20x-15=0.$
22. $\frac{1}{x}+\frac{1}{x+1}=\frac{1}{2}.$
23. $\frac{x}{x+2}=\frac{x+1}{2x-5}.$
24. $\frac{2x}{x+3}+\frac{3}{x+4}=1.$

SOLUTION BY FORMULA.

Since all quadratic equations may be regarded as special cases of the general quadratic equation $Ax^2+Bx+C=0$, if we can solve this equation we can solve any other by giving A, B and C particular values.

Solve $\mathbf{Ax^2+Bx+C=0.}$

Since $Ax^2+Bx+C=0,$

$$\therefore\ x^2+\frac{B}{A}x+\frac{C}{A}=0,$$

dividing by A.

$$\therefore\ x^2+\frac{B}{A}x+\left(\frac{B}{2A}\right)^2-\frac{B^2}{4A^2}+\frac{C}{A}=0,$$

adding $\left(\frac{B}{2A}\right)^2$ to complete the square and then subtracting it

$$\therefore\ \left(x+\frac{B}{2A}\right)^2-\left(\frac{B^2}{4A^2}-\frac{C}{A}\right)=0.$$

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[*] For additional examples, see E.P. 17, page 251/28.

$$\therefore\ \left(x+\frac{B}{2A}\right)^2-\frac{B^2-4AC}{4A^2}=0.$$

$$\therefore\ x+\frac{B}{2A}-\frac{\sqrt{B^2-4AC}}{2A}=0 \quad\text{or}\quad x+\frac{B}{2A}+\frac{\sqrt{B^2-4AC}}{2A}=0;$$

$$\therefore\ x=-\frac{B}{2A}+\frac{\sqrt{B^2-4AC}}{2A} \quad\text{or}\quad -\frac{B}{2A}-\frac{\sqrt{B^2-4AC}}{2A}.$$

These two results may be written more shortly

$$\mathbf{x}=\frac{-\mathbf{B}\pm\sqrt{\mathbf{B}^2-4\mathbf{AC}}}{2\mathbf{A}}.$$

Example VII. Solve $4x^2+1=7x$.

Writing the equation in the form $4x^2-7x+1=0$ and comparing with $Ax^2+Bx+C=0$, we see that

$$A=4,\quad B=-7,\quad C=1.$$

Substituting these values in $x=\dfrac{-B\pm\sqrt{B^2-4AC}}{2A}$, we get

$$x=\frac{7\pm\sqrt{49-16}}{8} \quad\text{or}\quad x=\frac{7\pm\sqrt{33}}{8}.$$

These can be worked to as many places of decimals as may be required.

EXERCISE X. c.

Solve by formula the following equations, working the roots to two places of decimals, if they are not rational :

1. $2x^2+5x+3=0$. **2.** $3x^2+4x-4=0$. **3.** $x^2-x-2=0$.

4. $5x^2=2x+7$. **5.** $2x^2-7x+6=0$. **6.** $2x^2+4x+1=0$.

7. $x^2+x=1$. **8.** $x^2-12x+25=0$. **9.** $2x^2+7=8x$.

10. $3x=3x^2-5$. **11.** $x^2+x=40$. **12.** $12x^2-1=6x$.

13. $x+\dfrac{1}{x}=11$. **14.** $\dfrac{x}{x-2}+\dfrac{x-2}{x}=4$. **15.** $\dfrac{x+1}{x-1}+\dfrac{3-x}{x+3}=1$.

Solve by formula the equations in Exercise X. b. 7–21, working out each root to the same degree of accuracy as in Exercise X. b.

PROBLEMS.

In the following examples it may be assumed that

(i) If the hypotenuse of a right-angled triangle is *a* units long and the other sides *b* and *c* units, then $a^2=b^2+c^2$.

(ii) If the radius of a circle is r inches, the area is πr^2 square inches, where $\pi = 3{\cdot}14159\ldots$, and so for any other corresponding units. Unless it is stated to the contrary, π may be taken as $\frac{22}{7}$.

(iii) If b is a mean proportional between a and c,

$$\frac{a}{b} = \frac{b}{c}.$$

Example VIII. The hypotenuse of a right-angled triangle is 10 cm. and one side is 5 cm.; find the third side, correct to one place of decimals.

Let the third side be x cm.;

$$\therefore\ x^2 \text{ sq. cm.} + 25 \text{ sq. cm.} = 100 \text{ sq. cm.};$$

$$\therefore\ x^2 + 25 = 100;$$

$$\therefore\ x^2 = 75;$$

$$\therefore\ x = 8{\cdot}7.$$

$$\begin{array}{r|l} & 75 \mid 8{\cdot}66 \\ & 64 \\ \hline 166 & 1100 \\ & 996 \\ \hline 1726 & 10400 \end{array}$$

The third side is **8·7 cm.**, to one place of decimals.

Example IX. The distance from London to Carlisle is 300 miles. One train goes 10 miles an hour faster than another, and takes $1\frac{1}{2}$ hours less on the journey. Find the speed of each.

Let x miles an hour be the speed of the first train;

$\therefore$ $(x-10)$,, is ,, ,, second ,,

$\therefore$ the first takes $\dfrac{300}{x}$ hours and the second $\dfrac{300}{x-10}$ hours on the journey.

$$\therefore\ \frac{300}{x-10} - \frac{300}{x} = \frac{3}{2},$$

$$\textit{i.e.}\ \frac{300\,.\,10}{x(x-10)} = \frac{3}{2}, \text{ simplifying the first side,}$$

$$\textit{i.e.}\ 3x(x-10) = 2\,.\,300\,.\,10,$$

$$\textit{i.e.}\ x^2 - 10x - 2000 = 0, \text{ dividing each side by } \mathbf{3},$$

$$\text{or } (x-50)(x+40) = 0;$$

$$\therefore\ x = 50 \quad \text{or} \quad -40.$$

Since a negative value of x is evidently unsuitable, we need only consider the value $x = 50$.

$$\therefore\ x - 10 = 40.$$

$\therefore$ the speeds of the trains are **50 and 40 miles an hour.**

EXERCISE X. d.

(NOTE.—*All these Examples should be worked by Algebra.*)

1. A number is equal to its square. Find the number.

2. The square of a number is equal to twice the number. Find it.

3. The square of a number added to the number itself is equal to 0. Find the number.

4. Find three consecutive numbers such that the square of one is equal to the sum of the squares of the other two.

5. Find two numbers, one of which is three times the other, such that their product is 243.

6. The diagonal of a square is 1 ft. long; find the length of each side in inches, to one place of decimals.

7. The hypotenuse of a right-angled triangle is 10 cm. and the other sides are equal. Find their lengths.

8. The hypotenuse of a right-angled triangle is twice one of the other sides and the third side is 3 in. Find the hypotenuse.

9. The area of a circle is 154 sq. ft.; find the radius. Take $\pi = \frac{22}{7}$.

10. Find two numbers in the ratio of 2 to 3 whose product is 294.

11. The area of a circle is 20 sq. cm. Find its radius in centimetres, correct to one place of decimals.

12. The length of a rectangle is 2 ft. more than its width and its area is 48 sq. ft. Find its length.

13. Find a mean proportional between 2 and 18.

14. If the area of a circle of radius 3 in., together with that of a circle of radius 4 in., is equal to that of a third circle, find the radius of the third circle.

15. Find a mean proportional between 2 and $\frac{1}{2}$.

16. AB is a straight line 8 cm. long. It is divided at a point C so that $AC \cdot CB = 12$. Find AC.

17. AB is a straight line 8 cm. long. A point C is taken in AB produced so that $AC \cdot CB = 33$. Find AC.

18. The sum of the first n natural numbers, *i.e.* 1, 2, 3, ..., n, is $\frac{n(n+1)}{2}$. How many numbers must be added to give 91 as the sum?

19. A falling body takes t sec. to fall a distance $16t^2$, if the resistance of the air is neglected. How long will it take to fall to the ground from a height of 144 ft.?

20. Looking out to sea from the top of a cliff, h ft. high, you can see a distance d miles, such that $2d^2=3h$. If the cliff is 300 ft. high, how far can you see?

21. The sum of the first n numbers 3, 7, 11, 15, ..., is $n(2n+1)$. How many numbers must I take so that this sum may be 210? What will be the last number?

22. Is it possible to find three consecutive whole numbers such that the sum of the squares of the greatest and least is equal to twice the square of the middle one?

23. A square courtyard, each side of which is 20 yd., consists of a square grass plot in the centre with a path round it. If the path cost £9 10s. at 2s. 6d. per sq. yd., what was the width of the path?

24. The hypotenuse of a right-angled triangle is 4 in. long and one of the other sides is 1 in. longer than the other. Find their lengths.

25. A circle just fits into a square. The area of the part of the square outside the circle is 300 sq. ft. Find, correct to a tenth of a foot, the radius of the circle. Take $\pi=3\frac{1}{7}$.

26. A flat circular ring, *i.e.* the space between two circles with the same centre, has an area of $16\frac{1}{2}$ sq. in. The radius of the inner circle is 5 in.; what is the width of the ring? Take $\pi=\frac{22}{7}$.

27. A stone is thrown up in such a way that after t sec. it is at a height of $80t-16t^2$ ft. When will it strike the ground again?

28. From London to Crewe is 160 miles. By increasing the speed of a certain train by 2 miles an hour the railway company shortens the time of the journey by 20 minutes. Find the new time.

29. The hypotenuse of a right-angled triangle is 25 in. and its perimeter is 56 in. Find the other sides.

30. A man spends £12 a year on coal. The price goes up 6s. a ton, and he reckons that he must use 2 tons less every year if he is to spend the same amount. How much did he use at first?

31. From A to B is 60 miles. If a man motors instead of cycling he goes 8 miles an hour faster and saves 2 hours. How fast does he cycle?

32. The sum of the ages of a class of boys is 252 years. If two boys, who are each 9 years old, join the class the average is reduced by 6 months. Find the number of boys in the class at first.

33. The distance that a ball rolls down a slope in t sec. is $10t+2t^2$ in. How long will it take to go 132 in. ?

34. On an architect's plan a room has an area of 400 sq. ft. The plan is altered so that the length is made 5 ft. less and the width 4 ft. more, without altering the area. Find the new length and breadth.

35. A Zeppelin starts from a base 250 miles from London and can travel at 60 miles an hour for 10 hours. What rate of wind will prevent it from returning to its base after a raid on London, if the wind blows continuously from the base to London ?

36. AB is 10 in. long. It is divided at C so that $AB \,.\, BC = AC^2$. Find AC to the nearest hundredth of an inch.

37. A stone is thrown up in such a way that after t sec. it is at a height of $120t-16t^2$ ft. After how long will it be at a height of 180 ft. ?

38. AB is 12 in. long. Find a point P in AB so that $AP^2+PB^2=100$, AP and PB being measured in inches.

39. A man rows 5 miles an hour in still water. He rows 7 miles up a river and back again in $3\frac{1}{3}$ hours. How fast does the stream flow ?

40. The seat of a tramcar is $16\frac{1}{2}$ ft. long. If it holds one person more than the proper number, each person has $1\frac{1}{2}$ in. of space less than he ought. How many is it supposed to hold ?

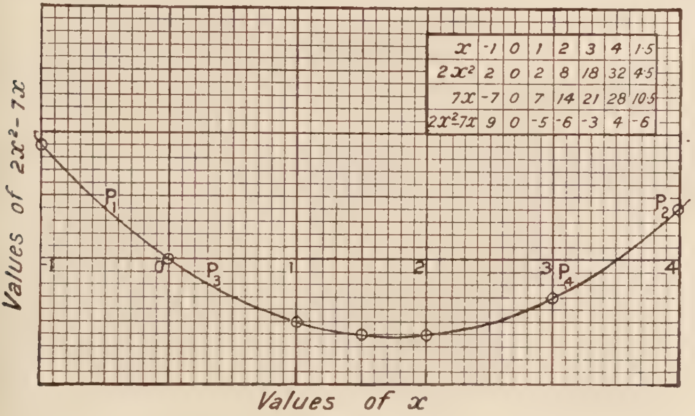
41. A circle of radius r is inscribed in a quadrant of a circle of radius 10 cm. Show that $r^2+r^2=(10-r)^2$. Hence find the radius of the inscribed circle to two places of decimals.

GRAPHICAL TREATMENT OF QUADRATIC FUNCTIONS

The expression Ax^2+Bx+C is called a quadratic function of x, or a function of the second degree in x.

We will represent one such function graphically (see Fig. 21).

If several such functions are represented graphically, A being always positive, it will be found that we always get a curve similar to this. Starting from the left it always slopes downwards to a lowest point and then slopes upwards. The shape will depend considerably both on the function and on the scales we use. If A is negative we get a curve of similar character turned upside down (see p. 61).



x	-1	0	1	2	3	4	1·5
$2x^2$	2	0	2	8	18	32	4·5
$7x$	-7	0	7	14	21	28	10·5
$2x^2-7x$	9	0	-5	-6	-3	4	-6

FIG. 21.

Fig. 21 is a graphical representation of the function $2x^2-7x$. The unit for values of x is one large division; the unit for values of $2x^2-7x$ is one small division.

We can use the curve given above to solve the equation $2x^2-7x+C=0$, where C is any number. *E.g.* to solve $2x^2-7x-4=0$, *i.e.* $2x^2-7x=4$, we find the points at which $2x^2-7x=4$. These are P_1 and P_2, and the values of x which make $2x^2-7x=4$ are $-0{\cdot}5$ and 4. These are the roots of the equation. Again, to solve $2x^2-7x+2=0$, *i.e.* $2x^2-7x=-2$, we consider the points P_3 and P_4. These give 0·31 and 3·2 for the roots of the equation.

It will be noticed that $2x^2-7x$ is never less than $-6{\cdot}2$ approximately. Consequently, $2x^2-7x=C$ has no roots, if C is less than $-6{\cdot}2$. If we take such an equation, *e.g.* $2x^2-7x=-10$, we find that the roots are $\dfrac{7 \pm \sqrt{49-80}}{4}$, *i.e.* they involve the square root of a negative number. If C is greater than $-6{\cdot}2$, we shall find that, no matter how large it is, we can always get two roots graphically, if we have a sheet of paper large enough.

EXERCISE X. e.*

1. Draw a graph of x^2-5x+6 from $x=0$ to $x=5$. Find the roots of $x^2-5x+6=0$.

2. Draw a graph of x^2-5x+7 from $x=0$ to $x=5$. (i) Show that $x^2-5x+7=0$ has no roots, as far as you can tell from the graph. (ii) What is the smallest value of x^2-5x+7? (iii) What value of x gives x^2-5x+7 its smallest value?

3. Draw a graph of $4x-x^2$ from $x=-1$ to $x=5$. (i) What values of x make $4x-x^2$ equal to 2? (ii) What are the roots of $4x-x^2=2$? (iii) What are the roots of $x^2-4x+2=0$? (iv) For what values of C has the equation $4x-x^2=C$ two roots? (v) For what values of C has it no roots? (vi) What is the greatest possible value of $4x-x^2$? (vii) What value of x gives $4x-x^2$ its greatest value? (viii) Prove the last two results by putting $4x-x^2$ in the form $4-(2-x)^2$.

4. Draw a graph of $5x-3x^2$. (i) What values of x make $5x-3x^2$ equal to -4? (ii) What are the roots of $5x-3x^2+4=0$? (iii) What are the roots of $5x-3x^2=1$? (iv) For what values of x is $5x-3x^2$ positive?

5. Plot $\frac{30}{x}-\frac{30}{x+5}$ for values of x between 7 and 10. Use your graph to solve the following: At how many miles an hour is a man cycling if, by increasing his speed by 5 miles an hour, he would take $1\frac{1}{4}$ hours less to cycle 30 miles?

6. Plot $x+\frac{1}{x}$ for values of x between 0·2 and 4. (i) What is the smallest value of $x+\frac{1}{x}$ for positive values of x? (ii) For what positive value of x is $x+\frac{1}{x}$ smallest? (iii) What values of x make $x+\frac{1}{x}$ equal to 3·5? (iv) What values of x make $x+\frac{1}{x}$ equal to $-3{\cdot}5$?

7. OA, OB are two straight lines at right angles. P and Q are two points, one in OA and one in OB, so that the sum of OP and OQ is 10 in. Find graphically the positions of P and Q when PQ is smallest.

Solve the following questions graphically:

Exercise X. d., 19, 33, 35, 36, 37, 38, 39.

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* For additional examples, see E.P. 18, page 251/28.

SIMULTANEOUS QUADRATICS.

If we are given two simultaneous equations, one of which is of the first degree in x and y and the other of the second degree, we can solve them as follows :

Example X. Solve the equations :

$$2x - 3y = 1, \quad \text{.........................(i)}$$

$$2x^2 - 2xy - 3y^2 = 1. \quad \text{.........................(ii)}$$

Express x in terms of y by means of (i).

$$2x = 1 + 3y, \text{ adding } 3y \text{ to each side.}$$

$$\therefore\ x = \frac{1+3y}{2}, \quad \text{.........................(iii)}$$

Substituting in (ii),

$$2\left(\frac{1+3y}{2}\right)^2 - 2\left(\frac{1+3y}{2}\right)y - 3y^2 = 1;$$

$$\therefore\ 2\left(\frac{1+6y+9y^2}{4}\right) - (y+3y^2) - 3y^2 = 1;$$

$$\therefore\ 1 + 6y + 9y^2 - 2(y+3y^2) - 6y^2 = 2,$$

multiplying each side by 2.

This reduces to $-3y^2 + 4y - 1 = 0$;

$$\therefore\ y = 1 \quad \text{or} \quad \tfrac{1}{3}.$$

Substituting in (iii), when $y = 1$, $x = 2$;

when $y = \frac{1}{3}$, $x = 1$.

$$\therefore \text{ the roots are } \begin{cases} x = 1, & y = \frac{1}{3}, \\ x = 2, & y = 1. \end{cases}$$

In working out problems we sometimes obtain two equations such as the following :

$$xy = 120,$$

$$(x+3)(y-5) = 75.$$

By subtraction we obtain an equation of the first degree, and can then solve by the method of Example X. above. Or, from the first equation, we get $y = \frac{120}{x}$ and substitute $\frac{120}{x}$ for y in the second.

We will work out Example IX., p. 159, by means of two equations.

The distance from London to Carlisle is 300 miles. One train goes 10 miles an hour faster than another, and takes $1\frac{1}{2}$ hours less on the journey. Find the speed of each.

Let the speed of the first train be x miles an hour and let it take y hours on the journey;

$$\therefore \; xy = 300. \quad \text{...(i)}$$

Also, since the second train goes $x-10$ miles an hour and takes $y+\frac{3}{2}$ hours,

$$(x-10)(y+\tfrac{3}{2}) = 300,$$

i.e.
$$xy + \tfrac{3}{2}x - 10y - 15 = 300.$$

Subtracting (i) from this,

$$\tfrac{3}{2}x - 10y - 15 = 0;$$
$$\therefore \; 3x - 20y - 30 = 0;$$
$$\therefore \; x = \frac{20y+30}{3}.$$

Substituting in (i), $$\left(\frac{20y+30}{3}\right)y = 300;$$

$$\therefore \; 20y^2 + 30y = 900;$$
$$\therefore \; 2y^2 + 3y - 90 = 0;$$
$$\therefore \; (2y+15)(y-6) = 0;$$

$\therefore$ $y = 6$, neglecting the negative value;

$\therefore$ $x = 50$, from (i);

$\therefore$ the speeds are **50 and 40 miles an hour.**

EXERCISE X. f.*

Solve the equations:

1. $x+y=7.$
$x^2+y^2=29.$

2. $x=y.$
$2x^2-y^2=1.$

3. $x-2y=1.$
$xy=3.$

4. $2x+2y=3.$
$x^2-xy=\frac{1}{2}.$

5. $xy=12.$
$(x-1)(y+2)=15.$

6. $x+y=0.$
$xy-y^2=-8.$

7. $x-y=1.$
$xy-2x=4.$

8. $3x+y=4.$
$xy=1.$

9. $3x-4y=4.$
$y^2=x.$

10. $x+y=6.$
$x^2+y^2=20.$

11. $x^2-y^2=40.$
$x+y=10.$

12. $(x+2)(y-2)=144.$
$xy=160.$

13. $2x-y=5.$
$xy=0$

14. $xy=6.$
$(x-\frac{1}{2})(y+1)=6.$

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* For additional examples, see E.P. 19, Nos. 1-20, page 251/29.

15. $x-y=10.$
$\frac{12}{y}-\frac{12}{x}=5.$

16. $2x^2+3xy-2x=21.$
$x=3y.$

17. $(x+10)(y-1)=180.$
$xy=100.$

18. $x+y=1.$
$\frac{x}{y}+\frac{y}{x}=\frac{5}{2}.$

Solve the following questions by means of two unknown quantities :

19. *AB* is a straight line 10 in. long. It is divided into two parts at *C*. Find their lengths, if $AC \,.\, CB=21$.

20. The hypotenuse of a right-angled triangle is 10 cm. and its perimeter is 24 cm. Find the other sides.

21. A man can either motor or cycle from *A* to *B*, a distance of 80 miles. If he motors he goes 12 miles an hour faster and takes 6 hours less. Find his speed by cycle.

22. The area of a right-angled triangle is 21 sq. in and its hypotenuse is $\sqrt{85}$ sq. in. Find the lengths of the other sides.

23. Two carpets have each the same area, 120 sq. ft. The first is 3 ft. longer and 2 ft. narrower than the other. Find the dimensions of each.

24. The perimeter of a rectangle is 13 ft. and its area is 10 sq. ft. Find its length and breadth.

25. If *AD* is perpendicular to the base *BC* of the triangle *ABC*, $BD^2-DC^2=BA^2-AC^2$. Find *BD* and *DC* if $AB=5''$, $AC=4''$, $BC=6''$.

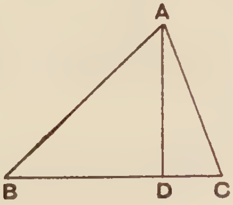

Fig. 22.

26. The hypotenuse of a right-angled triangle is 29 in. and its perimeter is 70 in. Find the other sides.

27. A man pays income tax on the part of his income above £160 ; he pays £20. Next year his income is the same and he pays tax on the part above £120 and the tax is 6d. in the pound more ; he pays £30. What was his income ?

CHAPTER XI.

FURTHER DEVELOPMENTS OF PREVIOUS CHAPTERS.

MULTIPLICATION, DIVISION AND SQUARE ROOT.

- *A.* Multiplication and Division.
- *B.* Squares, Square Roots, Etc.

FACTORS AND FRACTIONS.

- *C.* Factors, Etc.
- *D.* Simplification of Fractions

EQUATIONS AND IDENTITIES.

- *E.* Equations.
- *F.* Literal Equations.
- *G.* Identities.

MULTIPLICATION, DIVISION AND SQUARE ROOT.

A. Multiplication and Division.

Example I. Multiply $x+y+z$ by $x+y-z$.

$$\begin{array}{llllll} x & +y & +z \\ x & +y & -z \\ \hline x^2 & +xy & +xz \\ & xy & & +y^2 & +yz \\ & & -xz & & -yz & -z^2 \\ \hline x^2 & +2xy & & +y^2 & & -z^2 \end{array}$$

$\therefore$ the product is $\mathbf{x^2+2xy+y^2-z^2}$.

Notice that this could have been worked as follows:

The product of $(x+y)+z$ and $(x+y)-z$ is $(x+y)^2-z^2$,

i.e. $x^2+2xy+y^2-z^2$.

Example II. Divide $2x^2 - y^2 - z^2 + 2yz - xz + xy$ by $2x - y + z$.

The exact order of terms in the divisor and dividend is not important so long as they are arranged in descending powers of one letter, *e.g.* x.

$$\begin{array}{r|llllll|l}
2x - y + z & 2x^2 & +xy & -xz & -y^2 & +2yz & -z^2 & x + y - z \\
 & 2x^2 & -xy & +xz & & & & \\
\cline{2-4}
 & & 2xy & -2xz & -y^2 & +2yz & & \\
 & & 2xy & & -y^2 & +\ yz & & \\
\cline{3-6}
 & & & -2xz & & +\ yz & -z^2 & \\
 & & & -2xz & & +\ yz & -z^2 & \\
\cline{4-7}
 & & & \cdot & & \cdot & \cdot & \text{Quotient } \mathbf{x + y - z.}
\end{array}$$

Or it might have been worked as follows, if we had known that there was no remainder.

Arrange dividend and divisor in ascending powers of z.

$$\{(2x^2 + xy - y^2) + (-x + 2y)z - z^2\} = \{(2x - y) + z\}\{(\qquad) - \quad\}.$$

The first factor on the right side is the divisor, the second will be the quotient.

The product of the two last terms on the right side must be $-z^2$.

The product of the two first terms on the right side must be $2x^2 + xy - y^2$.

$\therefore$ the second factor must be $(x + y) - z$.

Now compare the z terms on the two sides and see that they agree.

It often saves trouble to pick out a particular term in the product of two quantities, without performing the whole of the multiplication.

Example III. What is the coefficient of x^3 in

$$(1 + x - 2x^2 + 3x^3)^2\ ?$$

Writing the expression in the form

$$(1 + x - 2x^2 + 3x^3)(1 + x - 2x^2 + 3x^3),$$

we see that the x^3 term is obtained by adding the following:

$$1 \times 3x^3, \quad x \times -2x^2, \quad -2x^2 \times x, \quad 3x^3 \times 1.$$

$\therefore$ the coefficient of x^3 is $3 - 2 - 2 + 3$, *i.e.* 2.

Example IV. Divide $2x^3 - x^2 + 6$ by $x - 2$, using the method of detached coefficients.

In this example both divisor and dividend are expressed in powers of a common letter, in this case x. If we have to multiply or divide such expressions, we can leave out the letter and write down only the coefficients. This is called **the method of detached coefficients**. It is important that all missing powers should be represented by a zero.

$$\begin{array}{r|rrrr|l}
1-2 & 2 & -1 & +0 & +6 & 2+3+6 \\
 & 2 & -4 & & & \\ \cline{2-3}
 & & 3 & +0 & & \\
 & & 3 & -6 & & \\ \cline{3-4}
 & & & 6 & +\ 6 & \\
 & & & 6 & -12 & \\ \cline{4-5}
 & & & & 18 &
\end{array}$$

Quotient $\mathbf{2x^2 + 3x + 6}$; remainder **18.**

If this is compared with the ordinary form of division, it will be easy to follow the method.

EXERCISE XI. a.

1. Multiply $a + b$ by $a^2 + ab + b^2$.

2. Multiply $a - 2b + 3c$ by $2a - b - 3c$.

3. Divide $x^3 + x^2 - x + 2$ by $x + 2$.

4. Divide $6x^3 - x^2 - 14x + 8$ by $3x - 2$.

5. Divide $x^5 - 1$ by $x - 1$.

6. Multiply $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

7. Divide $6x^3 + 13x^2 - 7x - 5$ by $2x + 5$.

8. Divide $a^3 - 8$ by $a^2 + 2a + 4$.

9. Divide $4x^3$ by $2x^2 - 2x + 1$, in descending powers of x.

10. Multiply $x^2 - ax + 2a^2$ by $a^2 - ax + 2x^2$.

11. Divide $a^6 - 8b^6$ by $a^2 - 2b^2$.

12. Divide $2 - a - 2a^2 + a^3$ by $1 - a$.

13. Multiply $x^2 + 1 - x$ by $3 - x^2 + x$.

14. Divide $a^2 - b^2 + 2bc - c^2$ by $a + b - c$.

15. Find the value of $(x + y + z)^2$.

16. Find the value of $(x - y + z)^2$.

17. Divide $a^3 - a^2b + 2b^3$ by $a + b$.

18. Divide $m^4 - n^4$ by $m - n$.

19. Divide $a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$ by $a + b + c$.

20. Divide $4a^2 - 12ab + 9b^2 - 16c^2$ by $2a - 3b + 4c$.

21. Multiply $2x^3 - x + 3$ by $x^3 - 2x^2 + 3$.

22. Divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.

23. Divide $3x^3 + 7x + 2$ by $2 - x$, in descending powers of x.

24. Find the value of $\left(x - \frac{1}{x}\right)^2$.

25. Divide $x^3 + 2x + 2$ by $x + 1$, (i) in descending powers of x, (ii) in ascending powers of x, as far as x^2 in the quotient.

26. What is the coefficient of x^2 in $(2x^2 - 7x + 6)(x^2 - x + 2)$?

27. Express $\frac{x^2 + 2x + 3}{x + 1}$ as a mixed fraction.

28. Express $\frac{x^2}{x - 2}$ as a mixed fraction.

29. Multiply $x^3 + 5x^2$ by $x - \frac{1}{x}$.

30. What is the coefficient of x^2 in $(2x^2 - x - 7)(x^2 + 3x + 1)$?

31. What is the coefficient of x^3 in $(2x^3 - x + 10)(x^3 + x^2 - 5x - 4)$?

32. What is the coefficient of x^2 in $(x^2 - ax + a^2)(x^2 + ax + a^2)$?

33. What is the constant term in $\left(2x^2 - x + 3 + \frac{5}{x} - \frac{1}{x^2}\right)^2$?

34. What is the coefficient of x in $(x^2 + bx + cx + bc)(x^2 + x + 1)$?

35. By dividing 1 by $1 + x + x^2$, show that

$$\frac{1}{1+x+x^2} = 1 - x + \frac{x^3}{1 + x + x^2}.$$

If $x = 0{\cdot}1$, show that if we take $1 - 0{\cdot}1$ for the value of $\frac{1}{1 + x + x^2}$ the error will be less than $0{\cdot}001$.

36. By dividing 1 by $1 + x$, show that

$$\frac{1}{1+x} = 1 - x + \frac{x^2}{1+x}.$$

Hence show that $\frac{1}{1{\cdot}02}$ is very nearly equal to $0{\cdot}98$.

37. By dividing 1 by $(1 - x)^2$, show that

$$\frac{1}{(1-x)^2} = 1 + 2x + \text{a fraction}.$$

Find the value of the fraction.

Use the method of detached coefficients (see p. 170) to work out the following :

38. Multiply x^3+2x^2-5x+1 by $2x^2+x-3$.

39. Multiply x^3-3x+4 by $3x+5$.

40. Multiply $2x^2+5x-4$ by $3x^2-2$.

41. Divide $2x^3+5x^2-8x-15$ by $x+3$.

42. Divide $6x^3+5x^2-2$ by $2x-1$.

43. Divide $18x^3+x-6$ by $3x-2$.

B. SQUARES, SQUARE ROOT, ETC.

By actual multiplication we can show that

$$(A+B+C)^2=A^2+B^2+C^2+2BC+2CA+2AB.$$

To get the right-hand side we add together the squares of A, B, C and twice the product of every two of them.

Instead of A, B, C we can put any expressions. *E.g.*

$$\begin{aligned}(2a-b+3c)^2&=4a^2+b^2+9c^2+2(-b)\,.\,3c+2\,.\,3c\,.\,2a\\&\qquad+2\,.\,2a(-b)\\&=4a^2+b^2+9c^2-6bc+12ca-4ab.\end{aligned}$$

From this we are often able to write down a square root of a perfect square at once. *E.g.* to find a square root of

$$4a^2+b^2+9c^2+6bc-12ca-4ab.$$

From the first three terms we see that, if there is a square root it will be $\pm 2a\pm b\pm 3c$.

We have to find the signs. By considering the signs of the last three terms, viz. $+6bc$, $-12ca$, $-4ab$, we see that

$$2a-b-3c \text{ is a square root.}$$

It will be found that $-2a+b+3c$ is also a square root.

Example V. Find a square root of

$$a^2+9b^2+25c^2-30bc-10ac+6ab.$$

$$\begin{array}{r|l|l}
 & a^2+9b^2+25c^2-30bc-10ac+6ab & a+3b-5c \\
 & a^2 & \\ \hline
2a+3b & 6ab+9b^2 & \\
 & 6ab+9b^2 & \\ \hline
2a+6b-5c & -10ac-30bc+25c^2 & \\
 & -10ac-30bc+25c^2 & \\ \hline
 & \cdot\qquad\cdot\qquad\cdot &
\end{array}$$

A square root is $\mathbf{a+3b-5c}$.

This process is the same as the corresponding process in Arithmetic. The first divisor is $2a+3b$. The first term of this is obtained by taking twice the quotient a. We then bring down a term, $6ab$, which is divisible by $2a$. By dividing $2a$ into $6ab$ we get $3b$, the second term of the quotient. This we add to the divisor.

N.B.—This process is hardly ever required, as we can nearly always find the square root by inspection, as cn p. 172.

EXERCISE XI. b.

Write down the squares of the following expressions:

1. $a+b+c$. **2.** $a+b-c$.
3. $a-b-c$. **4.** x^2+x+2.
5. $2a+3b-c$. **6.** x^2+2x-5.
7. $yz+zx+xy$. **8.** $x^2-2xy-y^2$.
9. $a^2-ab+2ac$. **10.** $5x+4y-z$.

Find, by the 'division' method (see p. 172, Ex. V.), square roots of the following:

11. $4x^4+12x^3+13x^2+6x+1$.
12. $x^4-2x^3y+11x^2y^2-10xy^3+25y^4$.
13. $9a^4-6a^3-23a^2+8a+16$.
14. $x^6-4x^4+14x^3+4x^2-28x+49$.
15. $9x^2+y^2+z^2+2yz+6xz+6xy$.
16. $4a^2+16b^2+9c^2+24bc-12ac-16ab$.
17. $a^4-2a^3b+a^2b^2-2a^2bc+2ab^2c+b^2c^2$.

Find, by inspection, square roots of the following. Show that each expression is a perfect square.

18. $a^2+4b^2+4c^2+8bc+4ac+4ab$.
19. $a^2+4b^2+4c^2+8bc-4ac-4ab$.
20. $x^4+y^4+z^4+2y^2z^2+2z^2x^2+2x^2y^2$.
21. $b^2c^2+c^2a^2+a^2b^2-2a^2bc+2ab^2c-2abc^2$.
22. $x^4+6x^3+11x^2+6x+1$. **23.** $x^4-6x^3+11x^2-6x+1$.
24. $25a^4-10a^3+41a^2-8a+16$.
25. $100x^4-40x^3-16x^2+4x+1$.

Given that $(A+B)^3=A^3+3A^2B+3AB^2+B^3$, write down the cube of each of the following:

26. $x+2y$. **27.** $x-y$. **28.** $x+1$.
29. $x-1$. **30.** $2a+3b$. **31.** x^2-y^2.

Write down by inspection the cube roots of the following. Show that each expression is a perfect cube.

32. $a^3+6a^2b+12ab^2+8b^3$. 33. $a^3-6a^2b+12ab^2-8b^3$.
34. $x^6+3x^4y^2+3x^2y^4+y^6$. 35. $8x^3+36x^2y+54xy^2+27y^3$.
36. $x^3-12x^2y+48xy^2-64y^3$.

FACTORS AND FRACTIONS.

C. FACTORS, ETC.

It can be shown that

$$A^3+B^3\equiv(A+B)(A^2-AB+B^2).$$
$$A^3-B^3\equiv(A-B)(A^2+AB+B^2).$$

It is easy to remember the first factor $A+B$ or $A-B$; if you forget the other, you can find it by division.

In these results A and B stand for any expressions.

The following rule may be of use:—To factorise the sum or difference of two cubes, write down the first factor as follows—cube root of first term, *same* sign, cube root of second term—build up the second factor from the first factor—square of first term, product of the two terms with the *sign changed*, square of second term.

E.g.
$$\begin{aligned}8a^3+125&=(2a)^3+5^3\\&=(2a+5)\{(2a)^2-2a\,.\,5+5^2\}\\&=(2a+5)(4a^2-10a+25).\end{aligned}$$

Also $(a+b)^3+c^3=\{(a+b)+c\}\{(a+b)^2-(a+b)c+c^2\}$.

Example VI. Factorise $x^2-a^2+2ab-b^2$.

$$\begin{aligned}x^2-a^2+2ab-b^2&=x^2-(a^2-2ab+b^2)\,*\\&=x^2-(a-b)^2\\&=\{x+(a-b)\}\{x-(a-b)\}\\&=\mathbf{(x+a-b)(x-a+b)}.\end{aligned}$$

Example VII. Factorise $a^4+a^2b^2+b^4$.

$$\begin{aligned}a^4+a^2b^2+b^4&=a^4+2a^2b^2+b^4-a^2b^2\\&=(a^2+b^2)^2-(ab)^2\\&=(a^2+b^2+ab)(a^2+b^2-ab),\end{aligned}$$

which may be written $\mathbf{(a^2+ab+b^2)(a^2-ab+b^2)}$.

---

* Whenever you build up an expression by means of brackets, check the work by simplifying the expression you have obtained. You ought to get the expression with which you started.

In this case, if you get rid of the brackets from $x^2-(a^2-2ab+b^2)$ you will get $x^2-a^2+2ab-b^2$.

EXERCISE XI. c.

1. Divide $a^3 - b^3$ by $a - b$. **2.** Divide $x^3 + 8y^3$ by $x + 2y$.

3. Multiply $x^2 + 2y^2 + 5xy$ by $x^2 + 2y^2 - 5xy$.

Find the factors of the following :

4. $x^3 + 1$. **5.** $x^3 - 1$. **6.** $8 - y^3$.

7. $x^3 + 8y^3$. **8.** $1000 - a^3$. **9.** $x^4 + x$.

10. $a^3b - ab^3$. **11.** $(a + b)^2 - c^2$. **12.** $4(a + b)^2 - 9$.

13. $a^4 - b^4$. **14.** $32a^3 - 4$. **15.** $8^3 + 5^3$.

16. $(a + 5b)^2 - (3a + 2b)^2$. **17.** $a^3b^3 + 1$.

18. $x^2 - (a + b)^2$. **19.** $(x^3 + y^3)^2$. **20.** $(x + y)^2 + (x + y)$.

21. $a^2 + 2ab + b^2 - x^2$. **22.** $x^2 - 2xy + y^2 - 1$.

23. $(x - y)^2 - 5(x - y) + 6$. **24.** $x^2 - y^2 - 2yz - z^2$.

25. $a^2 - 2a + 1 - b^2$. **26.** $x^2 - y^2 + x + y$.

27. $a^2 - b^2 + 2bc - c^2$. **28.** $a^3 + 3a^2b + 3ab^2 + b^3$.

29. $x(x^2 - y^2) + yz(x - y)$. **30.** $(x - y)^2 + (y - x)(z - x)$.

31. $n(n - 1)(n - 2) + (n - 1)^2$. **32.** $(5x - 3y + 2z)^2 - (3x + y - 4z)^2$.

33. $x^2 + 2xy + y^2 - a^2 + 2ab - b^2$.

34. Find the H.C.F. and L.C.M. of $x^3 + y^3$ and $(x + y)^2$.

35. Factorise $x^4 - 2x^3 - 8x + 16$, given that $x - 2$ is one factor.

36. Express the sum of $x^2(x - 3y)$ and $xy(x - 3y)$ in factors.

37. Write down the product of $m - 3n$ and $m^3 + 3mn + 9n^2$.

38. Divide $x^2 + 2xy + y^2 - z^2$ by $x + y - z$, by factors.

39. Find the L.C.M. of $y^3 - z^3$ and $2y^2 + 2yz + 2z^2$.

40. Simplify $\{n(n + 2)\}^2 - (n + 1)^4$.

41. Given that $a^3 - 3a^2b + 3ab^2 - b^3 \equiv (a - b)^3$, find the factors of $x^6 - 6x^4 + 12x^2 - 8$.

42. Show that 13 is a factor of $12^3 + 1$.

Factorise :

43. $x^4 + x^2 + 1$. **44.** $(a + b)^3 + c^3$. **45.** $a^3 - b^3 + a^2b - ab^2$.

46. $x^6 - y^6$. **47.** $x^3 - (y + z)^3$. **48.** $b^3 + c^3 + b + c$.

49. $x^4 - 7x^2y^2 + y^4$. **50.** $a^3 - \dfrac{1}{a^3}$. **51.** $x^4 - 12x^2 + 4$. **52.** $x^6 + 1$.

53. Show that $x + y - 2z$ is a factor of $x^3 + 3x^2y + 3xy^2 + y^3 - 8z^3$.

54. Show that $a+b$ is a factor of $(2a-b)^3-(a-2b)^3$.

55. Show that $a+b-c$ and $a-b+c$ are factors of $a^2-b^2+2bc-c^2$.

56. Find the H.C.F. of $x^4+x^2y^2+y^4$ and x^3-y^3.

57. Write down the product of $a^2+3ab+9b^2$ and $a^2-3ab+9b^2$.

58. Find the factors of $(x^3+3x^2y)(x+z)^3+(3xy^2+y^3)(x+z)^3$.

59. Find the L.C.M. of $(x+y+z)^2$ and $x^2+2xy+y^2-z^2$.

60. Show that $m+n$ is a factor of $(8m-3n)^3-(3m-8n)^3$.

D. SIMPLIFICATION OF FRACTIONS.

FRACTIONAL EQUATIONS.

Example VIII. Simplify $\dfrac{a^3+b^3}{a^3-b^3}-\dfrac{a^2-ab+b^2}{a^2+ab+b^2}$.

The L.C.M. of the denominators is a^3-b^3.

$$\therefore \text{ the given expression}=\frac{a^3+b^3-(a^2-ab+b^2)(a-b)}{a^3-b^3}$$

$$=\frac{a^3+b^3-(a^3-2a^2b+2ab^2-b^3)}{a^3-b^3}$$

$$=\frac{2a^2b-2ab^2+2b^3}{a^3-b^3}$$

$$=\frac{2b(a^2-ab+b^2)}{a^3-b^3}.$$

Notice that a^2-ab+b^2 is a factor of each numerator and a^2+ab+b^2 of each denominator, so that it might have been simplified by first writing it in the form $\dfrac{a^2-ab+b^2}{a^2+ab+b^2}\left(\dfrac{a+b}{a-b}-1\right)$.

Example IX. Simplify

$$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-c)(b-a)}+\frac{c}{(c-a)(c-b)}.$$

In the denominator we apparently have six different factors, but $a-c$ and $c-a$ are different only in the way that x and $-x$ are different; in fact, we may take $(b-c)(c-a)(a-b)$ as the L.C.M. For $a-c$ divides into $c-a$ and the quotient is -1; $\therefore$ $(a-b)(a-c)$ divides into $(b-c)(c-a)(a-b)$, and the quotient is $-(b-c)$.

$$\therefore \text{ the given expression}=\frac{-a(b-c)-b(c-a)-c(a-b)}{(b-c)(c-a)(a-b)}$$

$$=\frac{-ab+ac-bc+ab-ac+bc}{(b-c)(c-a)(a-b)}$$

$$=\mathbf{0}, \text{ since the numerator } =\mathbf{0}.$$

This expression is said to be symmetrical in a, b, c, since it will be found to be unaltered if we interchange any two of these quantities. It follows that if the answer to such an expression contains a, b, c, it also must be symmetrical.

Example X. Simplify $\dfrac{1}{2x(x-a)}+\dfrac{1}{2x(x+a)}-\dfrac{1}{x^2+a^2}$.

In questions of this kind, the amount of work may be reduced by simplifying the expression in parts.

If we add the first two fractions the result is $\dfrac{1}{x^2-a^2}$.

$$\therefore \text{ the given expression} = \frac{1}{x^2-a^2}-\frac{1}{x^2+a^2}$$
$$= \frac{2a^2}{x^4-a^4}.$$

EXERCISE XI. d.*

Simplify

1. $\dfrac{a^4-b^4}{a^2-2ab+b^2}\times\dfrac{a-b}{a(a+b)}$.

2. $\dfrac{a}{a-b}+\dfrac{b}{b-a}$.

3. $x-\cfrac{1}{x+\cfrac{1}{x}}$.

4. $\dfrac{x^3+y^3}{(x+y)^2}$.

5. $\dfrac{(x+y)^2-4z^2}{x^2-(y-2z)^2}$.

6. $\dfrac{4x^2-(y-1)^2}{6x-3y+3}$.

7. $(a^2+ab+b^2)\left(\dfrac{1}{a}-\dfrac{1}{b}\right)$.

8. $\dfrac{\dfrac{x}{y}+\dfrac{x+y}{x-y}}{\dfrac{y}{x}+\dfrac{x-y}{x+y}}$.

9. $\dfrac{x^2+\dfrac{y^3}{x}}{1+\dfrac{(x-y)^2}{xy}}$.

10. $\dfrac{\dfrac{u+1}{v}-\dfrac{v+1}{u}}{\dfrac{1}{v}-\dfrac{1}{u}}$.

11. $\left(x+\dfrac{xy}{x-y}\right)\div\left(x-\dfrac{xy}{x+y}\right)$.

12. $\dfrac{a+c}{(x-a)(b-a)}+\dfrac{b+c}{(x-b)(a-b)}$.

13. $\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}$.

14. $\dfrac{a^2-ab+b^2}{a^3+b^3}+\dfrac{a^2+ab+b^2}{a^3-b^3}$.

15. $\dfrac{1-\dfrac{x^2+y^2}{xy}}{\dfrac{x}{y}-\dfrac{y}{x}}\div\dfrac{x^3+y^3}{(x-y)^2}$.

16. $\dfrac{1}{(x-y)(x-z)}+\dfrac{1}{(y-z)(y-x)}+\dfrac{1}{(x-y)(x-z)}$.

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* For additional examples, see E.P. 15, page 251/25.

17. $\dfrac{2+x}{2(x+1)}+\dfrac{2-x}{2(x-1)}+\dfrac{x}{x^2+1}.$

18. $\left(\dfrac{a^2}{b}-\dfrac{b^2}{a}\right)\div\left(\dfrac{a}{b}+1+\dfrac{b}{a}\right).$

19. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-z)(y-x)}+\dfrac{x+y}{(z-x)(z-y)}.$

20. $\dfrac{x^2+2xy+y^2-z^2}{(x+y+z)^2}.$

21. $\dfrac{1}{x}-\dfrac{1}{x+1}-\dfrac{1}{x-1}+\dfrac{x}{x^2+1}.$

22. $\dfrac{x+a}{x-a}-\dfrac{x^2+a^2}{x^2-a^2}+\dfrac{2ax}{x^2+a^2}.$

23. $\dfrac{a^4-b^4}{a^2+b^2-2ab}\div\dfrac{a-b}{a^2+ab}.$

24. $\dfrac{x+a}{(a-b)(a-c)}+\dfrac{x+b}{(b-c)(b-a)}+\dfrac{x+c}{(c-a)(c-b)}.$

25. $\dfrac{3}{a-b}-\dfrac{1}{a-3b}+\dfrac{1}{a+3b}-\dfrac{3}{a+b}.$

26. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}.$

27. $\left(\dfrac{a}{a-b}+\dfrac{a}{a+b}\right)\div\left\{\dfrac{b}{a-b}+\dfrac{b^2}{(a-b)^2}\right\}.$

28. $\dfrac{1}{a^2-\dfrac{a^3-1}{a+\dfrac{1}{a+1}}}.$

29. $a-\dfrac{1}{b-\dfrac{1}{a-\dfrac{1}{b}}}.$

30. $\dfrac{x^4-4x^2+x+2}{x^3+x^2+x+6}.$

Solve the equations :

31. $\dfrac{x+20}{x+15}=\dfrac{x}{x-15}-\dfrac{10}{x}.$

32. $\dfrac{1}{x-2}+\dfrac{2}{x-4}=\dfrac{3}{x-1}.$

33. $\dfrac{2}{x+3}-\dfrac{2}{x+5}=\dfrac{1}{x-3}-\dfrac{1}{x+1}.$

34. $\dfrac{2}{x+2}+\dfrac{1}{x-1}=\dfrac{2}{x+4}+\dfrac{1}{x-5}.$

35. $\dfrac{6}{x-3}-\dfrac{9}{x-2}+\dfrac{4}{x-1}=\dfrac{1}{x-4}.$

36. $\dfrac{1}{(x+3)^2}-\dfrac{3}{x+3}=10.$

37. $\dfrac{x-1}{x^2+x}+\dfrac{x+1}{x^2-x}=\dfrac{x-3}{x^2-1}.$

38. $\dfrac{x+7}{x+6}+\dfrac{x+3}{x+2}=\dfrac{2x+11}{x+4}.$

39. $\dfrac{3x^2-4x+1}{2x^2-x+4}=\dfrac{3x}{2x+5}.$

40. $\dfrac{a}{x-b}+\dfrac{b}{x-a}=2.$

EQUATIONS AND IDENTITIES.

E. Equations.

Example XI. Solve the equations:

$$x+2y-z=5. \quad\text{(i)}$$
$$3x-y+3z=7. \quad\text{(ii)}$$
$$2x+3y+z=11. \quad\text{(iii)}$$

If we eliminate z from (i) and (iii), we get

$$3x+5y=16.$$

If we eliminate z from (i) and (ii), we get

$$6x+5y=22.$$

Solving these last two equations in the usual way, we get

$$x=2, \quad y=2.$$

Substituting these values in the first equation, $z=1$.

$$\therefore\ \mathbf{x=2}, \quad \mathbf{y=2}, \quad \mathbf{z=1}.$$

We can check our result by substituting in one of the other equations.

Example XII. Solve the equations:

$$x^2+y^2=26.$$
$$x+y=4.$$

These may be solved by the method of Example X. p. 165, or as follows:

Squaring each side of the second equation,

$$x^2+2xy+y^2=16.$$

Subtracting the first equation from this,

$$2xy=-10.$$

Subtracting this from the first equation,

$$x^2-2xy+y^2=36.$$
$$\therefore\ (x-y)^2=36.$$
$$\therefore\ x-y=6 \text{ or } -6.$$

∴ either	$x+y=4$,	or	$x+y=4$,
	$x-y=6$,		$x-y=-6$,
giving	$x=5, \ y=-1$,	giving	$x=-1, \ y=5$.

$$\textit{Answer}\begin{cases}\mathbf{x=5,\ -1}\\ \mathbf{y=-1,\ 5.}\end{cases}$$

This is called the *symmetrical* method, because we treat x and y alike. By the method of Example X. p. 165, we get rid of x and obtain an equation in y only, *i.e.* we do not treat x and y alike.

Example XIII. Solve the equations:

$$\frac{1}{x}+\frac{2}{y}=4.$$
$$\frac{2}{x}-\frac{3}{y}=1.$$

Regard these as equations in $\frac{1}{x}$ and $\frac{1}{y}$, and solve as simple simultaneous equations. Thus we regard the second equation as

$$2\left(\frac{1}{x}\right)-3\left(\frac{1}{y}\right)=1.$$

This gives $$\frac{1}{x}=2,\quad \frac{1}{y}=1.$$
$$\therefore\ \mathbf{x}=\tfrac{1}{2},\quad \mathbf{y}=\mathbf{1}.$$

Example XIV. Solve the equations:

$$2x^2-xy+y^2=11. \quad\text{.................................(i)}$$
$$x^2+xy+2y^2=4. \quad\text{.................................(ii)}$$

It will be noticed that every term on the left side of either equation is of the second degree in x and y.

Multiplying (i) by 4 and (ii) by 11, and subtracting, so as to make the right-hand side 0,

$$-3x^2-15xy-18y^2=0.$$

Dividing by -3, $$x^2+5xy+6y^2=0.$$
$$\therefore\ (x+2y)(x+3y)=0.$$

(A) Either $x=-2y$.

Substituting in (ii),

$$4y^2-2y^2+2y^2=4.$$
$$\therefore\ 4y^2=4.$$
$$\therefore\ y=1\quad\text{or}\quad -1.$$
$$\therefore\ \text{since}\qquad x=-2y,$$
$$\text{when}\quad y=1,\qquad x=-2;$$
$$y=-1,\quad x=2.$$

We can write these answers as follows: $x=\pm 2$, $y=\mp 1$, it being understood that the two upper signs are taken together and the two lower.

(B) Or $x=-3y$.

Substituting in (ii),

$$9y^2-3y^2+2y^2=4.$$
$$\therefore\ 8y^2=4.$$
$$y=\frac{1}{\sqrt2}\quad\text{or}\quad -\frac{1}{\sqrt2}.$$
$$\therefore\ \text{since}\qquad x=-3y,$$
$$\text{when}\quad y=\frac{1}{\sqrt2},\qquad x=-\frac{3}{\sqrt2};$$
$$y=-\frac{1}{\sqrt2},\quad x=\frac{3}{\sqrt2}.$$

$$\textit{Answer}\left\{\begin{array}{ll} \mathbf{x}=\pm 2, & \pm\frac{3}{\sqrt2}. \\ \mathbf{y}=\mp 1, & \mp\frac{1}{\sqrt2}. \end{array}\right.$$

EXERCISE XI. e.*

Solve the equations :

1. $x+y+z=4.$
$x+2y+3z=9.$
$3x+y+4z=12.$

2. $x+y-z=1.$
$2x+y+z=7.$
$x-5y+3z=3.$

3. $x+2y-3z=0.$
$3x+3y-z=5.$
$x-2y+2z=1.$

4. $x+y=5.$
$y+z=4.$
$x+z=5.$

5. $x-y=2.$
$y-z=-1.$
$x+y+z=9.$

6. $\frac{2}{x}+\frac{3}{y}=13.$
$\frac{3}{x}-\frac{1}{y}=3.$

7. $\frac{1}{x}+\frac{5}{y}=5\frac{1}{2}.$
$\frac{6}{x}+\frac{2}{y}=5.$

8. $\frac{1}{u}+\frac{1}{v}=50.$
$\frac{1}{u}-\frac{1}{v}=0.$

9. $2x-\frac{1}{y}=5.$
$x-\frac{4}{y}=-5.$

10. $\frac{1}{x^2}+\frac{1}{2y^2}=\frac{5}{6}.$
$\frac{3}{x^2}-\frac{2}{y^2}=-1.$

11. $x^2+y^2=58.$
$x+y=10.$

12. $x^2+y^2=122.$
$x-y=10.$

13. $x^2+y^2=34.$
$xy=-15.$

14. $x^3+y^3=65.$
$x+y=5.$

15. $x^2-y^2=7.$
$x-y=1.$

16. $x^3-y^3=98.$
$x^2+xy+y^2=49.$

17. $x^2-xy-20y^2=10.$
$x+4y=10.$

18. $x-2y=0.$
$x^2-3y^2=4.$

19. $x-y=1.$
$2x^2-xy=6.$

20. $2x-y=3.$
$xy=9.$

21. $2x+3y=1.$
$2xy+7y^2=3.$

22. $4x-y=6.$
$y^2=12x.$

23. $2x-y=2.$
$\frac{x^2}{3}+\frac{y^2}{4}=1.$

24. $x^2+y^2=30.$
$x=2y.$

25. $x^2+2y^2=3.$
$xy+y^2=2.$

26. $x^2-xy+y^2=3.$
$2xy-y^2=3.$

27. $5x^2-xy=42.$
$xy-5y^2=-2.$

28. $x^2-3xy+y^2=4.$
$x^2-y^2=4.$

29. $x^2+4y^2=41.$
$2x^2-3y^2=38.$

30. $\frac{1}{x-2}+\frac{2}{3x}+\frac{1}{x+2}=0.$

31. $\frac{x-1}{x}-\frac{x-2}{x-1}=\frac{1}{x+5}.$

32. $\frac{100}{x}+\frac{100}{x-2}=18\frac{1}{3}.$

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* For additional examples, see E.P. 19, Nos. 21-30, page 251/29.

F. LITERAL EQUATIONS.

Example XV. Solve the equations:

$$ax + by = c. \quad \text{......(i)}$$

$$px + qy = 0. \quad \text{......(ii)}$$

Eliminating y in the usual way,

$$aqx + bqy = cq.$$

$$bpx + bqy = 0.$$

$$\therefore\ (aq - bp)x = cq.$$

$$\therefore\ x = \frac{cq}{aq - bp}.$$

In this particular example y is most easily found by substituting in (ii). We get $y = -\frac{cp}{aq - bp}$.

Answer $\mathbf{x} = \frac{\mathbf{cq}}{\mathbf{aq - bp}}$, $\mathbf{y} = -\frac{\mathbf{cp}}{\mathbf{aq - bp}}$.

It is often simpler to find y in the same way as x was found, instead of substituting.

Notation.—If we have to consider, for example, several rectangles of given size, we can take the lengths and breadths of them to be a, b; c, d; e, f, etc.; but it is often better to call them a, b; a', b'; a'', b''; etc., inches, so that a, a', a'', etc., inches, all stand for lengths and b, b', b'', etc., inches for breadths. Or again, we may take a_1, b_1; a_2, b_2; a_3, b_3; etc., inches.

It should be noticed that a, a', a'' are quite different numbers, and are not connected in any such way as a and a^2. So a_1, a_2, a_3 are quite different numbers.

For instance, instead of the last example, we might have solved the equations:

$$ax + by = c,$$

$$a'x + b'y = 0,$$

and the result (putting a' for p and b' for q) would have been

$$x = \frac{cb'}{ab' - ba'}, \quad y = -\frac{ca'}{ab' - ba'}.$$

These values of x and y cannot be simplified.

EXERCISE XI. f.*

Solve the equations :

1. $x^2+px+q=0.$ **2.** $x^2+2px+q=0.$

3. $ax^2+bx+c=0.$ **4.** $(x+a)(x-b)+(x-a)(x+b)=0.$

5. $x^4-a^2x^2-b^2x^2+a^2b^2=0.$ **6.** $(x-a)(x-b)=(x-a-b)^2.$

7. Find the ratio of x to y from the equation

$$a(x+y)+b(x-y)=ax-by.$$

8. Find the ratio of x to y from the equation

$$(ax+by)(bx-ay)+(ax-by)(bx+ay)=0.$$

Solve the equations :

9. $x+y=1.$
$px+qy=r.$

10. $x=y.$
$ax+by=a^2-b^2.$

11. $ax+by=2ab.$
$bx-ay=b^2-a^2.$

12. $ax=by.$
$a'x+b'y=1.$

13. $\dfrac{x}{a}+\dfrac{y}{b}=1.$
$\dfrac{x}{a'}+\dfrac{y}{b'}=1.$

14. Solve the equations :

$$ax+by=c,$$
$$a'x+b'y=c',$$

and use the results to write down the solution of the equations :

$$4x+2y=8.$$
$$3x-y=4.$$

Solve the equations :

15. $x+y=0.$
$ax^2+by^2=a+b.$

16. $y=mx.$
$y^2=4ax.$

17. $y=mx+\dfrac{a}{m}.$
$y^2=4ax.$

18. $x+y=c.$
$x^2+y^2=r^2.$

19. $\dfrac{p}{x}+\dfrac{q}{y}=0.$
$px+qy=r.$

20. $x+yt_1t_2=c(t_1+t_2).$
$xy=c^2.$

21. In the diagram on p. 118 (Fig. 18), in which CT touches the circle, it can be proved that

$$\frac{BC}{CA}=\frac{TB}{TC}=\frac{TC}{TA}.$$

If $BC=p$, $CA=q$, $AB=r$, calculate TC and TB in terms of p, q, r.

22. The area of a washer of diameter $2a$ inches, the width of the washer being x inches, is $\pi\{a^2-(a-x)^2\}$. If the area is $\pi\,.\,\dfrac{3a^2-2ab-b^2}{4}$, find the width.

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* For additional examples, see E.P. 20, page 251/30.

G. IDENTITIES, ETC.

Example XVI. Prove that

$$4b^2c^2-(a^2-b^2-c^2)^2 \equiv (a+b+c)(-a+b+c)(a-b+c)(a+b-c).$$

$$\begin{aligned}\text{The first expression} &= \{2bc+(a^2-b^2-c^2)\}\{2bc-(a^2-b^2-c^2)\}\\ &=\{a^2-(b^2-2bc+c^2)\}\{(b^2+2bc+c^2)-a^2\}\\ &=\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}\\ &=(a+b-c)(a-b+c)(b+c+a)(b+c-a)\\ &=\text{the second expression.}\end{aligned}$$

Example XVII. If $x-\frac{1}{x}=1$, prove that

$$x^3-\frac{1}{x^3}=4.$$

$$\begin{aligned}x^3-\frac{1}{x^3}&=\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right) \quad \text{(see p. 174)}\\ &=\left(x-\frac{1}{x}\right)\left(x^2-2+\frac{1}{x^2}+3\right)\\ &=\left(x-\frac{1}{x}\right)\left(\overline{x-\frac{1}{x}}^2+3\right)\\ &=1(1+3), \quad \text{since } x-\frac{1}{x}=1,\\ &=4.\end{aligned}$$

Example XVIII. If

$$\frac{x+4}{(x+2)(x+3)}=\frac{a}{x+2}+\frac{b}{x+3}$$

is an identity, and a, b stand for numbers, find their value.

Multiply each side of the identity by $(x+2)(x+3)$.

$$\therefore\ x+4 \equiv a(x+3)+b(x+2).$$

$$\therefore\ x+4 \equiv (a+b)x+3a+2b.$$

Since this is an identity,

$$a+b=1, \quad 3a+2b=4.$$

If we solve these equations we get $\mathbf{a}=2$, $\mathbf{b}=-1$.

We can check this result by simplifying $\frac{2}{x+2}+\frac{-1}{x+3}$.

We shall get $\frac{x+4}{(x+2)(x+3)}$.

Example XIX. Prove that

$$a^3+b^3+c^3-3abc \equiv (a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$

This is proved by multiplying out the second expression.
The following deduction deserves notice.
If $a+b+c=0$, then $a^3+b^3+c^3-3abc=0$ or $a^3+b^3+c^3=3abc$.

EXERCISE XI. g.*

Prove the identities in Examples 1–12:

1. $(a+b)^3 \equiv a^3+b^3+3ab(a+b)$. **2.** $\frac{a^2}{b}+\frac{b^2}{a} \equiv (a+b)\left(\frac{a}{b}-1+\frac{b}{a}\right)$.

3. $(b-c)(a+x)+(c-a)(b+x)+(a-b)(c+x) \equiv 0$.

4. $bc(b-c)+ca(c-a)+ab(a-b) \equiv a^2(b-c)+b^2(c-a)+c^2(a-b)$.

5. $(b+c)^2+(c+a)^2+(a+b)^2 \equiv a^2+b^2+c^2+(a+b+c)^2$.

6. $a^3+\frac{1}{a^3}+3\left(a+\frac{1}{a}\right) \equiv \left(a+\frac{1}{a}\right)^3$. **7.** $\left(\frac{x+y}{2}\right)^2+\left(\frac{x-y}{2}\right)^2 \equiv \frac{x^2+y^2}{2}$.

8. $\frac{1}{(x-y)(x-z)}+\frac{1}{(y-x)(y-z)}+\frac{1}{(z-x)(z-y)} \equiv 0$.

9. $\frac{y+z}{(x-y)(x-z)}+\frac{z+x}{(y-z)(y-x)}+\frac{x+y}{(z-x)(z-y)} \equiv 0$.

10. $(b+c)(c+a)(a+b) \equiv (a+b+c)(bc+ca+ab)-abc$.

11. $(y+z)(y'-z')+(z+x)(z'-x')+(x+y)(x'-y')$
$\equiv (y'z-yz')+(z'x-zx')+(x'y-xy')$.

12. $(b-c)(b+c-a)+(c-a)(c+a-b)+(a-b)(a+b-c) \equiv 0$.

13. Prove that $(x-y)(x-z)+(y-z)^2 \equiv (y-z)(y-x)+(z-x)^2$. What similar expression is equal to either of these?

14. If $a+c=2b$, prove that $\frac{1}{a-b}+\frac{1}{c-b}=0$.

15. If $2s=a+b+c$, prove that

$$s^2+(s-a)^2+(s-b)^2+(s-c)^2=a^2+b^2+c^2.$$

16. If $a+b+c=0$, prove that

$$a^2+b^2+c^2=-2(bc+ca+ab).$$

17. If $x+\frac{1}{x}=a$, prove that $x^2+\frac{1}{x^2}=a^2-2$.

18. If $x+y=0$, prove that $\frac{x^2}{y}+\frac{y^2}{x}=0$.

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* For some miscellaneous examples, see E.P. 21, page 251/31.

19. If $2s = a + b + c$, prove that

$$(s - a) + (s - b) + (s - c) - s = 0.$$

20. If $a + b + c = 0$, prove that

$$(-a + b + c)(a - b + c)(a + b - c) = -8abc.$$

21. If $x^2 + y^2 + z^2 = yz + zx + xy$, prove that

$$(y - z)^2 + (z - x)^2 + (x - y)^2 = 0.$$

Show that this is only possible, if x, y, z are all equal.

22. If $X = x^2 - yz$, $Y = y^2 - zx$, $Z = z^2 - xy$, show that

$$X^2 - YZ = x(x^3 + y^3 + z^3 - 3xyz).$$

Write down a similar expression for $Y^2 - ZX$ and $Z^2 - XY$.

23. If $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$, prove that $\frac{a}{c} = \frac{a - b}{b - c}$.

24. Show that $(y - z)^2 + (z - x)^2 + (x - y)^2 + 2(y - z)(z - x)$

$$+ 2(z - x)(x - y) + 2(x - y)(y - z) = 0.$$

25. If $2s = a + b + c$, prove that

(i) $(s - b)(s - c) + (s - c)(s - a) + (s - a)(s - b) + s^2 = bc + ca + ab.$

(ii) $\frac{1}{(s - b)(s - c)} + \frac{1}{(s - c)(s - a)} + \frac{1}{(s - a)(s - b)} = \frac{s}{(s - a)(s - b)(s - c)}.$

(iii) $\frac{b + c}{(s - b)(s - c)} + \frac{c + a}{(s - c)(s - a)} + \frac{a + b}{(s - a)(s - b)} = \frac{a^2 + b^2 + c^2}{(s - a)(s - b)(s - c)}.$

In the following identities, A, B, C are numerical values. Find them. Check your results as in Example XVIII. p. 184.

26. $\frac{2x + 3}{(x + 1)(x + 2)} \equiv \frac{A}{x + 1} + \frac{B}{x + 2}.$

27. $\frac{2}{(x + 1)(x - 1)} \equiv \frac{A}{x - 1} + \frac{B}{x + 1}.$

28. $\frac{2x - 3}{(x + 1)^2} \equiv \frac{A}{x + 1} + \frac{B}{(x + 1)^2}.$ **29.** $\frac{5x + 5}{(2x + 3)(x - 1)} \equiv \frac{A}{2x + 3} + \frac{B}{x - 1}.$

30. $\frac{1}{x(x + 7)} \equiv \frac{A}{x} + \frac{B}{x + 7}.$ **31.** $\frac{2}{x(x + 1)(x + 2)} \equiv \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{x + 2}.$

32. $\frac{3x^2 - 1}{(x - 1)x(x + 1)} \equiv \frac{A}{x - 1} + \frac{B}{x} + \frac{C}{x + 1}.$

33. $\frac{3x^2 - 10x + 10}{(x - 1)^2(x - 2)} \equiv \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x - 2}.$

REVISION PAPERS.

A.

(CHAPTERS I. II. III.)

SOME of the later papers of this set, *e.g.* A. 20–A. 25, are rather difficult for a first course and can be used as revision papers when the pupils have reached a later stage. Such papers are marked with an asterisk.

In the papers A. 1 to A. 25 it is assumed that letters stand for positive numbers only.

A. 1.

1. Give a general statement by means of letters to include the following statements:

4×1^2 is a perfect square.
4×2^2 ,, ,,
4×7^2 ,, ,,

2. If n is an odd number, $n+2$ is the next odd number. Show that this is true when $n=1$ and when $n=87$.

3. If $x=5$, $y=1$, find the values of

(i) $3x$. (ii) xy^2. (iii) $2x-6y$.

4. A boy was 10 years old x years ago; how old is he now? how old will he be in y years?

5. Write down (i) the square of $4a^3$, (ii) the value of $2x^5 \div x^3$, (iii) the value of $4x^2y \div 2xy^2$.

6. Simplify the following:

(i) $\dfrac{x}{3}+\dfrac{2x}{3}$. (ii) $a^2b^2 \div \dfrac{a^2}{b^2}$. (iii) $1-\dfrac{p}{q}$.

A. 2.

1. If $x=6$, find the value of each of the following:

(i) $\frac{2x}{15}$. (ii) $\frac{1}{2}x^2$. (iii) x^2-5x.

2. A and B start to meet one another from two towns 8 miles apart. A walks u miles an hour and B v miles an hour. How far will they be apart after (i) an hour, (ii) half an hour?

3. Find the H.C.F. and L.C.M. of $3x^2$ and $6x^2y$.

Simplify (i) $\frac{3x^2}{6x^2y}$. (ii) $\frac{1}{3x^2}+\frac{z}{6x^2y}$.

4. Express by means of brackets (without simplifying) the total area of two carpets, each $x+y$ feet long and z feet wide.

5. Simplify $a(a-2b)+b(2a-b)$.

Subtract $a+b-c$ from $a-b+c$.

6. If 3 Russian versts = 2 miles, how many miles are equal to x versts? Show that the answer might also be given as x miles $-\frac{x}{3}$ miles. Use this form of the answer to turn 42 versts into miles.

A. 3.

1. What is the perimeter of an equilateral triangle, each side of which is $2a$?

2. If it costs $\frac{x}{6}$ shillings to telegraph x words to France, what will be the cost of a telegram of 10 words? How many words can be sent for a shilling?

3. Give a general statement by means of letters to include the following statements:

$$\frac{3\times 2}{5\times 2}=\frac{3}{5}.$$
$$\frac{7\times 2}{8\times 2}=\frac{7}{8}.$$
$$\frac{5\times 2}{11\times 2}=\frac{5}{11}.$$

4. Simplify (i) $\frac{2}{3}+\frac{3}{2}$. (ii) $\frac{3}{5}+\frac{5}{3}$. (iii) $\frac{a}{b}+\frac{b}{a}$.

5. Find the H.C.F. and L.C.M. of $6a^2bc$ and $8ab^2c$. Show that the product of the H.C.F. and L.C.M. is equal to the product of $6a^2bc$ and $8ab^2c$.

6. Multiply $a-b$ by c and $a-c$ by b. Find by how much the first is greater than the second.

A. 4.

1. In Feb. 1917 the Food Controller allowed each person 4 lb. of bread a week. How many pounds would a family of n people require for 5 weeks?

2. If $x=3$, what is the value of

$$\text{(i) } x(x-1), \quad \text{(ii) } 1-\frac{x}{x+1}, \quad \text{(iii) } \sqrt{2x^2-2}\ ?$$

3. What is the meaning of $x-(y+z)$? Which is the greater, $x-(y+z)$ or $x-y+z$, if x, y, z are all positive? Give reasons.

4. Add x^3-x^2+1, x^2-x+1, x^3-x+1.

5. Multiply $a-b$ by 3 and subtract the result from 4 times $a+b$.

6. If my yearly bill for *The Times* was x shillings when each copy cost 1½d., what was it when the price was raised to 2d.? How much extra do I now have to pay?

A. 5.

1. If a book is published at x shillings, the cash price is $x-\frac{x}{4}$ shillings. What is the cash price of a book published at 24s.?

2. During the summer months ordinary time is 1 hour ahead of Greenwich time. What is ordinary time when by Greenwich time it is (i) 3 o'clock, (ii) 9 o'clock, (iii) x o'clock? What is Greenwich time when by ordinary time it is (iv) 8 o'clock, (v) 10 minutes past 5 o'clock, (vi) 10 minutes past x o'clock, (vii) y minutes past x o'clock?

3. A side of a square is $a+b$ inches long. Express by means of brackets (i) the sum of the sides, (ii) the area, of the square. Do not simplify your answers.

4. Multiply $2ab$ by $3ac$ and divide the result by abc.

5. Simplify (i) a^2+3a^2. (ii) a^2+3a-a. (iii) $a-\frac{a}{2}$.

6. A man plants three potatoes to every square yard, and 8 potatoes weigh a pound. How many pounds of potatoes must he buy to plant x sq. yd. ?

A. 6.

1. Criticise the following statement: "If the price of 1 lb. of coffee is x, the price of 2 lb. is $2x$."

2. Fill in the blanks in the following:

$$\text{(i)}\ \frac{a}{x}=\frac{\quad}{2x}. \qquad \text{(ii)}\ \frac{a}{c}=\frac{a^2}{\quad}. \qquad \text{(iii)}\ x=\frac{\quad}{2}=\frac{\quad}{y}.$$

3. Simplify (i) $(x+a)+x+(x-a)$.
(ii) $4(x^2-2x+1)+2(x^2+2x+1)$.

4. A class of boys is told to work out examples 2, 5, 8, 11, 14, 17, 20, etc., in a set of examples. Find a formula which gives these numbers.

5. Add together x^2+1, $x+1$, x^2+x and subtract the result from x^3+2x^2+2x.

6. A man has on one of his bookcases a books of one volume, b books of two volumes and c books of three volumes. Each shelf holds x volumes, and the books fill four shelves and partly fill a fifth. How many volumes are there on the fifth shelf ?

A. 7.

1. If a rectangle is $(x+1)$ ft. long and $(x+2)$ ft. wide, the area is x^2+3x+2 sq. ft Show that this is true when $x=6$.

2 Find a formula giving all numbers that end in a 3.

3. If a is greater than b, show that $x-a$ is less than $x-b$.

4. Subtract $x-(y+z)$ from $x+(y-z)$.
Subtract $(b^2-2b)\div b$ from $(a^2-2a)\div a$.

5. If A sq. in. is the area of the base of a cone and h in. its height, its volume is $\frac{1}{3}Ah$ cu. in. What is the volume of a cone, if its base has an area of 60 sq. in. and its height is 1 ft. ?

6. There are x passengers in a train, of whom $2y$ travel with half-tickets. How many whole tickets will be needed to provide tickets for them all ? Half-tickets are made by cutting whole tickets in two.

A. 8.

1. Give a general statement by means of letters to include the following statements :

$$2+2+2=3 \text{ times } 2.$$
$$5+5+5=3 \text{ times } 5.$$
$$11\tfrac{1}{2}+11\tfrac{1}{2}+11\tfrac{1}{2}=3 \text{ times } 11\tfrac{1}{2}.$$
$$1{\cdot}7+1{\cdot}7+1{\cdot}7=3 \text{ times } 1{\cdot}7.$$

2. Show that any number between 10 and 100 which is divisible by 5 is either $10x$ or $10x+5$, where x is the ten digit.

3. Prove that $a^{22}\times a^2=a^{24}$. Write down, without proving, the value of (i) $a^{22}\div a^2$, (ii) $(a^{22})^2$.

4. Simplify (i) $\frac{a}{2}-\frac{b}{2}$. (ii) $\frac{ab}{abc}-\frac{b}{2bc}$. (iii) $\left(\frac{1}{2}-\frac{1}{x}\right)2x$.

5. A man pays a bill of £x ys. with £$(x+1)$. How much change does he get ?

6. A man cycles uphill from A to B at 6 miles an hour and back again at 15 miles an hour. How long does he take if the distance from A to B is $10x$ miles ?

A. 9.

1. Give a general statement by means of letters to include the following :

What must be added to 3 to obtain 20 ? $20-3$.
What must be added to 5 to obtain 20 ? $20-5$.
What must be added to 11 to obtain 20 ? $20-11$.

2. Show that £x is the same as $8x$ half-crowns.

3. Simplify (i) $10ab-ab$. (ii) $2x-\frac{x}{4}$. (iii) $x^2-(x^2-1)$.

4. Simplify $x^2(x-2)+x(x^2+x-1)-2(x^3-2x^2)$.

5. If $a=3$, what are the values of (i) $2a$, (ii) a^2, (iii) $3(a-2)$, (iv) a^2-4, (v) $(a-2)^2$?

6. A clock loses x seconds an hour ; how many minutes will it lose in a week ? Use your answer to find how many minutes it loses in a week if it loses 5 seconds an hour.

A. 10.

1. Three consecutive odd numbers are written down. The least is x; what is the greatest and what is their sum?

2. Simplify (i) $\frac{3}{11} \div 2$. (ii) $\frac{a}{b} \div c$. (iii) $\frac{2}{5} \div \frac{2}{7}$. (iv) $\frac{2}{x} \div \frac{2}{y}$.

3. State in words how you find the value of (i) $x - 3(y+z)$, (ii) $(x-3)(y+z)$. Do not try to find the value of them.

4. Express by means of brackets the number of chests of tea in a cargo of x tons, if each chest weighs a lb. when empty, and contains 56 lb. of tea.

5. Which of the following are factors of $6b^2c$:—$6b^3c$, $12b^2c^2$, 3, $6b^3$, $6b^2$, $6b^3c^3$, $2b$, $60b^3c^3$, $24bc^2$, $3bc$? Which of them are common factors of $6b^2c$ and $6bc^2$?

6. A man's salary for his first year is £180, and it is increased by £a at the end of each year. What will it be for his twentieth year?

A. 11.

1. Show that $x^2 + y^2$ is less than $(x+y)^2$ when $x = 2$ and $y = 5$.

2. The sum of the four angles of a quadrilateral is 360 degrees. If three of the angles contain x, y and z degrees, find a formula for the fourth angle, using brackets.

3. If $a = 4$, $b = \frac{1}{2}$, find the value of (i) $2a^3$, (ii) $(2b)^3$, (iii) $3a^2b$, (iv) $a - \left(\frac{a}{2} + b\right)$.

4. Add together $2b - c$, $2c - a$, $2a - b$, and multiply the result by abc.

5. A walks x ft. per sec., B walks $(x-1)$ ft. per sec. They are standing 100 yards apart, and walk towards one another. How far will they be apart after half a minute? If instead of walking towards A, B had walked in the opposite direction, after how long would A have overtaken B?

6. The area of the whole of a country is x square miles, and of this the capital occupies a square miles. In the capital there are 20,000 people to the square mile and in the rest of the country 500 to the square mile; what is the total population?

A. 12.

1. If x is an odd number, 24 is a factor of $x^3 - x$. Show that this is true when $x = 5$.

2. Write down (i) the square of $3ab^2$, (ii) a square root of $25a^2b^2$, (iii) the product of $x^2 - 2x$ and $3x^2$.

3. Write down a formula giving all numbers divisible by 7. Also a formula for all numbers which are divisible by 7 after 1 has been subtracted.

4. The temperature rises at the same rate from 8 a.m. till noon. At 8 a.m. it is 50°, at 10 a.m. it is 56°; what is it at noon? If at 8 a.m. it is $x°$ and at 10 a.m. it is $y°$, what is it at noon?

5. Simplify (i) $\frac{15abc}{15abc}$. (ii) $1 + \frac{3a}{a}$. (iii) $\frac{a}{2} + \frac{a}{3}$.

6. An estate contains a acres. On it are built 30 cottages, each with a garden. Each cottage takes b acres of ground and the garden c acres; the roads altogether take d acres. Express by means of brackets (without simplifying) the amount of ground that is left.

A. 13.

1. From the formula $F = \frac{W(v-u)}{gt}$ find F, if $W = 56$, $v = 44$, $u = 0$, $g = 32$, $t = 6$.

2. Multiply $10a^2b^3$ by $3a^3b^2$, and divide the result by $6a^4b^4$. Express as a fraction in its lowest terms the result of dividing $18xz$ by $42xyz$.

3. Write down (i) the square of $9a^3x^2$, (ii) the square of $\frac{3x^2}{5yz}$, (iii) a square root of $100m^6n^2$.

4. In a right-angled triangle, one acute angle is x degrees; what is the other?

5. A, B, C, D are four points in order on a straight line, such that $AB = CD$. If AD and BC are $2x$ and $2y$ inches long, what is the length of AC?

6. An author gets, for writing a book, £20 as soon as it is published and afterwards £$\frac{1}{12}$ for every copy after the first 500. Express by means of brackets the amount he gets (in pounds) for x copies, (i) if x is less than 500, (ii) if x is greater than 500.

A. 14.

1. A set of numbers is written down. The first is 1×2, the second 2×3, the third 3×4, etc. Write down the nth.

2. The 3rd, 10th, 17th, 24th, 31st days of October, 1915, were Sundays. Write down a formula which gives these days.

3. Add together $\frac{1}{2}(y+z-x)$, $\frac{1}{2}(z+x-y)$, $\frac{1}{2}(x+y-z)$.
Simplify $x^2(x^2+x+1)-x(x^2+x-1)+(x^2-x+1)$.

4. Which is the greater, $\frac{a}{2}$ or $\frac{3a}{7}$?

Simplify (i) $\frac{a^2}{b^2} \times ab$. (ii) $\frac{a+b}{2ab}-\frac{1}{a}$. (iii) $\dfrac{\frac{a^2}{b}}{\frac{b^2}{a}}$.

5. Tea costs $2x+y$ pence a pound and coffee $x+2y$ pence a pound. Express by means of brackets the total cost of 5 lb. of tea and 3 lb. of coffee. Also, by means of brackets, the change (in pence) out of a pound note after buying them. Do not simplify your results.

6. A block of stone, a ft. long, b ft. wide, c ft. high, weighs w tons. Find the weight (in pounds) of a cubic foot of it.

A. 15.

1. Give by means of letters a general statement which includes the following :

$\frac{3}{11}$ is the same as $\frac{1}{11}$ of 3 or $3 \times \frac{1}{11}$.
$\frac{7}{11}$,, ,, $\frac{1}{11}$ of 7 or $7 \times \frac{1}{11}$.
$\frac{18}{11}$,, ,, $\frac{1}{11}$ of 18 or $18 \times \frac{1}{11}$.

2. Subtract $a-b-2c$ from $a-(b-2c)$.

3. A sentry starts from A and walks backwards and forwards between A and B, a distance of 80 yards. How far is he from A when he has walked $(80n+15)$ yards, (i) when n is an even number, (ii) when n is an odd number ?

4. What is the value of the square of $\frac{x}{2}$ when $x=3$?

Divide the square of $2a^2b$ by the square of ab.

5. What fraction with 100 as denominator has the same value as $\frac{x}{20}$? How much per cent. does this fraction represent ?

6. A carriage has wheels of different sizes, the front wheels being 40 inches and the others 50 inches in diameter. How many times does a front wheel revolve while a hind wheel is revolving x times?

A. 16.

1. Rewrite the product 99×23, putting $100 - 1$ instead of 99. Now find the value of the product. Express $99x$ in the same way.

2. Write down the values of (i) $18a^5b \div 2ab$, (ii) a square root of $49a^2b^2c^2$, (iii) the H.C.F. of $6a^2$ and $6abc$, (iv) the L.C.M. of $2x^2$, $2y^2$ and $2xy$.

3. How many minutes are there between x minutes past 9 and 12 o'clock?

4. A, B, C are three points in order in a straight line. AB is $x - y$ inches long, AC is $2x + y$ inches long; what is the length of BC? If $x = 2y$, show that $BC = 4AB$.

5. Find the interest on £100 at r per cent. for 2 years.

6. Give, by means of letters, a general statement to include the following statements:

Twice the sum of the first 3 numbers $= 3^2 + 3$.
Twice the sum of the first 4 numbers $= 4^2 + 4$.
Twice the sum of the first 5 numbers $= 5^2 + 5$.

Show that your statement is true for the sum of the first 6 numbers.

A. 17.

1. The area of a triangle is

$$\tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)},$$

where a, b, c are the sides of the triangle. Find the area of a triangle, the sides of which are 5 in., 5 in., 6 in.

2. Add together $\dfrac{2a+b}{4}$, $\dfrac{a-2b}{12}$ and $\dfrac{2b-a}{3}$.

3. If $a = 2x - y$, $b = x - 2y$, prove that $a^2 + b^2 = 65$ when $x = 5$, $y = 2$.

4. Multiply the sum of a and b by their product. Add three times the result to $a^3 + b^3$, and arrange the answer in descending powers of a.

5. If, in a village, n per cent. are males, what is the percentage of females? If the population of the village is a, how many females are there?

6. For the journey from London to Horsham a passenger is allowed 100 lb. of luggage free, and has to pay $\frac{1}{2}$d. per lb. on the rest. Find the charge (in pence) for x lb.

A. 18.

1. Under the Derby Scheme of 1915, men of the same number of years of age were placed in the same groups; single men of ages from 18 to 40 were placed in groups numbered from 1 to 23 respectively; married men of ages from 18 to 40 were placed in groups numbered from 24 to 46 respectively. In what group was a man of age x years placed (i) if single, (ii) if married?

2. If $x=3$, $y=1$, find the value of

(i) $x+y$. (ii) $(x+y)^2$. (iii) x^2+y^2. (iv) $\dfrac{x+y}{x+3y}$.

3. If a figure has x sides, the sum of its angles is $2x-4$ right angles. Show that this makes the sum of the angles of a triangle equal to 180°.

4. Simplify (i) $2+\frac{1}{3}$. (ii) $a+\frac{1}{b}$. (iii) $3\div\frac{1}{5}$. (iv) $a\div\frac{1}{b}$.

5. A man buys a dog for £5 and sells it for £$(5+x)$. What is his gain per cent.?

6. A box is a inches long, b inches wide, c inches high. Show that the area of the outside is $2(bc+ca+ab)$ square inches.

A. 19.

1. The energy of a body which weighs W lb. and has a speed of v ft. per second is measured by $\dfrac{W\,.\,v^2}{2g}$. Find the energy of a body weighing 1 cwt. moving at a speed of 20 ft. per second. $(g=32.)$

2. Draw a rectangle $a+b$ inches long and a inches wide. By dividing it into two parts, show that $a(a+b)=a^2+ab$.

3. Simplify $a(a-b+c)+b(a+b-c)+c(-a+b+c)$.

4. ABC is a straight line; D is the middle point of AC. If $AB=x$, $BC=y$ inches, find AD.

5. How much per cent. is x cwt. of x tons ?

6. If each person eats $\frac{3}{4}$ lb. of sugar each week, find a formula giving the amount of sugar (in cwt.) required for n persons for a year of 52 weeks.

A. 20.*

1. In this century the first leap year is the year $1900+4$.
" second " $1900+2\times 4$.
" third " $1900+3\times 4$, etc.
Include all such statements in a general statement.

2. Show that

$$(a-b)(c-d)=ac+bd-(ad+bc)$$

when $a=10$, $b=8$, $c=5$, $d=3$.

3. The value of silver money is proportional to its weight, and 1 lb. of silver is made into 66 shillings. What should be the weight of £x of silver ? From your formula show that £33 of silver weighs 10 lb.

4. Which of the following are factors of $2ac$? Which are common multiples of $2ac$ and $4c^2$? 2, $6abc$, $8a^2c^2$, $2c$, ac, $4ac$, c, $4ac^2$, $2a$, $12a^2c^2x^2$, $8abc$, $4a^2c$.

5. Simplify (i) $\frac{p}{p+q}+\frac{q}{p+q}$. (ii) $\frac{p^2}{q}\times\frac{q^2}{p}+pq\left(\frac{p}{q}+\frac{q}{p}\right)$.

6. A hectare $=100$ ares. An are is a square, each side of which is 10 metres. Express A hectares in square metres. Express x francs per hectare in centimes per square metre.

A. 21.*

1. Write down an algebraic formula which includes the following statements :

(i) 35 shillings per cwt. is the same price as $\left(\frac{3+5+\frac{5}{2}}{14}+3\right)$ pence per lb.

(ii) 67 shillings per cwt. is the same price as $\left(\frac{6+7+\frac{7}{2}}{14}+6\right)$ pence per lb.

(iii) 94 shillings per cwt. is the same price as $\left(\frac{9+4+\frac{4}{2}}{14}+9\right)$ pence per lb.

Do not prove the truth of these examples, but prove that the formula you write down is true.

2. In a tennis match between A and B, the scores of the three sets are

A	6	$3x-2y$	6
B	$x+y$	6	y

By how many games does A defeat B ?

3. Simplify $\left(\frac{x^{12}}{x^3}\right)^3 \div \left(\frac{x^6}{x^3}\right)^3$.

4. A road rises steadily x feet per mile of road ; how many inches does it rise in 100 yards ?

5. What is the perimeter of a rectangle of area A sq. in. and length l in. ?

6. In using a screw-driver, I find that x complete turns are needed to drive a screw through a 2-inch plank. Through what distance will the screw-head advance when the screw-driver is turned through an angle of 120° ?

A. 22.*

1. The fines for returning books late to a library are half-a-crown for any time during the first week and one shilling a day afterwards. Give a formula for a fine on a book returned x days late. (x greater than 7.)

2. Simplify $x(y-1)+y(x-1)+xy\left(\frac{1}{x}+\frac{1}{y}\right)$.

3. Multiply the square of $6ab$ by the square root of $\frac{a^2}{9b^2}$ and subtract a^3b from the result.

4. Through how many degrees does the hour hand of a clock turn between x minutes past six and x minutes to seven ?

5. A tank, x feet long, y feet wide, contains water to a depth of h feet. The water is drained off into an empty tank, a feet long, b feet wide. Find the depth of water in the second tank.

6. A man pays £x entrance fee and £y a year subscription for membership of his golf club ; in the first q years he plays k games ; what is the average cost of each game ?

A. 23.*

1. The following table shows the fee required for registering a letter and the corresponding limit of compensation allowed :

Fee in pence	2	3	4	5	6	7	8	9	10	11	12	13	14
Compensation in £	5	10	20	30	40	50	60	70	80	90	100	110	120

(i) Write down a formula for the limit of compensation allowed when the fee paid is x pence, where x is a whole number not less than 3 nor greater than 14.

(ii) Write down a formula for the fee that must be paid to obtain compensation up to £$10y$, where y is an integer less than 13.

2. Simplify $\frac{1}{2}a+\frac{3}{4}a-a$, and divide the result by $\frac{1}{2}$.

3. Fill in the blanks in

$$\text{(i)}\ \frac{1}{a}=\frac{a}{\quad}.\qquad \text{(ii)}\ x=\frac{\quad}{\sqrt{4x^2}}.\qquad \text{(iii)}\ \frac{a^2-1}{a}=\frac{\quad}{2}-\frac{1}{a}.$$

4. $ABCD$ is a straight line and B is the mid point of AC; $AB=2x+y$ inches, $BD=3x-y$ inches ; find the length of CD.

5. The edges of a skeleton cube are made from a wire x inches long ; what is the volume of the cube ?

6. A box containing 12 golf balls weighs x lb., and when empty weighs y oz. ; what is the weight of a golf ball in oz. ?

A. 24.*

1. The rates for letter post were : not more than one ounce 1d., more than one ounce and not more than two ounces 2d., for each additional two ounces $\frac{1}{2}$d. Find the cost of a letter weighing x ounces, where x is an odd integer, greater than unity.

2. Simplify
$$\frac{x}{\frac{2}{3}}-\frac{3x}{2\frac{1}{2}}.$$

3. Find the L.C.M. of 12, $2a^2$, $9a^3b$, ab^2.

What is the least expression this L.C.M. must be multiplied by, in order to make the result (i) a perfect square, (ii) a perfect cube ?

4. If $x=a+2b$, $y=3a-b$ and $a=2b=6$, prove that $\frac{x}{3}-\frac{y}{5}=1$.

5. A line AB is $4x$ inches long; a circle centre A, radius $3x$ inches, cuts AB at P; a circle centre B, radius $2x$ inches, cuts AB at Q; what is the length of PQ?

6. A train is travelling x miles an hour, when the brakes are put on; these reduce its speed by y miles an hour every second. After how many seconds will it stop?

A. 25.*

1. Draw two figures composed entirely of straight lines, such as the three given figures. Write down for each of the five figures, (1) the number of compartments, (2) the total number of corners, (3) the total number of edges; arrange your results in tabular form. Suppose any such figure to contain x compartments, y corners, z edges; by examining your table, discover the simple formula that connects x, y, z. Verify it from a new figure containing at least four compartments.

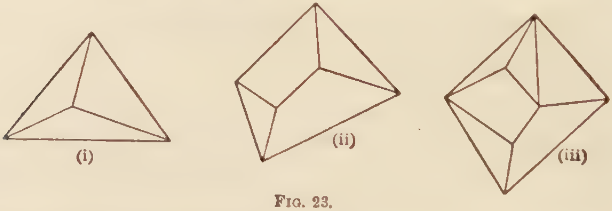

(i) (ii) (iii)

FIG. 23.

2. Subtract (i) $2a-3(b-c)$ from $5a+(b+c)$.
(ii) $2x^2$ from $(2x)^2$.

3. Simplify (i) $\frac{15xy}{12y^2} \div \frac{25x^3}{2xy}$. (ii) $x-\frac{x}{100}$.

4. An inland telegram containing 12 words or less costs 9d., each additional word costs ½d.; find the cost of a telegram containing x words, where $x > 12$.

5. A box is x feet long, y feet wide, z feet high; what length (in feet) of cord is needed to bind it once round in each of the 3 directions, allowing 1 foot extra for knots?

6. Two metre rules are divided into a and b equal parts respectively; they are placed side by side so that the tenth points of division in each coincide. What is the distance in cms. between their extremities?

B.

(CHAPTERS I.-VII.)

B. 1.

1. The temperature at 9 a.m. is a degrees and at 9 p.m. is b degrees. How many degrees has it risen?

Substitute $a=10$, $b=8$ in your answer. Substitute $a=-5$, $b=2$ in your answer.

2. If a third-class railway fare, at a penny per mile, from A to B is $2a$ shillings and sixpence, what is the distance of B from A? What would the second-class fare be at $1\frac{1}{2}$d. per mile?

3. Simplify

(i) $\frac{3}{5}\times 10$. (ii) $\frac{a}{b}\times ab$. (iii) $r-\frac{2r}{3}$. (iv) $\sqrt{x^2-\frac{3x^2}{4}}$.

4. Simplify $2(x+y)-4(x-y)$.

Multiply $b-a$ by $-ab$, and subtract the result from a^3-b^3. Give your answer in descending powers of a.

5. Express the following statement in the form of an equation, without simplifying: The difference between the squares of two consecutive whole numbers is 39.

6. Solve the equation $x=2(12-2x)$, explaining each step. Check your answer.

B. 2.

1. The base of a triangle is $2x$ inches and the height $1{\cdot}7x$ inches; what is its area?

2. What are the values of (i) $3+(-5)$, (ii) $(-5)+3$, (iii) $1-(-2)$, (iv) $(-6)-2$?

3. Express the following statement as an equation, in two forms, (i) as a relation between quantities, (ii) as a relation between numbers. Do not solve the equation. A journey of 150 miles takes me 5 hours; I travel a part of the distance in a train going at 40 miles an hour, the rest in a train going at 25 miles an hour.

4. Solve the equations:

$$7x-5y=4.$$
$$y=2(x-1).$$

5. By means of a diagram show the changes in the value of $3-2x$, as x changes from -2 to 3. For what values of x between -2 and 3 is $3-2x$ negative ? For what other values of x, not shown on the diagram, is it negative ?

6. $ABCD$ is a rectangular enclosure ; $AB=50$ yards, $BC=40$ yards ; P is a point on AB 10 yards from A ; find a point Q on BC, such that the distance of P from Q *via* B is half the distance of P from Q *via* A, D and C.

B. 3.

1. By how much per cent. is $5x$ greater than $4x$?

2. Simplify (i) $\dfrac{2a}{2ab}$. (ii) $\dfrac{2ab}{2a}$. (iii) $\dfrac{2}{3+\frac{1}{4}}$. (iv) $\dfrac{a}{b+\frac{1}{c}}$.

3. If the radius of the base of a cone is r inches and its height h inches, the volume may be taken as $\frac{22}{21}r^2h$ cu. in. Find the volume of a cone, if the radius of the base is $4\frac{1}{2}$ in. and the height 10 in.

4. $(a+b)^2=a^2+2ab+b^2$. Show that this is true (i) when $a=1$, $b=-2$, (ii) when $a=0$, $b=-3$.

5. The following table gives the number of shipbuilders in every 1000 out of work during the different months of the year. Represent this information graphically ; state the month in which (i) it is least, (ii) it is greatest. Explain roughly how it alters.

Jan., 80 ; Feb., 68 ; Mar., 61 ; Apr., 63 ; May, 60 ; June, 62 ; July, 61 ; Aug., 68 ; Sept., 81 ; Oct., 92 ; Nov., 99 ; Dec., 98.

6. A tank which holds 900 gallons is filled by two pipes, one of which can supply 40 more gallons per minute than the other. When both pipes are turned on, the tank is filled in ten minutes. How long would the smaller pipe take to fill the tank by itself ?

B. 4.

1. Find (in shillings) the Simple Interest on £x for a month at 5 per cent.

2. What is the reciprocal of

$$\text{(i) } a, \quad \text{(ii) } \frac{1}{x}, \quad \text{(iii) } \frac{a}{b}, \quad \text{(iv) } 1+\frac{p}{q}\ ?$$

3. Write down the value of (i) $1-5$, (ii) $-a+4a$, (iii) $a-4a$, (iv) $-a-(-4a)$.

4. A workman works a hours every morning and b hours every afternoon for 5 days in the week, c hours on Saturday morning and takes a holiday on Saturday afternoon. If a full week's work is 54 hours, express by means of brackets the number of hours he loses each week.

5. Solve the equations :

$$\text{(i) } 3(7-x)+2(5-x)=1.$$

$$\text{(ii) } \begin{aligned} 7x+y&=9.\\ x+7y&=15.\end{aligned}$$

6. A first-class and third-class ticket from Horsham to Henfield are 1s. 10d. and 1s. respectively ; to Lavant 5s. 4d. and 2s. 8d. ; to Bexhill 7s. 10d. and 4s. Which of these satisfy the rule that if third-class fare is xs., first-class is $2x$s. ? Which satisfy the rule that if third-class fare is xs., first-class is $2x$s. $-\,2x$d. ?

B. 5.

1. A right angle is divided into two parts. One part is $(45+x)$ degrees ; what is the other ?

2. If the average death rate for the United Kingdom per 1000 is 14·8, write down (using $+$ and $-$) how much the following are above the average : Hampstead 10·8, Shoreditch 18·4, Greenwich 13·2, Wandsworth 10·9.

3. Add together $1-2x+x^2$ and $2x^2-x-1$, and subtract the result from $3x-2$.

4. If $2x+y=z$, what is the value of x when $y=1$ and $z=7$?

5. Express the following statement in the form of an equation. " I cycle for 2 hours, then increase my speed 2 miles an hour for half an hour, and altogether I have gone 24 miles."

6. A first-class ticket costs xd. per mile and a third-class yd. per mile. What will be the cost in shillings of 3 first-class tickets and 1 third-class from London to Carlisle (300 miles) ?

B. 6.

1. Show that $(2a+2b)^2=4(a+b)^2$.

Express $\dfrac{a+b}{2}$ as the sum of two fractions; also $\dfrac{2a+b}{2}$ as the sum of two quantities.

2. A penny weighs $\frac{1}{3}$ oz., a halfpenny $\frac{1}{5}$ oz. What is the weight of (i) x shillings' worth of pennies, (ii) a shilling's worth of coppers, n of which are halfpennies?

3. Multiply $a-b-c$ by $-b$, $b-c-a$ by $-c$, $c-a-b$ by $-a$ and add the results.

Divide $a^2bc+ab^2c+abc^2$ by $-abc$.

4. Solve the equations:

$$\text{(i)}\quad x(x-3)+5=x(x+1)-2.$$

$$\text{(ii)}\quad 5x-12y=1.$$

$$x+y=7.$$

5. The duty on a bottle of wine is trebled, and this increases the cost of a bottle from 1s. 9d. to 2s. 2d. If the increase in the cost of a bottle is the same as the increase in the duty, find how much the duty was at first.

6. If the sum of two numbers is equal to twice their difference, prove that their product is equal to three times the square of the smaller number.

B. 7.

1. The temperature of the air is taken every two hours for a day, and the results are given below. Plot these values, and draw a smooth curve through the plotted points. Describe the way in which the temperature changes, and give to the nearest hour the time at which it is highest and lowest.

12	2 a.m.	4	6	8	10	12	2 p.m.	4	6	8	10
53	51½	51	55	61	67	71	73	73	69	60	56

2. What are the values of $3-x$ when $x=1, 2, 3, 4, 5$? What are the values of $(3-x)^2$ for the same values of x?

3. Solve the equations: (i) $\dfrac{2x-7}{2}+x=\dfrac{1}{2}$.

$$\text{(ii)}\quad \frac{3-x}{11-x}=\frac{1}{10}.$$

4. A parcel is just small enough to go by post if $l+c=6$, where l is its length and c its girth, in feet. What is the greatest girth of a parcel, if its length is 2 ft.? What is the greatest length of a box, the end of which is 9 in. square?

5. If three-fifths of a number is 36, what is the number?

6. The energy of a moving body, weighing W lb., which has a speed of v ft. per sec., is measured by $\frac{Wv^2}{64}$. If a body weighing 56 lb. has its speed reduced from 20 ft. per sec. to 10 ft. per sec., how much energy does it lose?

B. 8.

1. A stone moves $5x+2x^2$ ft. in x seconds. How far will it go in (i) 3 seconds, (ii) 4 seconds, (iii) the fourth second?

2. Plot the values of $2+3x$ from $x=-4$ to $x=0$. How does the value of $2+3x$ change as x increases from -4 to 0?

3. Divide x^2-2x by x. Divide $-x^2+2x$ by $-x$.

Show that $\frac{x^2-2x}{x}=\frac{-x^2+2x}{-x}$. Is $\frac{y-z}{x}$ the same as $\frac{z-y}{-x}$?

4. What is the value of $x^2-\{x(2x-3)-7\}$? Express the answer in ascending powers of x.

5. Solve the equations: (i) $0{\cdot}2x+1{\cdot}5=2{\cdot}2$.
(ii) $x=2y,\ 7x-9y=55$.

6. A boy buys 15 dozen papers at 8d. a dozen. He sells some of them at a penny each and sends back the rest, getting 7d. a dozen for them. If he gained 3s. $6\frac{1}{2}$d., how many did he sell?

B. 9.

1. If a ship has unloaded $\frac{a}{b}$ of its cargo, how much remains? Can a and b have any values?

2. Solve (i) $\frac{x}{3}+\frac{x}{5}=2$. (ii) $\frac{2}{x}=\frac{3}{5}$. (iii) $\frac{x}{3}=-\frac{1}{5}$.

3. A boat is sailing at x miles an hour against a current flowing at y miles an hour. How fast is it advancing? What does this become if $x=4$ and $y=6$? Explain your answer.

4. Subtract d from $a+(n-1)d$.
Add $a+d$ to $a+(n-2)d$.

5. The census has been taken every tenth year, viz. in the years 1801, 1811, 1821 ... 1911. Give a formula for the census years.

6. If the telegraph posts at the side of a railway are 60 yd. apart, and a train passes x each minute, show that its speed is roughly $2x$ miles per hour.

B. 10.

1. The height of the water at a post above mean sea level is $+16$ ft. at high water and -10 ft. at low water. What is its height when half-way between high and low water?

2. Rewrite $38 \times \frac{7}{8}$, putting $1-\frac{1}{8}$ instead of $\frac{7}{8}$. Use this to find the value of $38 \times \frac{7}{8}$. By putting x instead of 38, show that any number can be multiplied by $\frac{7}{8}$ in this way.

3. Is $\left(\frac{217}{344}\right)^2$ greater or less than $\frac{217}{344}$?
For what values of x is x^2 less than x?

4. Subtract $x-(y+z)$ from $(x+y)-z$.
Subtract $3-5x+x^2$ from $2x^2-x-2$.

5. For a parcel of any weight up to 1 lb. the postage is 3d.; from 1 lb. up to 2 lb., 4d.; from 2 lb. up to 3 lb., 5d.; from 3 lb. up to 5 lb., 6d. Draw a graph giving the postage for any weight up to 5 lb. Show that if it weighs exactly 2 lb., you cannot tell the postage from the graph.

6. The hour hand of a clock is half the length of the minute hand, and its tip moves through 2 in. every hour; how many feet does the minute hand move through in x hours?

B. 11.

1 A is a inches east of O, B is b inches east of O. Find how much B is east of A. Substitute $a=7$, $b=5$ in the result, and explain your answer. Substitute $a=-2$, $b=3$, and explain your answer.

2. If $a=-1$, $b=2$, find the value of (i) a^2, (ii) $2a-b$, (iii) $\frac{a}{b}-\frac{b}{a}$.

Simplify $\frac{1}{3}(2x-3y)-\frac{1}{2}(x-y)$.

3. Draw a graph of x^3 for values of x between -4 and 4. From your curve read off the values of $2{\cdot}6^3$ and $\sqrt[3]{40}$.

4. Show that 27 guineas is the same as £28 7s. Show that if x guineas is the same as £$(x+1)$, ys., then $y=x-20$.

5. (i) Solve the equation : $\frac{x}{200-x}=\frac{3}{5}$.

(ii) Explain why you cannot solve the equations :

$$6x-8y=5.$$
$$9x-12y=7\tfrac{1}{2}.$$

Also explain why you cannot solve

$$6x-8y=5.$$
$$9x-12y=8.$$

6. A bookseller takes off a quarter of the published price for cash. At what price must he publish a book so as to get 6s. cash for it?

B. 12.

1. If 1 inch $=x$ mm. and 1 mm. $=y$ inches, show that $xy=1$.

2. The following table gives the minimum temperature in degrees Centigrade for each month of 1914 in London. Plot these graphically. Also write down the increase in the minimum from each month to the next.

Jan.	$-6{\cdot}7$	Apr.	$-0{\cdot}2$	July	$8{\cdot}9$	Oct.	$1{\cdot}5$
Feb.	$-1{\cdot}6$	May	$3{\cdot}1$	Aug.	$7{\cdot}9$	Nov.	$-0{\cdot}2$
Mar.	$-2{\cdot}3$	June	$4{\cdot}6$	Sept.	$1{\cdot}0$	Dec.	$-2{\cdot}2$

3. Find the H.C.F. of $16abc^2$ and $24a^2bc$.

Add $\frac{1}{16abc^2}$ and $\frac{1}{24a^2bc}$. Simplify $\frac{16abc^2}{24a^2bc}$.

4. When it is x o'clock at Greenwich it is $\left(x+\frac{l}{15}\right)$ o'clock by local time in longitude l degrees east (so long as $x+\frac{l}{15}$ is not greater than 12). What is the time in Bombay (long. 74° E.) when it is 4 p.m. at Greenwich ? Use the formula to find the time at Bombay when it is 10 o'clock at Greenwich. Explain your answer.

5. By means of brackets show that to multiply a number by $\frac{20}{19}$ is the same as to add $\frac{1}{19}$th of the number.

6. Solve the equations :

$$\text{(i)}\ \frac{8x}{15}-\frac{7x}{6}=\frac{5}{2}.$$

$$\text{(ii)}\ \frac{8x}{3}+2y-5=x+\frac{y}{2}+8=\frac{6x}{5}+3y+\frac{9}{5}.$$

B. 13.

1. The charges for parcels by passenger train between London and Hastings are given below. Find a formula giving the charge for n lb. (n not greater than 10).

Weight	2	3	4	5	6	7	8	9	10 lb.
Charge	4	5	6	7	8	9	10	11	12 pence.

2. Multiply $2x^2+xy-y^2$ by $-x$, and subtract from the result $-y(x^2+xy-2y^2)$.

3. Express as equivalent fractions with the same denominator

$$\text{(i)}\ 1,\ \frac{x}{y},\ \frac{y}{x}. \qquad \text{(ii)}\ \frac{a+b}{a},\ \frac{a+b}{b}.$$

4. The real weight of a body is A gm. and the error made in weighing it is not more than a times its weight. Between what two values does the incorrect weight lie ?

What are the answers if $a=\frac{1}{100}$?

5. Solve the equations :

$$\text{(i)}\ \tfrac{1}{2}(x+5)-\tfrac{2}{3}(x-2)=\tfrac{7}{2}.$$

$$\text{(ii)}\ y=2x+5,\ y=x-7.$$

6. The perimeter of a semicircular area is 8 in. ; find its diameter to the nearest tenth of an inch.

B. 14.

1. Explain the difference between 35 in Arithmetic and xy in Algebra when $x=3$ and $y=5$. Also the difference between $2\frac{3}{5}$ and az when $a=2$, $z=\frac{3}{5}$.

2. The expression $ax+b$ is equal to 2·9 when $x=7$, and is equal to 7·1 when $x=14$. If a and b are constants (see Glossary), find their values.

3. Coal costs £x per ton. At this rate how many pence would be paid for 14 lb.? If the selling price is 25 per cent. more than this, what is it?

4. In an experiment in Mechanics the formula $a = \frac{w - F}{W + w} \cdot g$ is used. Find F if $W = 900$, $w = 100$, $g = 980$ and $a = 70$.

5. Simplify (i) $\frac{8x^3 - 6x^2}{-2x}$. (ii) $\frac{ab}{a^3b + a^2b^2 + ab^3}$.

6. In the House of Commons, 378 members vote, and the Government has a majority of 62. How many voted for the Government?

B. 15.

1. Find the value of (i) $3x$, (ii) x^3, (iii) $3x^3$, (iv) $(3x)^3$ when $x = 2$.

2. A ship starts from O and steams 8 miles an hour due E. in a current which flows E. at 3 miles an hour. What is the speed at which it is moving eastwards? After 2 hours it turns round and steams at 8 miles an hour due W. What is now the speed at which it is moving eastwards? How far will it be E. of O x hours after leaving O if x is greater than 2? Substitute 7 for x in the result, and interpret your result.

3. A cask containing 6 gallons of cider weighs 78 lb. The same cask containing $3\frac{1}{2}$ gallons weighs 53 lb. Find the weight of the cask.

4. Solve the equations:

$$\text{(i) } 0{\cdot}25x - 0{\cdot}4(x - 3) = 1{\cdot}02.$$

$$\text{(ii) } 3x - 10y = -19, \quad y = \frac{x - 2}{5}.$$

5. A quantity of an alloy of copper and tin weighs 112 lb. Another alloy is made which contains $\frac{1}{4}$ less copper and $\frac{1}{3}$ less tin, and weighs 82 lb. What fraction of the first alloy was tin?

6. The top and bottom marks of a form are 125 and 34, and these are reduced to 100 and 50. Represent graphically the relation between the reduced and unreduced marks. If the reduced mark corresponding to an unreduced mark x is $ax + b$, find a and b. Show that this result agrees with your graph when $x = 90$.

B. 16.

1. The following table gives the sizes of copper wire and their diameters in inches :

Size -	1	2	3	4	5	etc.
Diameter	0·001	0·002	0·003	0·004	0·005	etc.

What is the diameter of a wire of size 13 ? of size n ?

2. What day of the week is the $(7n+4)$th day after a Sunday, if n is a positive integer ?

3. The following table gives the number of South Wales miners (in thousands) absent from work on each day for a fortnight :

Mon., 16 ; Tu., 13·4 ; Wed., 11·5 ; Th., 10·9 ; Fr. 9.8 ; Sat., 11·3.

Mon., 17·5 ; Tu., 14·1 ; Wed., 12·4 ; Th., 11·3 ; Fr., 9·9 ; Sat., 11·2.

Plot these numbers, and explain how they change each week.

4. Fill in the bracket in the following identity :

$$2x-3\equiv 2(\qquad)+1.$$

5. If $a=-2$, $b=-1$, $c=0$, find the value of

(i) $(a-b)-(a-c)$.

(ii) $(a-b)^2-(a-c)^2$.

6. The Food Controller in Feb. 1917 allowed per week to each person 4 lb. of bread ; instead of any part of the bread three-quarters of its weight of flour might be used. If a man's allowance weighs $3\frac{3}{4}$ lb., how much bread does he have ?

B. 17.

1. The number of which

the ten digit is 3 and the unit digit 5 is $3\times 10+5$.

" " 4 " " 7 is $4\times 10+7$.

" " 2 " " 9 is $2\times 10+9$.

Write down a general statement to include all such statements as these.

2. Subtract $\frac{3}{4x^2}$ from $\frac{4}{5x}$. From your answer show that the second fraction is the greater if x is greater than 1.

3. The product xy is constant. If $y = 1\frac{1}{2}$ when $x = 2\frac{1}{2}$, what is the value of y when $x = 3$? What is the value of x when $y = 3$?

4. If a stone is thrown up with velocity u ft. per second, its height after t seconds is $ut - 16t^2$. If $u = 80$, find its height after (i) 5 seconds, (ii) 6 seconds. Explain the answers.

5. Using the formula of Question 4, find the velocity with which the stone is thrown up, if it is at the same height after 4 sec. and after 6 sec.

6. If a railway cutting x ft. high is faced with a brick wall, the thickness of the wall is $\left(\frac{x}{3} + 3\right)$ ft. for walls less than 18 ft. high and $\left(\frac{2x}{3} - 3\right)$ ft. for walls more than 18 ft. high. Draw a graph showing the thickness of the wall for values of x between 6 and 40.

B. 18.

1. If a ship is L ft. long and B ft. broad, its tonnage, according to the formula, is $\frac{(L - \frac{3}{5}B)B^2}{188}$. What is the tonnage of a ship 100 ft. long and 24 ft. broad, to the nearest whole number?

2. Find the H.C.F. of $12a^2b^2$ and $18abc$.

What is the H.C.F. of $12a^2b^2$ and $18abc$ when $a = 1$, $b = 2$, $c = 2$?

3. What is the value of (i) $(a - b)^2$, (ii) $(b - a)^2$ when $a = -2$, $b = -5$?

Simplify $(a - b) + (b - a)$. From your answer show that if the value of $a - b$ is x, the value of $b - a$ must be $-x$.

4. Solve the equations:

$$\text{(i)}\quad \frac{x}{3} - \frac{x-2}{4} = 1.$$

$$\text{(ii)}\quad 3x = 7y;\ 11x - 9y = 20.$$

5. A train arrives at x o'clock and is y hours late. If it had been $(y + 1)$ hours late, when would it have arrived? At what time was it due? x is less than 12 and greater than y.

6. The following table gives the normal weight of men of various heights. Plot these values and draw a smooth curve through the plotted points.

Height .	62	63	64	65	66	67	68	69	70	71	72 in.
Weight -	126	133	139	142	145	148	155	162	169	174	178 lb.

By drawing another graph, find roughly the greatest error in taking the weight of a man of height x in. to be $(5x - 182)$ lb.

B. 19.

1. If a hemp rope, C inches in circumference, will carry a load up to W tons, $W = 0{\cdot}25C^2$. Show that $C = 2\sqrt{W}$.

2. A gardener draws a roller x feet wide at p miles an hour. How long will he take to roll half an acre of grass?

3. The temperature is taken each morning for a week as follows: Sun., $-10°$; Mon., $-6°$; Tu., $3°$; Wed., $0°$; Thurs., $-4°$; Fr., $5°$; Sat., $-7°$ C. Find the average for the week to the nearest whole number. State how much each of these temperatures is *above* the average for the week. The average for the month is $-1°$ C. State how much each of these temperatures is below the average for the month.

4. By means of brackets show that to multiply a number by $\frac{9}{5}$, one can multiply it by 2 and subtract a tenth of the result.

5. Add $a^2 - a - 1$ and $1 - a - a^2$; subtract $1 - a - a^2$ from $a^2 - a - 1$. Multiply the two results.

6. A man would pay the same income tax whether he paid 1s. 6d. in the pound on all his income above £130 or 2s. in the pound on all his income above £160. Find his income.

B. 20.

1. If an inch of rain falls, the weight of water that falls is 100 tons per acre. If there is a fall of x tenths of an inch, what is the weight per sq. mile? 1 sq. mi. = 640 acres.

2. Solve the equation:

$$\frac{1-2x}{5} = \frac{3x+1}{4} - \frac{5x-1}{3}.$$

3. If $2A+3B=21$ and $7A-4B=1$, show that $3\mathrm{A}+5B=34$.

4. Remove the bracket from $2xy(3x^2y-4xy^2)$.

Express $36a^3b-18a^2b^2+36ab^3$ as a product, by means of brackets.

5. Show that $4a$ is greater than $3a$, if $a=2$, but that $4a$ is less than $3a$, if $a=-2$.

6. A man has 22 min. to get to the station, a distance of two miles. If he takes a motor bus which travels a mile in 8 min., at what distance from the station can he get out and walk, if he walks a mile in 16 min.?

B. 21.*

1. The following table shows the distance in which trains travelling at various rates should be capable of being stopped by the brakes:

Speed of train, in miles per hour	30	35	40	45	50	55	60
Distance travelled before train stops, in yards	99	135	176	223	275	333	396

Exhibit this graphically; and find the greatest speed of the train which can be stopped in 200 yards.

2. If $x=-1$, $y=-3$, find the values of

(i) $(1-x)^2$. (ii) xy^2. (iii) $\dfrac{y-1}{x-1}$.

3. Solve $$\frac{x}{0 \cdot 5}-\frac{x+1}{0 \cdot 8}=1.$$

4. If $y=ax+b$, where a and b are constants, and if $x=4$ when $y=1$ and $x=7$ when $y=7$, find the values of a and b; find also the value of x when $y=0$.

5. An oak beam, supported at the ends, will just carry at the middle a load of $\dfrac{3bd^2}{2l}$ cwt., where b, d are the breadth and depth in inches, l the length in feet. What is the greatest load that such a beam 12 ft. long, 1 ft. broad and 18 in. deep will carry?

6. A circle is inscribed in the triangle ABC, touching BC, CA, AB at X, Y, Z; $BC=5$, $CA=4$, $AB=7$; use the fact that the two tangents from a point to a circle are equal to calculate AY and BX.

B. 22.*

1. (i) Subtract $1-(a-b)$ from $a-(1-b)$.

(ii) Divide $(-6ab^2)^3$ by $3a^2b^2$.

2. (i) What number exceeds 23 by the same amount that 23 exceeds 17 ?

(ii) What number exceeds a by the same amount that a exceeds b ?

3. A lift starts from the ground floor, ascends x floors, descends y floors and finally ascends z floors ; each floor is 12 feet high ; how high is it now above the ground ?

4. (i) Find a value of x that makes $4(x-11)$ equal to $x-1$.

(ii) What values of x, y make the expressions $x+y$, $5(2x-y)$, $x+2(y-1)$ equal ?

5. A heavy uniform beam AB, of length $2a$ inches, is supported at its ends. C is the mid point of AB and P is any point on AB, x inches from C. The height, y inches, of P above C is given by $y=kx^2(6a^2-x^2)$. where k is a constant. If D is the mid point of BC, compare the depths of C and D below B.

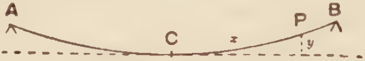


FIG. 24.

6. A chain is composed of n circular links, each of which is d inches in external diameter and is made of material $\frac{1}{2}$ inch thick. What is the total length of the chain, when taut ?

B. 23.*

1. The following table gives the relation between the velocity of the wind and the pressure it exerts :

Speed of wind, in miles per hour	5	10	20	30	40	60	70	80	100
Pressure, in lbs. per sq. foot	0·12	0·49	1·97	4·43	7·87	17·7	24·1	31·5	49·2

Represent these facts graphically.

If the wind blows at not less than 50 miles an hour, it is said to be a "storm"; what is the least pressure exerted by the wind in a storm?

2. Solve the equation: $\frac{1}{3x}-\frac{1}{4x}=\frac{1}{60}$.

3. Simplify (i) $2x(x-3x^2)-x^2(1-2x)-(-2x)^2$.

(ii) $1+\frac{4x-2y}{2x}-\frac{9y-x}{3y}$.

4. (i) Multiply the reciprocal of $\frac{1}{x^2}$ by $2x$.

(ii) Find a square root of the product of $12xy^2$ and $3x^3$.

5. If T is the tonnage of a vessel, L ft. its length, B ft. its breadth, $T=\frac{(L-\frac{3}{5}B)B^2}{188}$. Make L the subject of this formula.

6. ABC is a triangle; a circle, centre A, cuts AB, AC at P, Q; a circle, centre B, radius BP, cuts BC at K; a circle, centre C, radius CQ, cuts CB at H; $BC=x$, $CA=y$, $AB=z$ in. Calculate the length of AP (i) if $HK=\frac{1}{2}BC$, (ii) if H coincides with K.

B. 24.*

1. (i) Subtract $a-1$ from $1-a$.

(ii) Multiply the sum of $3x^2-2x+1$ and x^2+x+2 by the excess of $2(x-1)$ over $x-2$.

2. Simplify (i) $(ab)^2-(-ab)^2$. (ii) $\frac{x-1}{-1\frac{1}{2}}+\frac{2x-1}{4\frac{1}{2}}$.

3. Solve the equations: $\frac{x+y}{3}=\frac{2x}{21}=\frac{x-y-2}{15}$.

4. (i) If $3a+2b-c=7$, what is the value of b when $a=2$, $c=-1$?

(ii) If $5x+2y=47$, what is the least integral value of x which makes y a negative integer?

5. The tangents at two points P, Q of a circle meet at T; H is a point on TP; the other tangent from H to the circle touches it at N, cuts TQ at K; $TH=5''$, $TK=6''$; $HK=3''$. Calculate HN, using the fact that the tangents from a point to a circle are equal.

6. Three loops of string are made once round a box in the three different directions: their lengths are 7″, 9″, 12″; find the volume of the box.

B. 25.*

1. One sort of candle burns for $2x$ hours and another sort for $3x$ hours; how many candles of the first kind will last as long as y candles of the second kind, if I used one at a time in each case?

2. Simplify

$$\text{(i)}\ x-\frac{x^2}{2x}.\quad \text{(ii)}\ \sqrt{a^2+\frac{9a^2}{16}}.\quad \text{(iii)}\ (3x)^3\div(-2x)^2.$$

3. If $3x+4y=1$, $2x-5y=16$, $ax-4y=2$, find a.

4. ABC is a triangle such that $AB=4x$, $BC=7x$, $CA=5x$ inches; D is a point on BC such that $BD=\frac{1}{2}DC$; find how much longer it is from A to D *via* C than from A to D *via* B.

5. In a factory, 340 people are employed. The men are paid 4s. 6d. a day and the women are paid 3s. a day; the total wages amount to £60 a day. How many men are employed?

6. One end A of a light rod of length l inches is built into a wall and a weight is suspended from the other end. The consequent deflection at any point P, x inches from A, is

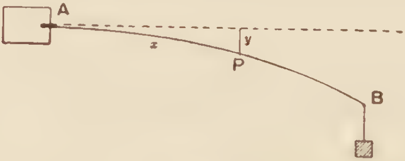


FIG. 25.

y inches, where $y=kx^2(3l-x)$ and k is a constant. If C is the mid point of AB and D, E the mid points of AC, CB, compare the deflections at D, C, E.

C.

(CHAPTERS I.–X.)

C. 1.

1. What are the factors of (i) $4a^2+4ab$, (ii) $6ab+6b^2$? What is their L.C.M.? Simplify

$$\text{(i) } \frac{4a^2+4ab}{6ab+6b^2}. \qquad \text{(ii) } \frac{1}{4a^2+4ab}+\frac{1}{6ab+6b^2}.$$

2. PQ, RS are parallel and the distance between them is h. $PQ=a$, $RS=b$. Show, by dividing $PQRS$ into two triangles, that its area (A) is given by

$$A=\tfrac{1}{2}ah+\tfrac{1}{2}bh \quad \text{or} \quad A=\tfrac{1}{2}h(a+b).$$

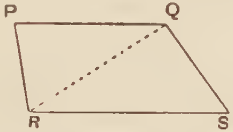


FIG. 26.

3. Solve $$\frac{x+3}{y+5}=\tfrac{1}{2}, \quad \frac{x-1}{y-1}=\tfrac{1}{3}.$$

4. Express as a compound quantity twice x shillings $+y$ pence, where y is between 0 and 6; also express the same as a compound quantity, if y is between 6 and 12.

5. What values of x will make $(x+12)(x-12)=0$? Show that if x is greater than 12, $(x+12)(x-12)$ must be positive.

6. The surface of a cone of slant height l, the radius of the base being r, is given by $S=\pi r(l+r)$. Change the subject so that it gives l in terms of S and r.

C. 2.

1. A is a village on a mountain side. A climber is 6500 ft. above A at 2 o'clock and descends 2500 ft. an hour. How much will he be above A at x o'clock? Put 5 for x in your answer. Explain the meaning of the result you get.

2. Give a general statement which includes the following statements:

3×5 is less than the square of 4 by 1.
6×8 is less than the square of 7 by 1.
10×12 is less than the square of 11 by 1.
37×39 is less than the square of 38 by 1.

3. Factorise (i) $10x^2-10$. (ii) $2x^2-4x+2$.

Find the value of

(i) $(10x^2-10)\div 5(x+1)$. (ii) $(2x^2-4x+2)\div(2x-2)$.

4. By means of an equation find how to divide a hundredweight into two parts, so that one is three-fifths of the other.

5. Find a number which, when divided by p, gives the same result as when p is divided by q. Check your result by putting $p=6$, $q=3$.

6. Solve $(x+7)(x-1)=180$.

Use your answer to write down two numbers which differ by 8 and whose product is 180.

C. 3.

1. A discount of 1s. in the pound is taken off a bill of £a. What is the bill now (in pounds)?

2. Plot values of $2-3x$ from $x=-3$ to 4. For what values of x is it (i) positive, (ii) negative, (iii) zero?

3. Solve the equations:

$$\text{(i)}\ \frac{2x}{3}-y=4\tfrac{1}{2}. \qquad \text{(ii)}\ ax+a=bx+b.$$
$$\frac{5x}{6}+y=9.$$

4. Factorise $12x^2+5x-2$. What values of x make $12x^2+5x-2$ equal to 0? What values of x make it equal to -2?

5. Factorise (i) a^9-a^8, (ii) a^4-a^3. Write down the L.C.M. of a^9-a^8 and a^4-a^3. Simplify $\dfrac{a^4-a^3}{a^9-a^8}$.

6. Show by Algebra that the difference between the squares of two consecutive whole numbers must be an odd number.

C. 4.

1. Make n the subject of the following formula: $I=\dfrac{E}{\dfrac{r}{n}+R}$.

2. A picture frame is 22 in. long and 16 in. wide. The width of the frame is x in. Find the area of the space occupied by the frame. Use your answer to find the area of this space if the frame is 1 in. wide.

3. The cost of sending x gallons of milk 50 miles is $a+bx$ pence, where a and b are the same for all quantities. If it costs 1s. $3\tfrac{1}{2}$d. for 10 gallons and 6s. 6d. for 60 gallons, find the cost for x gallons.

4. Simplify (i) $\dfrac{x^3-x}{x^3-x^2}$. (ii) $\dfrac{1}{x^3-x}-\dfrac{1}{x^3-x^2}$.

5. Show that $x=2$ makes $x^3-x^2+7x-18$ equal to 0. Write down one factor of $x^3-x^2+7x-18$. Find another factor.

6. The total weight of 20 bronze coins is 5·6 oz. If a penny weighs $\frac{1}{3}$ oz. and a halfpenny $\frac{1}{5}$ oz., how many of the coins were pennies ?

C. 5.

1. Write down the square of $5a-3x$.
Add a term to $4x^2-28xy$ so as to make it a perfect square. Express $4x^2-28xy$ as the difference of two squares.

2. Simplify
$$\frac{3}{x-4}+\frac{4}{x-3}-\frac{7x}{x^2-7x+12}.$$

Solve the equation:
$$\frac{x+(a-b)}{x+(a+b)}=\frac{a+b}{a-b}.$$

3. If d, c, a are the diameter, circumference and area of a circle, $c=\pi d$ and $a=\frac{1}{4}\pi d^2$. Find a in terms of c.

4. If $2x+5y$ is multiplied by $x-y$ and the product divided by $x+2y$, what is the remainder ?

5. If a, b, c are the sides of a triangle and $(b-c)(c-a)(a-b)=0$, what do you know about the triangle ?

6. Find a number such that, whether divided into two equal parts or into three equal parts, the product of the parts shall be the same.

C. 6.

1. If a body starts with a speed of u ft. per sec., increases it a ft. per sec. every second, and after going s ft. has a speed of v ft. per sec., then $v^2=u^2+2as$. Find s in terms of v, u, a. If $v=u+at$, show that $s=\dfrac{u+v}{2}\,.\,t$.

2. Divide $x(a+b)+y(a+b)+z(a+b)$ by $a+b$.
Find whether $ax+x^2-a+x$ has any factors. If so, state what they are.

3. Prove that $xy=\left(\dfrac{x+y}{2}\right)^2-\left(\dfrac{x-y}{2}\right)^2$. Use this to find 291×73, with the help of a table of squares which gives $182^2=33124$ and $109^2=11881$.

4. Solve
$$\frac{5x-9}{2x-1}-2=\frac{2x-5}{x-1}.$$

If $12x+7y=29$ and $6x-5y=23$, find the value of $2(x+y)$.

5. Simplify $\frac{1}{(x-y)^2}+\frac{1}{(x+y)^2}-\frac{2}{x^2-y^2}$.

6. The effect of quadrupling the duty on wine is to increase the price of a bottle of wine by 50 per cent. (increase of cost is the same as increase of duty). Find what percentage of the original price was duty.

C. 7.

1. One man sells eggs at x a shilling, another at yd. per dozen. If the prices are the same, show that $xy=144$.

2. Simplify $\frac{4x-4}{6x-6}$. For what values of x is $\frac{4x-4}{6x-6}$ equal to $\frac{2}{3}$?

Simplify $1-\frac{3x}{4x-4}-\frac{5}{6x-6}$.

3. Draw a graph giving the value of x^2-x for values of x between -2 and 3. Use your graph (i) to find the roots of $x^2-x=5$, (ii) to find the values of x for which x^2 is equal to x.

4. If $v=at$ and $s=\frac{1}{2}at^2$, prove that $v^2=2as$. Also prove that $s=\frac{1}{2}vt$.

5. A chain 9 ft. long hangs over a peg. If one end is 18 inches below the other, how much of the chain is there on each side of the peg?

A chain a ft. long hangs over a peg. If one end is x ft. below the other, how much of the chain is there on each side of the peg?

6. Solve the equations:

$$\text{(i) } ax=1, \qquad \text{(ii) } 2x^2-x-4=0,$$

giving the roots in (ii) to one place of decimals.

C. 8.

1. Write down the value of (i) $5-(-3)$, (ii) $-3+(-1)$, (iii) $-x^2-(-2x^2)$, (iv) $-2x\times(-3x)$, (v) $-2x\div(-3x)$.

2. Factorise (i) $x(2x+5)+3$, (ii) $x(x+7)-6(x+2)$.

Find the value of 104×96 by expressing 104 as $100+4$ and 96 as $100-4$.

3. Solve the equations:

$$\text{(i) } \frac{3x-5}{2x+1}=\frac{6x-11}{4x+3}.$$

$$\text{(ii) } \frac{x}{2}+y=x+\frac{y}{2}=2.$$

4. At a shooting range, one pays twopence for a miss and receives sixpence for a hit. A man has 27 shots, and has to pay 1s. 10d. to the keeper of the range. How many hits did he make? (C.S.C.)

5. Find the number which is half-way between $a-b$ and $b-a$.

6. State three special cases of the following theorem and verify them. If x is any whole number the last digit of x^5 is the same as the last digit of x.

C. 9.

1. Find the value of $(x^2+12x+35)(x^2-x-20)\div(x^2-25)$.

Also of $$\left(x+12+\frac{35}{x}\right)\left(x-1-\frac{20}{x}\right)\div\left(x-\frac{25}{x}\right).$$

2. Find the remainder when x^3+7x^2-15x is divided by $x-2$. What number added to x^3+7x^2-15x will make it divisible by $x-2$?

3. A formula in Optics is $\frac{1}{x_1}+\frac{1}{x_2}=\frac{2}{r}$. Show that this is the same as $2x_1x_2-rx_1-rx_2=0$. Obtain x_1 in terms of x_2 and r, and *write down* x_2 in terms of x_1 and r.

4. Draw a graph giving the value of $4x-3x^2$ for values of x between -1 and 4. Use it to solve the equation $4x-3x^2=1$. Also to solve $3x^2-4x+1=0$. Find from your graph the value of $4x-3x^2$, when $x=3{\cdot}4$.

5. Solve the equations:

$$\text{(i) } \frac{x}{2}-\frac{x-1}{3}=17.$$

$$\text{(ii) } \frac{x}{a}+\frac{x}{b}+\frac{x}{c}=bc+ca+ab.$$

6. What is the average weight of a things weighing x lb. each and b things weighing y lb. each? Use your result to find the average weight per pot of 4 three-pound and 5 one-pound pots of marmalade.

C. 10.

1. A copper wire is 130 in. long. Various weights are hung at the end of it, and the corresponding increase in length is given below. Find a formula for the length corresponding to a weight w lb.

Weight, -	2	4	6	8	10	12 lb.
Increase, -	0·04	0·08	0·12	0·16	0·20	0·24 in.

2. Simplify $\frac{x^4-1}{(x^2+1)^2}$. Show that the answer may be put in the form $1-\frac{2}{x^2+1}$.

3. ABC is a semi-circle, AC being the diameter. If it is 50 yd. shorter to go from A to C along AC than along ABC, find the diameter of the circle to the nearest yard.

4. Solve the equations:

$$\text{(i)}\ x+2y=2x-3y+7=0.$$

$$\text{(ii)}\ \frac{x}{a}+\frac{x}{b}=a+b.$$

5. The population of a town increases n per cent. each year. Show that it is multiplied by $1+\frac{n}{100}$ each year. Show that this expression is the same as $\frac{100+n}{100}$. Give this as a decimal, when $n=2$.

6. A man spends 25s. a week on rent and food. His rent increases 10 per cent. and the cost of food 20 per cent. He now spends 29s. a week. Find his original rent.

C. 11.

1. Find a formula for expressing in feet per sec. a speed given in miles per hour. If you take x miles per hour to be the same as $\frac{3x}{2}$ ft. per sec., how much per cent. of the correct value is the error?

2. Multiply x^2-xy+y^2 by $x+y$. From your result deduce (i) the result of dividing x^3+y^3 by x^2-xy+y^2, (ii) the factors of x^3+y^3. Simplify $\frac{x+y}{x^2-xy+y^2}+\frac{2}{x+y}$.

3. Plot values of $x+\frac{2}{x}$ for values of x between 0·3 and 5. Find the values of x for which $x+\frac{2}{x}$ is equal to 4. Show that these values are the roots of the equation $x^2-4x+2=0$.

4. For a certain quantity of gas $\frac{pv}{273+t}=10$, where p is the pressure in gm. per sq. cm., v the volume in c.c., t° C. the temperature. Change the subject so as to give t in terms of p and v.

5. A man buys 1000 bulbs for 19s., some of them at the rate of 50 for 9d., and the rest at 100 for 2s. 6d. How many did he buy of each kind?

6. Any number of two digits is subtracted from the number formed by reversing the digits. Show that the difference is divisible by 9.

C. 12.

1. If $a+b=5$ and $x-y=1$, write down, if possible, the value of (i) $b+a$, (ii) $y+x$, (iii) $\frac{b+a}{y-x}$, (iv) y^2-x^2.

2. I spend £1 in laying in a stock of sugar and another £1 when the price has gone up 1d. per lb. Altogether I got 108 lb. What were the prices ?

3. Solve the equations :

$$\text{(i)}\quad 1-\frac{x}{x+2}=\frac{x+2}{x+1}-1.$$

$$\text{(ii)}\quad (x+1)(y+4)=48.$$
$$xy=24.$$

4. Show that $x-1\equiv 5(x+3)-4(x+4)$. Express $\frac{x-1}{(x+4)(x-3)}$ in the form $\frac{A}{x+4}-\frac{B}{x+3}$ when A and B are numbers.

5. Simplify

$$\text{(i)}\quad \left(\frac{a}{b}-\frac{b}{a}\right)\div\left(\frac{1}{a}+\frac{1}{b}\right).$$

$$\text{(ii)}\quad \frac{y-z}{yz}+\frac{z-x}{zx}+\frac{x-y}{xy}.$$

6. A flat ring is bounded by two circles having the same centre. The radius of the inner circle is r in. and of the outer circle is $r+h$ in. Show that the area of the ring is $\pi h(2r+h)$ sq. in.

C. 13.

1. If a body weighing w lb. is describing a circle of radius r ft. at the end of a string with a velocity v ft. per sec., and the pull on the string is P lb., then $P=\frac{wv^2}{gr}$. Change the subject so as to give (i) r, (ii) v, in terms of the other quantities.

2. What common factors have $2ax+x^2$ and $x(x^2-4a^2)$?

Simplify
$$\frac{a}{2ax+x^2}-\frac{a}{2ax-x^2}-\frac{2a}{x^2-4a^2}.$$

3. Solve the equations :

(i) $xy=48$, $(x-3)(y+3)=78$. (ii) $6x^2=x+3$,

giving each root to two significant figures.

4. Show that $(ax+by)^2+(bx-ay)^2=(a^2+b^2)(x^2+y^2)$.

5. A piece of new flannel is a yd. long and b yd. wide. After being soaked it loses one-twelfth of its length and one-fifteenth of its width. Find its new area.

6. A number consists of two digits, the sum of which is 9. Show by Algebra that it must be divisible by 9.

C. 14.

1. Find the factors of $x(x+z)-y(y+z)$.

Simplify $$\left(\frac{x}{y+z}-\frac{y}{x+z}\right)\cdot\frac{(y+z)+(x+z)}{x+y+z}.$$

2. If $\frac{1}{u}-\frac{1}{v}=\frac{1}{k}$, find v in terms of u and k.

3. Soldiers walk at 3·6 miles an hour for 50 minutes, rest 10 minutes, again walk for 50 minutes, and so on. Draw a graph to show their position after any time up to 5 hours. A cycle gives them a start of $1\frac{2}{3}$ hours and then goes 12 miles an hour. When does it catch them up? Are they walking or resting when it overtakes them? Solve graphically.

4. 1 dollar + 1 franc = 5s.; 1 franc + 1 shilling = 0·44 dollar; 1 shilling + 1 dollar = 6·2 francs. Find the value of a dollar and a franc in English money. Are all these equations necessary? Do they contradict one another?

5. Solve the equations:

$$\text{(i)}\quad \frac{2x-2}{y-5}=\frac{x-4}{2y-6}=2.$$

$$\text{(ii)}\quad \frac{x-a}{b}=\frac{x+b}{a}-2.$$

6. If the angles of a triangle are A, B, C degrees, show that $90-\frac{A}{2}$, $90-\frac{B}{2}$, $90-\frac{C}{2}$ degrees are possible angles of a triangle.

C. 15.

1. If you write down the whole numbers beginning with m and going up as far as n, how many will there be?

2. Solve the equations:

$$\text{(i)}\quad (x+1)(x-2)-(x-3)(x+4)=(x-2)(x-3).$$

$$\text{(ii)}\quad x=y+6,\quad x^2=y+48.$$

3. If a is divided by b, the quotient being q and the remainder r, what is the relation between a, b, q and r? Use your result to find the number which, when divided by 11, gives quotient 8 and remainder 4.

4. Draw a graph of x^3 for values of x between -2 and 2. Also draw a graph of $2x-1$ for the same values. By considering their points of intersection, find roots of the equations $x^3=2x-1$.

5. A motor cycle is bought for £45 and sold at a gain of x per cent. What was the selling price? If the selling price was £40 10s., write down an equation for x. Solve it. Interpret your answer.

6. AB is 12 in. long. Find the position of a point C in AB so that $AC\,.\,CB=27$ sq. in.

C. 16.

1. Arrange in descending powers of x, expressing each power as a single term : $x^3 + a(x^2 + bx + cx) + bc(x - a)$.

2. Find the value of y from

$$\frac{x}{1{\cdot}2} + 0{\cdot}4y = 5, \quad \frac{x}{2{\cdot}7} - 0{\cdot}3y = 0{\cdot}75,$$

giving the result to the nearest tenth. Taking the approximate value of y, substitute it in each equation and find the corresponding value of x. (C.S.C.)

3. Factorise (i) $12a^2 + 3ad - 8ac - 2cd$.

(ii) $x(x+1) - 3y(3y+1)$.

4. Solve (i) $2x - (a - x + a + x) = 2b$.

(ii) $p + \frac{x}{q} = q + \frac{x}{p}$.

5. Simplify $1 + \frac{3}{x-1} + \frac{2}{(x+1)^2}$.

6. From Bristol to London is 120 miles ; an express train starts 6 min. late from Bristol, but by travelling 2 miles an hour faster than its ordinary rate arrives punctually in London. Find its ordinary rate.

C. 17.

1. Solve $2x - 1 = \frac{1}{2}(x - \frac{1}{3})$; also $2(x - 1) = \frac{1}{2}(x + \frac{1}{3})$.

2. The force required to pull m tons by rail up a slope of 1 in n may be taken to be $\left(\frac{2240}{n} + 10\right)m$ lb. Find the force (in tons) required to pull 200 tons up a slope of 1 in 35.

3. A man arrives at Charing Cross and wants to catch a train at Paddington ; a four-wheeler at 8 miles an hour will make him 4 min. late, a motor bus at 10 miles an hour will give him 2 min. to spare ; how long has he for the journey ?

4. Simplify $\frac{(a+b)x}{(x+a)(x-b)} - \frac{(b+c)x}{(x+c)(x-b)}$.

5. Solve $x^2 + y = 2{\cdot}27$; $x = y + 1{\cdot}3$ (C.S.C.)

6. Two thermometers A and B are both graduated, but on A the freezing point is marked 32° and the boiling point 212°, while on B neither of these points is indicated. The following pairs of corresponding readings of the two thermometers are observed :

A	59°,	77°.
B	35°,	45°.

Show by means of a graph the readings of B corresponding to those of A between $-25°$ and $212°$, and from your graph ascertain

(1) the freezing point and the boiling point on B;

(2) the reading of each thermometer corresponding to zero on the other;

(3) the temperature for which the readings of A and B are the same. (C.S.C.)

C. 18.

1. If a yacht is L ft. long and B ft. broad, its tonnage may be taken to be $\frac{(L-B)B^2}{188}$. A 60-ton yacht is to be built, and the length is to be 4 times the breadth. Show that the length will be $4\sqrt[3]{3760}$ ft.

2. Solve
$$\frac{3}{x-2}-\frac{x+1}{4-x}=1.$$

3. Simplify
$$\frac{a-\frac{b(2a-b)}{a+b}}{\frac{a^2}{a-b}-b}.$$

4. A and B notice that at 9 o'clock the temperature is $t_1°$; A also observes that it rises $t_2°$ between 9 and 2 and falls $t_3°$ between 2 and 6. What will be the temperature at 6? B does not look again till 6; what rise will he notice between 9 and 6? Use these values to show that $t_1+(t_2-t_3)=t_1+t_2-t_3$. Work out the question again, supposing a fall of $t_2°$ and then a rise of $t_3°$, and supposing that B notices a *fall* between 9 and 6.

5. The resistance R, in pounds per ton, to the motion of a railway train is roughly given by the formula $R=a+bv^2$, when v is the speed in miles an hour, and a, b are numbers which are the same for all values of R and v. When $v=20$, $R=9{\cdot}6$, and when $v=40$, $R=20{\cdot}4$. Find the values of a and b, and find R when $v=35$. (C.S.C.)

6. Show that the sum of any four consecutive numbers is equal to the difference between the product of the two greatest and the two least.

C. 19.

1. A rainfall of 1 inch is equivalent to 100 tons per acre. If you take $1''=2{\cdot}5$ cm. and 1 acre $=0{\cdot}4$ hectare, what will be the weight per hectare corresponding to a rainfall of x cm.?

2. Add $\frac{a^2+1}{a^2-1}$ and $\frac{a^2-1}{a^2+1}$.

3. Put $\frac{1}{y}$ instead of x in the fraction $\frac{x-2}{x^2-1}$ and simplify the result.

4. A bankrupt can just pay a dividend of 7s. 6d. in the pound; if his debts had been £1200 more, he could only have paid 6s. 8d. in the pound. What are his debts?

5. Express $\frac{2x+3}{x+6}-\frac{x}{x+1}-\frac{x+1}{x+2}$ as a single fraction, and find the values of x for which it vanishes. Give any value that is not integral as a decimal to two places. (C.S.C.)

6. A number of two digits is $5\frac{1}{2}$ times the sum of its digits. Show that the digits must be equal.

C. 20.

1. If a train is travelling at v miles an hour round a curve of r ft. radius, the outer rail should be h inches above the inner, where $h=3{\cdot}75\frac{v^2}{r}$. If $h=3$ and $r=2000$, find the value of v.

2. Find the H.C.F. of $3x^4+3x^3-6x^2$ and $4x^3-2x^2(x^2+1)$.

3. Solve the equations:

$$ax^2=by, \quad bx=ay^2.$$

4. If $y=\frac{ax+1}{x-a}$, prove that $x=\frac{ay+1}{y-a}$.

From either of the above formulae determine (to two decimal places) the value of a, in order that x may be equal to $3{\cdot}2$ when $y=2{\cdot}6$, and, using this value of a, find (to one decimal place) the value of x when $y=10$. (C.S.C.)

5. Find, correct to two places of decimals, the roots of the equation $x=\frac{x-1}{x-3}$.

6. Twenty pounds of gold are coined into 934 sovereigns and one half-sovereign. What will be the total weight of x sovereigns and y half-sovereigns? (A sovereign weighs twice as much as a half-sovereign.)

C. 21.

1. Make P the subject of the following formula for pneumatic transmission through a tube: $t=0{\cdot}0005\sqrt{\frac{l^3}{Pd}}$, where t is the number of seconds a carrier takes to go through a tube l feet long and d feet in diameter under a pressure of P lbs. per sq. inch.

How is the time affected, if the length of the tube is doubled?

2. Multiply $3 - \frac{1}{x+1}$ by $x^2 + x$.

3. What values of x make $2x^2 + 3$ equal to $8x$? Give each of them correct to one decimal place.

4. Find the factors of

(i) $x^4 - 7x^2y - 18y^2$.
(ii) $2ab - a^2 - b^2$.
(iii) $(x^2 + 1)^2 - 4x^2$.

5. Gold weighs 1200 lb. to the cubic foot, copper 550 lb. How much copper must be added to 1 lb. of gold so that the alloy may weigh 1130 lb. to the cubic foot?

6. A hollow cube is made out of a rectangular piece of thin cardboard $6x$ inches long, $4x$ inches wide. What is its volume?

C. 22.

1. The following formula is given for the speed of a paddle-steamer: if P = indicated horse-power, S = mid-sectional area in sq. feet, V = velocity in knots, then $V = \sqrt[3]{\frac{620P}{S}}$; (i) make P the subject of this formula; (ii) in what ratio must P be altered if V is to be doubled?

2. Prove the identity:

$$(x - 1)(2x + 3) + (2x + 1)(x - 5) \equiv (3x + 2)(x - 3) + (x - 2)(x + 1).$$

3. Solve the equations:

(i) $x + \frac{x}{a} = a$.

(ii) $\frac{2}{x} - \frac{x}{2} = \frac{15}{4}$.

4. If $V = \pi r^2 h$ and $S = 2\pi r^2 + 2\pi rh$, find a formula for V in terms of π, r, S.

5. (i) What number must be added to $x^2 - 5x - 3$ to make the result a perfect square?

(ii) Express $x^2 - 6x - 7$ as the difference of two squares.

6. At a certain election there were two rival candidates, and their supporters were conveyed to the polling booths in carriages capable of accommodating 8 and 12 voters respectively. If the voters, 740 in all, just filled 75 carriages, find by what majority the election was won. (C.S.C.)

C. 23.

1. Prove that 1, 3, $-\frac{1}{2}$ are the solutions of the equation

$$2x(x^2+1)-7(x^2-1)=4.$$

2. By what must $\frac{1}{x-1}$ be multiplied to give $\frac{1}{x}-1$?

3. Find the numerical value of a which satisfies

$$5(x-3)+3(x+7)\equiv a(x+1)-2,$$

and check your answer by simplification of each side.

4. If $V=\frac{4}{3}\pi r^3$ and $S=4\pi r^2$, find an equation between V and S which does not contain r.

5. A man has x pounds, $5x$ shillings and $30x+3y$ pence. Express the total amount in pence and also in half-crowns. If $x=7$ and the total sum is £19. 5s., what is the value of y? (C.S.C.)

6. A train travels k miles from A to C, and passes a station B on the way. The whole journey occupies p hours. Its average speed from A to B is s miles an hour and from B to C is t miles an hour. How far is B from A?

C. 24.

1. If $\frac{1}{x}-\frac{2}{y}=1$, prove that $\frac{2}{x+1}-\frac{1}{y+1}=1$.

2. A gives B x^2-y^2 miles start; A travels x miles an hour, B travels y miles an hour; how long will A take to catch up B, and how far from the starting-point will he be?

3. Solve the equation:

$$(x+5)^2-4(x+5)-12=0.$$

4. If $A=\pi r^2$ and $C=2\pi r$, express A in terms of C, π.

5. Simplify $\left(1+\frac{2}{x-1}\right)\div\left(x-\frac{2x}{x+1}\right)+\frac{1}{1-x}$.

6. In the case of a shell fired with a flat trajectory, the height, h feet, of the shell t seconds after it is fired is given by the formula $h=16t(T-t)$, where T seconds is the total time of flight; (i) make T the subject of this formula; (ii) if the shell attains its greatest height, a feet, after half the total time of flight, find a in terms of T; (iii) if the total time of flight is 6 seconds, find the greatest height it attains and the times at which it is at a height of 95 feet.

C. 25.

1. Two telegraph posts of equal height are at distance $2l$ feet apart, and the connecting wire is of length s feet; the sag at the mid point is d feet, where $d = \frac{1}{2}\sqrt{3ls - 6l^2}$ approx. (i) Make s the subject of this formula; (ii) if $x = s - 2l$, prove that $x = \frac{4d^2}{3l}$; (iii) if the poles are 30 yards apart and the sag is 6 inches, by how much approximately will the length of the wire exceed the distance between the poles ?

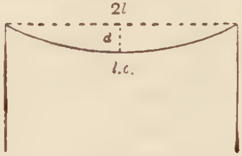


FIG. 27.

2. Prove that the triangle, whose sides are $2n+1$, $2n(n+1)$, $2n^2+2n+1$ inches, is right-angled: what special case is obtained by putting $n=3$?

3. Solve the equation:

$$3x - 7 = \frac{1}{3x-7}.$$

4. Take any number; double it; add 4; divide by 2; multiply by the original number; add 1; take the square root. Prove that the result always exceeds by unity the original number.

5. A line AB, 5 inches long, is produced to a point C such that $AC \,.\, CB = 6AB^2$. Calculate the length of BC.

6. In a mill the men earn a shillings a day and the women b shillings a day; there are n people employed and the average wage is c shillings a day. How many men are employed ?

HARDER REVISION PAPERS.

D.

(CHAPTERS I.-XI.)

D. 1.

1. Divide $15 - 13x + 7x^2 - x^3$ by $x^2 - 2x + 3$.

2. Factorise (i) $8a^6 + a^3b^3$.

(ii) $(a + x)^2(b + x) - (a + x)(b + x)^2$.

3. Solve the equations:

(i) $x^2 - ax - 90a^2 = 0$.

(ii) $ax + by = c^2$.

$$\frac{a}{b + y} - \frac{b}{a + x} = 0.$$

4. Find the value of $\frac{a - x}{b - x}$ when $\frac{2}{x} = \frac{1}{a} + \frac{1}{b}$.

5. Simplify $1 + x + x^2 + \frac{x^3}{1 - x}$.

6. A copper wire a inches long is stretched $\frac{1}{n}$ inch. Find by how much per cent. its length is increased. If the diameter before and after stretching was d_1 and d_2 inches, find how much per cent. the volume was reduced.

D. 2.

1. Multiply together $x + y$ and $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

2. Solve the equations:

(i) $x + y = c$.

$\frac{x}{a} = \frac{y}{b}$.

(ii) $y + z = 25$.

$z + x = 27$.

$x + y = 32$.

3. Find by inspection a square root of the following perfect square: $x^4+4x^2y^2+4x^2z^2-4x^3y-4x^3z+8x^2yz$.
Is $a^2+b^2+c^2-2bc-2ca-2ab$ a perfect square?

4. Simplify $\dfrac{1}{(x+y)^2}+\dfrac{1}{(x-y)^2}-\dfrac{2}{x^2-y^2}$.

5. By drawing a diagram showing the changes in the value of $2x-\dfrac{x^2}{2}$, find (i) its greatest value, (ii) the values of x for which it is positive, (iii) the values of x for which it is increasing as x increases.

6. A line AB is 10 inches long. It is divided into two parts so that one part is n times the other. Find the length of each part. Test your answer by putting $n=4$.

D. 3.

1. Divide $a^2(b-c)+b^2(c-a)+c^2(a-b)$ by $b-c$. Find the other factors.

2. Solve the equations: $a_1x+b_1y=c_1$,
$a_2x+b_2y=c_2$,
and use your results to find the solution of
$3x+8y=20$,
$4x-7y=3$.

3. Show that $a^2+\dfrac{1}{a^2}-2\left(a+\dfrac{1}{a}\right)+3$ may be written in the form $\left(a+\dfrac{1}{a}\right)^2-2\left(a+\dfrac{1}{a}\right)+1$. Write down its square roots.

4. Simplify

$$\frac{x^3-a^3}{x^2-ax+bx-ab}\times\frac{x^3-b^3}{x^2+ax-bx-ab}\div\frac{x^2-ax-bx-ab}{x^2+ax+bx+ab}.$$

5. Find two numbers such that their sum, product and quotient are proportional to 40, 192, 3.

6. A ball, of radius r in., is lying on a table, and a cone just fits over the ball and rests on the table. If the height of the cone is 12 in. and the radius of the base 5 in., show that $\dfrac{r}{12-r}=\dfrac{5}{13}$. Hence find r.

D. 4.

1. Factorise (i) x^6+5x^3+4.
(ii) $(1+a)^2(1+b)+(1+a)(1+b)^2$.

2. Solve the equations: $\frac{5}{x}+2y=\frac{3}{2}$.

$$\frac{3}{2x}-y=-\frac{1}{12}.$$

Find the ratio of x to y from the equation

$$2x^2-11xy-21y^2=0.$$

3. Simplify

$$\left(\frac{1}{x-a}-\frac{1}{x-b}\right)\left(\frac{1}{x+a}-\frac{1}{x+b}\right)\div\frac{(a-b)^2}{x^2-(a-b)x-ab}.$$

4. Find a term which, added to $4a^4-12a^3+5a^2+1$, will make it a complete square.

5. Find the remainder when x^3+3x^2+ax+b is divided by x^2+x+4. What values of a and b will, make x^2+x+4 a factor of x^3+3x^2+ax+b ?

6. A chord of length $2a$ is placed in a circle of radius r, and the perpendicular from the centre on to the chord is p. Find a relation between a, r and p. If a chord 10″ long is placed in a circle of radius 6″, how far is it from the centre ?

D. 5.

1. Find whether the following statement is correct: "The sum of the first $2n$ prime numbers, regarding 1 as a prime number, is always a prime number."

2. Find all the integral values of a for which x^2+ax-6 has rational factors.

3. Simplify (i) $\left(\frac{a}{1+a}+\frac{1+a}{a}\right)\div\left(\frac{a}{1+a}-\frac{1-a}{a}\right)$.

(ii) $\frac{x^2-y^2-z^2+2yz}{x^2+y^2-z^2+2xy}$.

4. A triangle has side x, y and z inches long, where x, y and z are given by $x+y=10$, $2x=3y$, $z=3$.

Calculate their values. (C.S.C.)

5. In a right-angled triangle, if a, b, c are the sides and p is the perpendicular on to the hypotenuse, $a^2=b^2+c^2$ and $ap=bc$. Prove that $\frac{1}{p^2}=\frac{1}{b^2}+\frac{1}{c^2}$.

6. A bamboo 10 ft. high breaks. The top part falls over without being detached and the top touches the ground 3 ft. away from the foot of the bamboo. At what height was the break ?

D. 6.

1. A rectangle is a ft. long and b ft. wide. Each of these is increased by one half. What is the new area? By how much per cent. has the area increased?

2. Prove that

$$(a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$
$$=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4.$$

3. If a, b, c are the squares of three consecutive whole numbers, prove that $(c-b)-(b-a)=2$.

4. The sum of the squares of the first n numbers is

$$\frac{n(n+1)(2n+1)}{6}.$$

What is the sum of the squares of the first $2m+1$ numbers?

5. The velocity v (measured in miles an hour) of a stream flowing in a straight channel of uniform breadth and depth is given by the formula $\frac{1}{v^2}=a+\frac{b}{x}$, where the depth of the water is x inches. If $v=2$ when $x=4$ and $v=3{\cdot}3$ when $x=12$, find, to one decimal place, the value of v when $x=8$. (C.S.C.)

6. Three circles whose centres are $C_1C_2C_3$ are such that each touches the other two and each is outside either of the others. If the sides of the triangle $C_1C_2C_3$ are 4″, 5″, 7″, find the radii of the circles.

D. 7.

1. Find the value of $1-3x+3x^2-x^3$ when $x=0{\cdot}3$.

2. Factorise (i) x^2-x-y^2-y.
(ii) $x(x-2)-(y^2-1)$.

3. Find the remainder when $x^5-ax+30$ is divided by x^2-3x+2. For what value of a is $x^5-ax+30$ exactly divisible by x^2-3x+2? (C.S.C.)

4. Find the L.C.M. of the two expressions

$$3x^2-11x+6 \text{ and } 6x^2-7x+2.$$

Find also a third expression containing x^2 such that the L.C.M. of all the three expressions will be the same as that of any two of them. (C.S.C.)

5. Show that $a-c$ is a factor of $(a-b)^3+(b-c)^3$.

6. ABC is a triangle. If the bisector of the exterior angle at A meets BC produced in D, $\frac{BD}{DC}=\frac{BA}{AC}$. Find the lengths of BD and DC, if $BC=a$, $CA=b$, $AB=c$.

D. 8.

1. Find one of the factors of $27(x+a)^3-8(x+b)^3$.

2. Simplify $\dfrac{(x+y)^2-z^2}{(x+z)^2-y^2}$.

3. The hypotenuse of a right-angled triangle is $\sqrt{65}$ and the sum of the other two sides is 11. Find them.

4. If a, b, c are the sides of a triangle and $2s=a+b+c$, the area of the triangle is $\sqrt{s(s-a)(s-b)(s-c)}$. Use this formula to find the area of an equilateral triangle, each side of which is $2a$.

5. Solve the equation:

$$\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c}.$$

6. AB is a railway 220 miles long, and three trains (P, Q, R) travel upon it at the rate of 25, 20, and 30 miles an hour respectively; P and Q leave A at 7 a.m. and 8.15 a.m. respectively, and R leaves B at 10.30 a.m. When and where will P be equidistant from Q and R?

D. 9.

1. Solve the equations:

$$12(x+y)-(2x+3y)=11.$$
$$7(x+y)+2(2x+3y)=9.$$

2. Simplify $\left\{\dfrac{x}{a}+\dfrac{2x^2}{a(b-x)}\right\}\left\{\dfrac{a}{x}-\dfrac{2ax}{x(b+x)}\right\}$.

3. If $\left(a+\dfrac{1}{a}\right)^2=3$, prove that $a^3+\dfrac{1}{a^3}=0$.

4. Find the H.C.F. and L.C.M. of

$$a^4-x^4 \text{ and } a^3-a^2x-ax^2+x^3.$$

5. A mixture of x lb. of tea at a pence per lb. and of y lb. of tea at b pence per lb. costs c pence per lb. Prove that

$$\frac{x}{y}=\frac{c-b}{a-c}.$$

6. Find the two times between 7 and 8 o'clock when the hands of a watch are at right angles.

D. 10.

1. Solve the equations:

$$\text{(i)}\quad 3(x-a)-2(y+a)=5-4a.$$
$$2(x+a)+3(y-a)=4a-1.$$

$$\text{(ii)}\quad \frac{3}{x+1}+\frac{5}{x}-\frac{6}{x-1}=0.$$

2. Find the value of $\dfrac{x^2-y^2+x}{y^2-x^2+y}$, when $x=\dfrac{a-b}{a+b}$ and $y=\dfrac{a+b}{a-b}$.

3. What must be the value of a if 1 is a root of the equation $x^3-7x^2+5x+a=0$? When a has this value, what are the other roots (to the nearest hundredth) ? (C.S.C.)

4. Show that any whole number can be expressed in one of the forms $3n$, $3n-1$, or $3n+1$, where n is a whole number.

5. A man receives $\dfrac{x}{y}$ of £1 and afterwards $\dfrac{y}{x}$ of £1 ; he then gives away £2. Show that he does not lose by the transaction.

6. What number must be added to $4x^2-20x$ to form a complete square ?

Show that $4x^2-20x+36$ is never less than 11, whatever value is given to x.

D. 11.

1. Solve the equations :

$$\text{(i)}\quad \frac{x-4}{2}+\frac{2x-3}{3}=\frac{6x+1}{8}.$$

$$\text{(ii)}\quad x^2-xy+y^2=39.$$
$$x+y=9.$$

2. Find an integral value of x for which the expression

$$x^5+10x^4+30x^3+32x^2+12x-4$$

shall be exactly divisible by x^2+3x+1. (C.S.C.)

3. Simplify $\quad$ (i) $x+\dfrac{1}{1+\dfrac{x+1}{3-x}}$.

$$\text{(ii)}\quad \frac{a^4-b^4}{a^2-2ab+b^2}+\frac{a^4-b^4}{a^2+ab}.$$

4. Factorise $ab(ab+c^2)+c(a^3+b^3)$.

5. A house cost £750. At what rent must it be let to pay 5 per cent. on the cost after allowing 10 per cent. of the rent for repairs ?

6. OA, OB are two roads at right angles, and OB is 3 miles. If an armoured motor car starts from B towards O at 30 miles an hour and a cyclist from O along OA at 20 miles an hour, when will the cyclist first be within range of a gun on the car if it has a range of 2 miles ? Now work out the same question with a range of 5 miles, and interpret your answer.

D. 12.

1. In the expression $4x^2-7x-2$ substitute y^2-y for x; arrange your result in powers of y, and resolve it into its factors. (C.S.C.)

2. Transpose the equation $x(x+1)(x+2)(x+3)=\frac{105}{16}$ by putting $x=y-\frac{3}{2}$, and thence solve it. (C.S.C.)

3. Solve the equations:

(i) $x^2+x=4840$, giving the roots to the nearest integer.

(ii) $\frac{a}{x}-\frac{b}{y}=0, \quad ax+by=c.$

4. Simplify $\frac{1}{a(a-b)(a-c)}+$ two similar terms.

5. A number, when divided by 7, leaves a remainder 5; prove that the square of this number, when divided by 7, leaves a remainder 4.

6. A pound of tea and three pounds of sugar cost 6s., but if the price of sugar were to rise 50 per cent. and of tea 10 per cent., they would cost 7s. Find the price of tea and of sugar.

D. 13.

1. Solve the equations: $ax+by=c$, $a^2x+b^2y=c^2$.

2. Multiply together $x+a$, $x+b$, $x+c$; and deduce from your result the coefficient of x in the product $(x+2)(x-5)(x-11)$. (C.S.C.)

3. If $x+\frac{1}{x}=p$, prove that $x^3+\frac{1}{x^3}=p^3-3p$.

4. Prove that, if x is very small, the square root of $1+x$ is very nearly $1+\frac{x}{2}$. Use your result to find approximately the square root of $1{\cdot}03$.

5. For what values of x is $(5-x)(x-2)$ positive?

6. A pint of milk weighs a lb., a pint of water b lb., and a pint of a mixture of the two c lb. What fraction (by volume) of the mixture is milk?

If $b=1{\cdot}25$, $a=1{\cdot}288$, and $c=1{\cdot}276$, what fraction is milk? (C.S.C.)

D. 14.

1. If $x-y=a$ and $y-z=b$, what is the value of $z-x$? Hence show that the three equations $x-y=a$, $y-z=b$, $z-x=c$ have either no solution or an infinite number of solutions for x, y, z. (C.S.C.)

2. Find the areas of the rectangles with the following sides: (*a*) 10 in. and 8 in., (*b*) 9·5 in. and 8·5 in., (*c*) 9·1 in. and 8·9 in., (*d*) $9+x$ in. and $9-x$ in. For what value of x is the area of (*d*) greatest? If the perimeter of a rectangle is 36 in., what must the sides be so that it may have the greatest area?

3. Solve the equations:

$$2x^2+5xy=3.$$
$$4x^2-xy=17.$$

4. Prove that the sum of the three fractions

$$\frac{b-c}{1+bc},\quad \frac{c-a}{1+ca},\quad \frac{a-b}{1+ab}$$

is equal to their product.

5. If x, y, z are consecutive multiples of 3, prove that

$$x^2-2y^2+z^2=18.$$

6. Find the time between 4 and 5 o'clock when the hands are in the same direction.

D. 15.

1. Simplify

$$\left(x+\frac{1}{x}\right)^2-\left(y+\frac{1}{y}\right)^2-\left(xy-\frac{1}{xy}\right)\left(\frac{x}{y}-\frac{y}{x}\right). \qquad \text{(C.S.C.)}$$

2. Add a constant to $25x^2-15x$ so as to make it a perfect square. Show that $25x^2-15x$ can never be less than $-2\frac{1}{4}$.

3. Solve the equations:

(i) $y+z-x=4, \quad z+x-y=0, \quad x+y-z=-4.$

(ii) $x^2+y^2=2(a^2+b^2), \quad xy=a^2-b^2.$

4. Factorise $x^2+xy-6y^2+5x+6.$

5. If $x=0·01$ find the value of $\dfrac{3-5x}{1-4x+3x^2}$ to four decimal places (*i.e.* to the nearest ten-thousandth).

Also express $\dfrac{3-5x}{1-4x+3x^2}$ in ascending powers of x as far as the term involving x^4, and verify the above result. (C.S.C.)

6. The rates for first-class season tickets between London and Balcombe are given below. They are in the ratio of 30 : *a* : *b* : *c*. Find *a*, *b*, *c* graphically by plotting £8 5s. and 30 and joining the point to the origin.

Three Months.	Two Months.	One Month.	Fifteen Days.
£8s. 5s. 0d.	£6. 17s. 6d.	£4. 2s. 6d.	£3. 3s. 3d.

D. 16.

1. Solve the equations:

$$\text{(i)}\ \frac{x+1}{y}=7, \quad \frac{x}{1+y}=6.$$

$$\text{(ii)}\ x(1+y)=70, \quad y(1+x)=72. \qquad \text{(C.S.C.)}$$

2. Simplify $$\frac{2+x}{2(1+x)}-\frac{2-x}{2(1-x)}+\frac{x}{1+x^2}.$$

3. Show by means of a graph that $\frac{6x}{6+x}$ increases, as x increases, for positive values of x.

Also prove the same thing by showing that $\frac{1}{y}=\frac{1}{6}+\frac{1}{x}$, where y stands for $\frac{6x}{6+x}$.

4. Find the factors of $(a+b)^2+(c+d)^2-2(a+b)(c+d)$.

5. Show that the product of any two consecutive numbers can be put in the form n^2-n or n^2+n, where n is an even number.

6. If $A=m \,.\, a$, $B=m \,.\, b$, where m is the H.C.F. of A and B, show that

(i) $\frac{A}{B}$, when reduced to its lowest term, is $\frac{a}{b}$.

(ii) The L.C.M. of A and B is $\frac{AB}{m}$.

D. 17.

1. Show that

$$\tfrac{1}{2}\{(x-y)^2+(y-z)^2+(z-x)^2\}=x(x-y)+y(y-z)+z(z-x),$$

and find what either expression becomes when multiplied by $x+y+z$.

2. Find a square root of

$$x^4+12x^3+22x^2-84x+49.$$

3. Find one value of x for which $(x-1)^3+(x-3)^3$ is zero.

4. Simplify $$\left(\frac{x+\frac{1}{4}}{x-\frac{1}{2}}-\frac{x+\frac{1}{2}}{x-\frac{1}{4}}\right)\div\frac{1\frac{1}{2}}{4-\frac{1}{x}}.$$

5. An elastic string AB of natural length a inches is fastened at the end A and is stretched so that its length becomes b inches; through what distance does the mid point of the string move? Find also the distance through which a point of the string, originally c inches from A, moves.

6. The base of a segment of a circle of radius r is $2a$ and its height is h. Show that $2hr=h^2+a^2$.

If a segment of a circle has a base 10″ long and is 3″ high, find the radius of the circle.

D. 18.

1. Solve the equations :

$$\text{(i)}\ x(1+y)+3=0=y(1+x)+8.$$

$$\text{(ii)}\ \frac{a}{x}+\frac{x}{a}=2\tfrac{1}{2}.$$

2. What is the value of a, if the coefficient of x in the product $(2x+3)(x-2)(x+a)$ is zero ?

3. Prove that

$$(8x+9y)^2+(3x+2y)^2-(2x+3y)^2-(3x+4y)^2$$
$$\equiv 2(3x+4y)(2x+3y)+2(3x+2y)(8x+9y).$$

4. If $V=\frac{1}{3}\pi r^2h$, $S=\pi rl$, $l^2=r^2+h^2$, prove that

$$S^2=\frac{9V^2}{h^2}+3\pi Vh.$$

5. The maker of an article sells it to A at a gain of p per cent. ; A sells it to B for k pounds at a gain of q per cent. What is the cost of making it ? (C.S.C.)

6. If c, d are the inside diameters of the ends of a cask and b the inside diameter at the bung and l its length, all in inches, the capacity of the cask is approximately g gallons, where

$$g=\frac{\sqrt{2}}{1000}\,l(b^2+cd).$$

(i) Make b the subject of this formula ; (ii) how is the volume affected if the length remains unaltered, but if the diameter of every circular section, perpendicular to the length, is doubled ?

D. 19.

1. Divide x^4+4y^4 by $x^2-2xy+2y^2$.

2. Prove that

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}+\frac{n(n-1)(n-2)}{1.2.3}=\frac{(n+1)n(n-1)(n-2)}{1.2.3.4}.$$

3. Solve the equations :

$$xy-x^2=2.$$
$$xy-y^2=-3.$$

4. If $x=y+1$, prove that

$$x^2-2xy+y^2=x-y \quad \text{and} \quad x^2-x=y^2+y.$$

5. Before the war, I could get £1 more for 275 francs than I can now. For £1, I can now get $2\frac{1}{2}$ francs more than formerly. What is the present rate of exchange, in francs per £ ?

6. Calculate the values of $x(9-x)^2$ for x equal to 0, 1, 2. 3, 4, 5, 6, 7, 8, 9. Draw a curve showing the values of $x(9-x)^2$ from $x=0$ to $x=9$.

If a very thin rod, 9 in. long, swings like a pendulum, the expression $x(9-x)^2$ measures the tendency to break at a place x in. from the point of suspension. From your graph find where the rod is most likely to break. (C.S.C.)

D. 20.

1. The following formula is taken from a book on magnetism:

$$m = \frac{r^2h}{1 - \dfrac{r^3}{\sqrt{(r^2+d^2)^3}}}.$$

Make d the subject of this formula.

2. Solve the equations:

$$x^3 + \frac{x}{y} = 222.$$

$$x^2 + \frac{1}{y} = 37.$$

3. Find the factors of

(i) $3x^2 - 48$.

(ii) $(x-1)^2 - (1-x)(3-5x)$.

4. Prove that

(i) $(3x-4)^2 + (3x-5)^2 - 2(3x-4)(3x-5) \equiv 1$.

(ii) $x^3 - 11x = 2(6-x^2)$ if $x = -1$, and find two other values of x for which these expressions are equal.

5. Two boys A, B start together to run from school to a shop, 560 yards away, and back. A returning meets B 40 yards from the shop and gets back to school one minute before him. How long was he absent?

6. Prove that $\left(\frac{a+b}{2}\right)^2 > ab$, unless a is equal to b.

D. 21.

1. Find the values of the constants a and b so that the relation $y = ax^2 - bx$ may be satisfied by $x=1$, $y=-2$, and by $x=6$, $y=3$. Plot the graph of this function of x from -3 to $+8$, and determine approximately the values of x for which $x^2 - 5x = 10$. Check by calculation.

2. If $\frac{1}{u} + \frac{1}{v} = \frac{2}{q}$ and $u - p = r = v - q$, find a relation between p, q and r, and find the value of r when $p = 3{\cdot}5$ and $q = -4$. (C.S.C.)

3. Find the coefficient of x^3 in

$$(1+x+x^2+x^3+x^4)(1-4x+6x^2-10x^3+6x^4-4x^5).$$

4. Find numerical values of A and B such that

$$(x-1)(2x+3) \equiv (Ax+B)(1-x).$$

5. $ABCD$ is a quadrilateral such that $AB+CD=BC+AD$; P is any point on AB; a circle, centre A, radius AP, cuts AD at Q; a circle, centre B, radius BP, cuts BC at R; prove that the circles, centre D, radius DQ, and centre C, radius CR, will cut CD at the same point.

6. A beam of length L feet, supported at each end, is just strong enough to carry a load of M tons placed at a point which is x feet from one end and y feet from the other end. M is found from the formula, $M=\dfrac{0{\cdot}36L^2}{4xy}$. (i) Find the greatest load it can carry at the centre of the beam. (ii) Prove that $x=\dfrac{L}{2}\left[1 \pm \sqrt{1-\dfrac{9}{25M}}\right]$. (iii) If the beam is 10 feet long, find how near the centre a load of 1 ton can be placed.

D. 22.

1. Show that the Interest on £P, at r per cent. Compound Interest, for 3 years, is $P\left(1+\dfrac{r}{100}\right)^3-\text{P}$ pounds.

2. If $2s=a+b+c+d$, and if $a+c=b+d$, prove that

(i) $(s-a)(s-b)(s-c)(s-d)=abcd.$

(ii) $(s-a)^2-(s-b)^2+(s-c)^2-(s-d)^2=2(bd-ac).$

3. Solve the equations:

(i) $(x^2-2x)^2-11(x^2-2x)+24=0.$

(ii) $2x+\dfrac{1}{y+1}=14, \qquad 3x-\dfrac{2}{y+1}=7.$

4. If $x=1+\dfrac{a}{y}$, $y=1+\dfrac{b}{z}$, $z=1+\dfrac{c}{d}$, prove that

$$x=\frac{(1+a)(c+d)+bd}{(1+b)d+c};$$

and if also $d=1+x$ and $a=b=c=2$, prove that $x^2=\dfrac{11}{3}$. (C.S.C.)

5. A ruler graduated in cms. and one graduated in inches are placed side by side; the readings 10, x, $2x+1$ on one correspond to the readings 3, 5, y on the other. Find y in terms of x.

6. Find the side of a square inscribed in a semicircle of radius 10 cms., two corners of the square being on the diameter and the other two on the arc of the semicircle.

D. 23.

1. Prove that

$$16(x-2)(x-4)(x-6) = (x-1)(x-3)(x-5)(x-7)\left\{\frac{5}{x-1}+\frac{3}{x-3}+\frac{3}{x-5}+\frac{5}{x-7}\right\}.$$

2. A runs x yards in y seconds ; B runs 100 yards in z seconds more than A ; how long does B take to run 120 yards ?

3. If $x+y+z=0=ax+by+cz$, prove that
$(x^2+y^2+z^2)(bc+ca+ab-a^2-b^2-c^2)=2(bx+cy+az)(cx+ay+bz)$.

4. Find an integral value of x between 0 and 4 for which the expression x^3-x^2-x-2 vanishes. Factorise this expression.

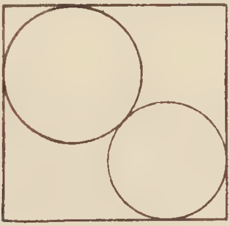


FIG. 28.

5. The sides of a rectangle are in the ratio 8 : 9, and two circular discs of radii 4″, 6″ are fitted into it, as in the figure. Find the length of the rectangle.

6. A balance is badly made, so that a body really weighs $a+bQ$ lb. when the balance shows Q lb. ; a and b being numbers, the same for all weights. If the balance shows 1 lb. for a real weight of 1·1 lb. and 2 lb. for a real weight of 1·9 lb., what are the values of a and b, and what will the balance show for a real weight of 2 lb. ? (C.S.C.)

D. 24.

1. Simplify

$$\left\{2x+1+\frac{x-1}{x+1}-\frac{x+2}{x}\right\}\left\{\frac{1}{2x+1}+\frac{x+1}{x-1}-\frac{x}{x+2}\right\}.$$

2. Substitute $\frac{a}{a-1}$ for x and $\frac{a+1}{a}$ for y in $\frac{xy}{\frac{1}{x}-\frac{1}{y}}$ and reduce the result to its simplest form.

3. Solve the equations :

$$\text{(i)}\quad \frac{1}{x}-\frac{2}{y}=\frac{2}{x+1}-\frac{1}{y+1}=3\tfrac{1}{3}.$$

$$\text{(ii)}\quad \frac{a}{x-2a}=\frac{3a}{x-12a}.$$

4. Write down the constant term in the expansion of

$$\left(2x^2-3x-1-\frac{4}{x}+\frac{1}{x^2}\right)\left(3x^2+x+2-\frac{1}{x}-\frac{1}{x^2}\right).$$

5. Express $x^3-6x^2+12x-8$ in terms of y if $x=y+2$.
Hence show that x^3+12x is greater or less than $6x^2+8$ according as x is greater or less than 2.

6. The figure is a graph of $a+bx$, where a and b are numbers. Find them.

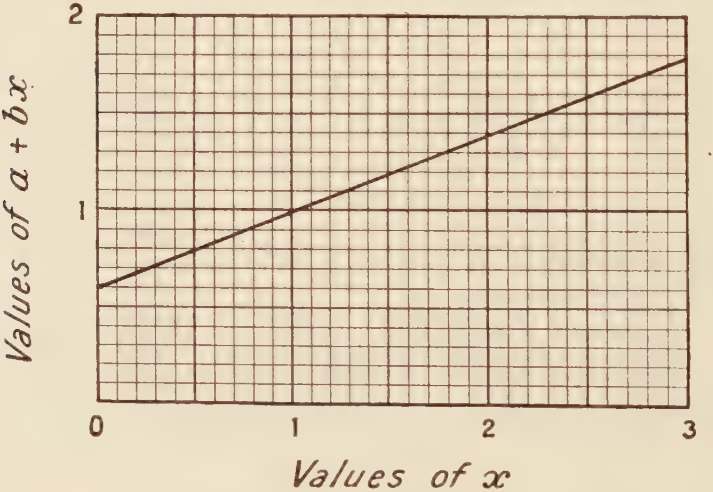


FIG. 29.

D. 25.

1. Prove that the product of $6x^2+x-2$ and $4x^2-11x-3$ is equal to the product of $8x^2-2x-1$ and $3x^2-7x-6$.

2. Find the greatest value of xy, if x and y satisfy the equation $3x+2y=35$.

3. If the expression px^2+qx+r takes the values $\frac{1}{c}, \frac{1}{c+1}, \frac{1}{c+2}$ when x takes the values 0, 1, 2, find the values of p, q, r and prove that when $x=c+2$ the expression is equal to $\frac{1}{c+1}$.

4. Solve the equations:

$$\text{(i)}\quad \frac{x}{a^2}=\frac{y}{b^2}=\frac{z}{c^2}=xyz.$$

$$\text{(ii)}\quad a(x^2+1)=x(a^2+1).$$

5. If $(x^2+ax+b)^2 \equiv x^4+2ax^3+px^2+qx+r$, prove that

$$\text{(i)}\ p=a^2+2b. \qquad \text{(ii)}\ (4pr-q^2)^2=64r^3.$$

6. B is k miles due east of A; A walks a miles an hour due south, and B walks b miles an hour due west. Find their distance apart after x hours; and prove that their distances apart after $\left(\frac{bk}{a^2+b^2}+t\right)$ hours and $\left(\frac{bk}{a^2+b^2}-t\right)$ hours are equal.

D. 26.

1. Prove that the addition of the term $(p+1)^3$ to the expression $\frac{1}{4}p^2(p+1)^2$ has the same effect as substituting $p+1$ for p in this expression.

2. Simplify

$$\frac{x-\dfrac{x+y}{xy+1}}{y+\dfrac{x-y}{1-xy}} \times \frac{\dfrac{1}{x}-\dfrac{x+y}{xy-1}}{\dfrac{1}{y}-\dfrac{x-y}{1+xy}} \times \frac{1+\dfrac{1}{y^2}}{1+\dfrac{1}{x^2}}.$$

3. Find y from the equations

$$y^2=4ax, \quad y+mx=2am+am^3.$$

4. Find the equations whose roots are

(i) 1, 0·1. (ii) $a-1$, $1-a$. (iii) 0, -1, 2.

5. Two trains start at the same time to run from X to Y and Y to X respectively; they meet after p hours, and the first train arrives at Y q hours later; assuming that they travel at uniform rates throughout, how long does the second train take over the whole journey?

6. Prove that $(y-z)^3+(z-x)^3+(x-y)^3$ is equal to zero when y is equal to z. Hence prove that $y-z$ is a factor of the expression, and find the remaining factors. (C.S.C.)

D. 27.

1. Resolve into factors

(i) $3(x^3+y^3)-xy(1+9xy)$.

(ii) $2a^2+a(c-3b)+b^2-bc$. (C.S.C.)

2. If x lies between 95 and 105 and y lies between 110 and 130, find the limits within which (i) $x+y$, (ii) $x-y$, (iii) $\frac{x}{y}$ must lie.

3. Solve the equation:

$$x^2-ax=\frac{a^2}{4b}(4-b).$$

Also find the value of the positive root, to the nearest hundredth, when $a=2{\cdot}3$, and $b=1{\cdot}75$. (C.S.C.)

4. What does $\frac{n(n+1)}{2}$ become if you put $n+1$ for n in it? Show that you would get the same quantity by adding $n+1$ to $\frac{n(n+1)}{2}$.

5. Solve the equations:

$$\text{(i)}\quad x^2-2y=y^2-2x=3.$$

$$\text{(ii)}\quad x^2y=32,\ yz=-6,\ zx=-12.$$

(C.S.C.)

6. AB is the diameter of a circle, centre O; PN is the perpendicular from a point P on the circumference to AB: the tangent at P meets AB produced in T. If $PN=5''$, $NB=3''$, calculate OB and BT.

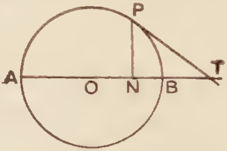


FIG. 30.

D. 28.

1. A man's pace is taken as a inches; in this estimate there may be a small error of e inches. Prove that the number of paces he takes to go x yards lies between two limits which differ by about $\frac{72ex}{a^2}$.

2. Express the relation $y-b_1=\frac{b_2-b_1}{a_2-a_1}(x-a_1)$ in the form $Ax+By+C=0$, where A, B, C are independent of x and y.

3. Solve

$$\text{(i)}\quad \frac{x+7}{x+6}+\frac{4x}{4x-1}=\frac{2x+5}{x+2}.$$

$$\text{(ii)}\quad (x-y)^2-48=8(y-x);\ xy=45.$$

4. If $y=x+\frac{2}{x}$ and $z=x+\frac{1}{x}$, prove that $3yz=1+y^2+2z^2$.

5. Multiply out

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

Show that when $a^2=b^2+c^2$ the given expression is equal to $4b^2c^2$. (C.S.C.)

6. A room is a feet long and the end walls are b feet wide; two points P, Q are marked, one on each end wall, at distances x feet, y feet from the ceiling, and each is equidistant from the side walls; if the shortest distance from P to Q *via* a side wall is equal to the shortest distance *via* the ceiling (the broken line being on the wall or ceiling throughout), prove that $(2x+a)(2y+a)=(a+b)^2$.

D. 29.

1. (i) Find the L.C.M. of x^3+y^3, $x^2+2xy+y^2$ and x^2y+xy^2.

(ii) Find the factors of $(ab+c)^2-c(a+b)^2$.

2. Solve the equations:

$$\text{(i)}\ xy-5x+3y=15,\ x^2+y=54.$$

$$\text{(ii)}\ (x-a)(x+c)=b(a+b+c).$$

3. Simplify

$$(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b),$$

and factorise the result.

4. Obtain the square of $1-2x+3x^2-4x^3+3x^4-2x^5+x^6$ in ascending powers of x as far as x^3, and also in descending powers as far as x^9. (C.S.C.)

5. If x is equal to the product of two numbers a, b divided by half their sum, prove that

$$\frac{1}{x-a}+\frac{1}{x-b}=\frac{2}{x}. \qquad \text{(C.S.C.)}$$

6. In the diagram OA, OB are two *fixed* perpendicular lines; $OA=a$ inches, $OB=b$ inches, two variable circular arcs touch OA, OB at A, B respectively and touch each other; if P is the point of contact and if x inches, y inches are the lengths of the radii of the arcs AP, BP, prove that $(x+a)(y+b)$ is constant.

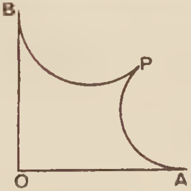


FIG. 31.

If $a=8$, $b=6$, $x=4$, calculate y.

D. 30.

1. Divide $a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc$ by $bc+ca+ab$.

2. Simplify

(i) $\left(\dfrac{a^3-b^3}{a^2+ab+b^2}+\dfrac{a^3+b^3}{a^2-ab+b^2}\right)\div\left(\dfrac{a^2-b^2}{a+b}+\dfrac{a^2-b^2}{a-b}\right)$.

(ii) $\sqrt{x^4+\dfrac{1}{x^4}+2}-\sqrt{x^4+\dfrac{1}{x^4}-2}$.

3. Reduce to its simplest form the equation

$$\frac{x-am^2}{a(m^2-n^2)}=\frac{y-2am}{2am-2an},$$

and then find x, y from this equation and the equation $y^2=4ax$.

4. If $a^2x+b^2y=c^2$ and $bx+ay=d$, express a^5x+b^5y in terms of a, b, c, d in as simple a form as possible.

5. If D is the mid point of the base BC of the triangle ABC, it may be assumed that $AB^2+AC^2=2AD^2+2BD^2$. If $AB=2x$, $BC=2x+2$, $CA=2x+1$, $AD=x+4$, find the perimeter of the triangle.

6. A solid bounded by plane faces has C corners, E edges, F faces. Arrange in tabular form the values of C, E, F for (i) a cube, (ii) a pyramid on a square base, (iii) a triangular prism, (iv) a solid shaped like a barn whose roof is formed by four sloping parts, (v) a solid shaped like an H. By examining your table discover the simple formula which connects C, E, F. Verify it by taking (1) a solid shaped like an L, (2) a solid obtained by slicing off one corner of a rectangular block.

D. 31.

1. For what value of λ does the following expression in x contain no constant term: $ax^2+bx+c+\lambda(a'x^2+b'x+c')$?

2. A water bottle full of water holds m c.c.; an ink bottle full of ink holds n c.c. Their contents are mixed and the ink bottle is filled with the mixture. How much of the ink is in it?

3. The following table gives the time required for seasoning oak of various cross sections:

Cross section -	8	12	16	20	24 sq. in.
Time - -	6	10	14	18	22 months.

Find a formula for timber of cross section $4n$ sq. in.

4. The distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$. Find the distance between the points

$$\left(am, \frac{a}{m}\right) \text{ and } \left(-\frac{a}{m}, -am\right).$$

5. The hot water tap of a bath will fill it in a minutes, the cold water tap in b minutes. In how many minutes will it be filled if both are on ? Find the time if $a=15$, $b=10$.

6. Plot the graph of $\frac{1}{2}x+\frac{2}{x}$ for values of x between 0·3 and 10. Employ it to find the roots of the quadratic $\frac{1}{2}x+\frac{2}{x}=3{\cdot}7$. Also find, from the graph, the values of m for which the quadratic $\frac{1}{2}x+\frac{2}{x}=m$ has equal roots. (C.S.C.)

D. 32.

1. At sea-level water boils at 212° F. At a height H ft. above sea-level the boiling point is lowered B degrees below 212°, where B and H are connected by the approximate formula $H=520B+B^2$. At what temperature will water boil at the top of a mountain. 3500 ft. high ? (C.S.C.)

2. The area of the triangle formed by the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)].$$

Show that it can also be written

$$\tfrac{1}{2}[y_1(x_3-x_2)+y_2(x_1-x_3)+y_3(x_2-x_1)].$$

3. Solve the equations :

$$ax+by=a^2-b^2.$$
$$a(x+y)+b(x-y)=a^2+b^2.$$

4. Plot values of $x+\frac{1}{x}$ for values of x between 0 and 5. What is its value when (i) $x=0{\cdot}02$, (ii) $x=0{\cdot}0001$? Why cannot you say what its value is when $x=0$? Why cannot you say what its greatest value is ? Sketch roughly the graph of $x+\frac{1}{x}$ for values of x between 0 and -5.

5. A squadron is steaming due N. at 12 knots. A light cruiser leaves the flagship, steams due W. at 30 knots and returns to the flagship at the same speed. If it is away 50 min., how far did it go, before changing its course ?

1 knot = 1 sea mile per hour.

6. Plot values of $\frac{10}{1+x^2}$ for positive values of x. Find a point for which $\frac{10}{1+x^2}=x$. Show that this point gives a value of x that satisfies $x^3+x-10=0$.

D. 33.

1. If $y=b\,.\,\frac{x+a}{a-x}$, find $x+y$ in terms of x, and also in terms of y.

2. If incomes are taxed at the rate of x pence in the pound, but y pounds out of the total are exempt from the tax, how much does a man pay whose income is m pounds ? To what rate in the pound would this be equivalent if his whole income were taxed ?

Find x and y if the tax is equivalent to 1s. 2d. in the pound on the whole of an income of £3000 and to 1s. 1d. in the pound on £2625 ; and find, to the nearest pound, the incomes on which the proposed tax will then be equivalent to 1s., to 11d., and to 10d. respectively in the pound on the whole income. (C.S.C.)

3. Solve $(x-a)^2(x+4a)=(x+2a)^2(x-2a)$.

4. Prove that

$$\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\equiv\frac{-(b-c)^2-(c-a)^2-(a-b)^2}{2(b-c)(c-a)(a-b)}.$$

5. Show that $2(x^2+1)^2-(x^2-2x-1)^2$ is a perfect square.

6. A cubic inch of aluminium weighs 0·092 lb. and a cubic inch of copper 0·31 lb. Find the percentage composition by weight of a mixture of copper and aluminium of which a cubic inch weighs 0·276 lb. (C.S.C.)

D. 34.

1. Simplify $$\frac{a}{a+b}-\frac{a+b}{2b}+\frac{a^2+b^2}{2b(a-b)}.$$

2. The area of the triangle formed by the three points (x_1, y_1), (x_2, y_2), (x_3, y_3), is

$$\tfrac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2].$$

Show that it is equal to the area formed by the three points

$$(x_1+h, y_1),\quad (x_2+h, y_2),\quad (x_3+h, y_3).$$

3. Solve the equations : $lx+my=1.$
$mx+ly=1.$

4. Prove that $\frac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}=0$, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}$. Show also that it is equal to 0, if $\frac{a_1}{b_1}=\frac{a_2}{b_2}$.

5. Find a square root of

$$9x^2 - 12x - 2 + \frac{4}{x} + \frac{1}{x^2}.$$

6. In order to reach a certain place a railway transports passengers x miles by train and y miles by a service of motor cars. The train costs the company 34·43d. per mile and the provision of the necessary motor cars costs 26·20d. for each mile of the remaining distance. If 32 tickets for the journey are taken, at a through rate of one penny a mile, the receipts are £11 1s. 4d., and the railway just clears expenses. Find x and y correct to the nearest mile. (C.S.C.)

D. 35.

1. The volume of a solid cylinder of length l cm. and radius r cm. is $\pi r^2 l$ c.c. Find a formula for the volume of a glass tube of length l cm., outside radius r cm. and thickness h cm.

2. Find the condition that the values of x and y obtained from the equations

$$a_1x + b_1y = 1$$

and

$$a_2x + b_2y = 1$$

should satisfy the equation $a_3x + b_3y = 1$.

3. If $x = a^2 - bc$, $y = b^2 - ca$, $z = c^2 - ab$, show that

$$(x + y + z)(a + b + c) = ax + by + cz.$$

4. Solve the equations:

$$x^2 + y^2 + 2x + y = 5.$$
$$x^2 + y^2 + x + 2y = 5.$$

5. If the speed of an aeroplane is v miles an hour, what is its average speed from A to B and back, if the wind is blowing from A to B at x miles an hour all the time? Show that its average speed is greatest when there is no wind.

6. A plays a billiard match of 1000 points against B and gives him 250 points. At a certain stage in the game B is 124 points ahead, and if A continues to gain at the same rate he will lose the game by 40 points. What are their scores?

EXTRA PRACTICE EXERCISES.

Note.—The following "drill" exercises (taken from *Durell's Practical School Algebra*), may be used either to give extra practice at the first reading or in a revision course. They are grouped by chapters, as follows:

CHAPTER.		
I.	E.P. 1,	Generalisations.
	E.P. 2,	Substitution.
	E.P. 3,	Substitution in Formulae.
II.	E.P. 4,	Like and Unlike Terms.
	E.P. 5,	Products and Quotients.
	E.P. 6,	H.C.F., L.C.M., Fractions.
	E.P. 7,	Use of Brackets.
	E.P. 8,	Removal of Brackets.
III.	E.P. 9,	Easy Problems in Generalised Arithmetic.
	E.P. 10,	Harder Problems in Generalised Arithmetic.
V.	E.P. 11,	Positive and Negative Numbers.
VII.	E.P. 12,	Simultaneous Equations and Problems.
VIII.	E.P. 13,	Fractions and Equations.
	E.P. 14,	Factors.
	E.P. 15,	Fractions.
X.	E.P. 16,	Quadratics with rational Roots.
	E.P. 17,	The general Quadratic.
	E.P. 18,	Graphs of Quadratic Functions.
XI.	E.P. 19,	Simultaneous Quadratics.
	E.P. 20,	Literal Equations.
	E.P. 21,	Miscellaneous Examples.

E.P. 1.

Generalisations.

Give general statements by means of letters, to include the statements in each of the following examples :

1. $\frac{3}{3}=1$; $\frac{5}{5}=1$; $\frac{12}{12}=1$.

2. $4+4+4=3\times 4$; $5+5+5=3\times 5$; $12+12+12=3\times 12$.

3. $\frac{5}{2\times 5}=\frac{1}{2}$; $\frac{8}{2\times 8}=\frac{1}{2}$; $\frac{11}{2\times 11}=\frac{1}{2}$.

4. $\frac{2}{3}\times 3=2$; $\frac{4}{7}\times 7=4$; $\frac{12}{5}\times 5=12$.

5. $7\times 0=0$; $4\times 0=0$; $10\times 0=0$.

6. $5\times 5\times 5=5^3$; $7\times 7\times 7=7^3$; $8\times 8\times 8=8^3$.

7. $\frac{3}{4}\times\frac{4}{3}=1$; $\frac{8}{11}\times\frac{11}{8}=1$; $\frac{7}{5}\times\frac{5}{7}=1$.

8. $1+\frac{2}{3}=\frac{3+2}{3}$; $1+\frac{7}{11}=\frac{11+7}{11}$; $1+\frac{5}{14}=\frac{14+5}{14}$.

9. $0\div 3=0$; $0\div 9=0$; $0\div 14=0$.

10. $1\div 3=\frac{1}{3}$; $1\div 7=\frac{1}{7}$; $1\div 12=\frac{1}{12}$.

11. $1\div\frac{3}{4}=\frac{4}{3}$; $1\div\frac{8}{11}=\frac{11}{8}$; $1\div\frac{13}{9}=\frac{9}{13}$.

12. $5^3\div 5=5^2$; $8^3\div 8=8^2$; $10^3\div 10=10^2$.

13. The cost of 7 lb. of sugar at 3d. per lb. is 3×7 pence. The cost of 7 lb. of sugar at $3\frac{1}{2}$d. per lb. is $3\frac{1}{2}\times 7$ pence.

14. If golf balls cost 2s. each, 4 balls cost 2×4 shillings, 7 balls cost 2×7 shillings.

15. In £3 there are 3×20 shillings ; in £7 there are 7×20 shillings.

16. There are 20 chairs in a room. When 8 people are seated, $20-8$ chairs are empty. When 13 people are seated, $20-13$ chairs are empty.

17. A man is 27 years old ; in 5 years his age will be $27+5$, in 8 years his age will be $27+8$.

18. Seven inches $=\frac{7}{12}$ feet ; 15 inches $=\frac{15}{12}$ feet.

19. The number of $1\frac{1}{2}$d. stamps sold for 12 pence is $\frac{12}{1\frac{1}{2}}$, the number sold for 30 pence is $\frac{30}{1\frac{1}{2}}$.

20. A tin contains 100 cigarettes. If 20 are used, $100-20$ remain ; if 35 are used, $100-35$ remain.

21. I have to make a journey of 45 miles ; after going 12 mi., I still have to go 45 – 12 miles more ; after going 30 mi., I still have to go 45 – 30 miles more.

22. One brick weighs 2 lb., 7 bricks weigh 2 × 7 lb.

23. 1 cwt. $=\frac{1}{20}$ ton, 9 cwt. $=\frac{9}{20}$ ton.

24. There are 50 questions in an exercise ; when 10 have been done, 40 remain ; when 20 have been done, 30 remain.

25. I walk 100 yards a minute ; in 7 minutes, I walk 700 yards.

In the following questions, the generalisation should contain TWO *unknowns :*

26. If a boy bicycles 9 miles an hour, he goes 9 × 2 miles in 2 hours. If a boy bicycles 10 miles an hour, he goes $10 \times 1\frac{1}{2}$ miles in $1\frac{1}{2}$ hours.

27. If an empty coal scuttle weighs 3 lb. and if the scuttle, when full, weighs 14 lb., the coal in it weighs 14 – 3 lb.

28. From 10 a.m. to 5 p.m. is $(12 - 10) + 5$ hours.

29. A man saves £40 a year. After 3 years he has saved £40 × 3.

30. If a boy scores 55 runs in 5 completed innings, his average is $\frac{55}{5}$ runs.

31. A man buys a horse for £70 and sells it for £90. His gain is £20.

32. A soldier's stride is 30 inches ; in 100 yards he takes $\frac{100 \times 36}{30}$ paces.

33. A hotel contains 200 beds. If 150 are occupied, 50 are empty.

34. If a clock loses 5 seconds each hour, it loses 5 × 24 × 7 seconds in 7 days.

35. A man is 40 years old when his son is 10 years old. When the son is 15, the father is 45.

36. If a car uses a gallon of petrol every 24 miles, it requires $\frac{36}{24}$ gallons for a journey of 36 miles.

37. When I have read 12 chapters of a book, there still remain 18 chapters and therefore the book contains 30 chapters.

38. If a railway fare is 18 pence, 4 whole tickets and 5 half-tickets cost $(18 \times 4 + 18 \times \frac{5}{2})$ pence.

39. I earn £300 a year and spend £20 a month. I therefore save £60 a year.

40. I burn half a ton of coal a week. Therefore 7 tons of coal last me 14 weeks.

E.P. 2.

Substitution.

1. If $a=3$, $b=5$, what are the values of

(i) ab; (ii) $a+b$; (iii) $2a-b$; (iv) ba^2;
(v) $2a^2$; (vi) $\frac{2a}{b}$; (vii) $(b-a)$; (viii) $1+ba$?

2. If $u=3$, $s=1$, $t=0$, what are the values of

(i) us; (ii) us^2; (iii) uts; (iv) $u^2+s^2+t^2$;
(v) $2u-3s$; (vi) $us+st$; (vii) $3t$; (viii) $4s^3-3ut$?

3. If $x=6$, $y=0$, $z=3$, what are the values of

(i) $x+y+z$; (ii) $2x-3y-2z$; (iii) $xy+yz$;
(iv) $\frac{x}{z}$; (v) $\frac{1}{z}-\frac{1}{x}$; (vi) xz^2;
(vii) $xz-2xy$; (viii) $\frac{yz}{x}$?

4. If $N=4$, $n=\frac{1}{2}$, what are the values of

(i) Nn; (ii) $\frac{1}{n}$; (iii) Nn^2; (iv) $N-5n$;
(v) $\frac{N}{n}$; (vi) $\frac{n}{N}$; (vii) $2n^2$; (viii) $\frac{1}{2}N^2$?

5. If $p=4$, $q=9$, $r=0$, what are the values of

(i) $\sqrt{p}$; (ii) $\sqrt{q}$; (iii) $\sqrt{(pq)}$; (iv) pqr;
(v) q^2-pr; (vi) $6p-2q+3r$; (vii) $\frac{q-1}{p}$; (viii) $\frac{r}{q}$?

6. If $m=1$, $n=2$, $t=3$, what are the values of

(i) m^3; (ii) mnt; (iii) $m+n-t$; (iv) $t^2-m^2-n^2$;
(v) n^2-mt; (iv) n^3; (vii) $\frac{m+n}{t}$; (viii) $\frac{nt}{m}$?

7. If $e=3$, $f=5$, $g=0$, what are the values of

(i) $4ef$; (ii) $2f^2$; (iii) $3fg+ge$; (iv) $5e^2-6f$;
(v) $3e^2-f^2$; (vi) fge^2; (vii) e^3+2g^2; (viii) $\frac{g}{ef}$.

8. If $a=b^2-b$ and $b=5$, what is $\frac{a}{b}$?

9. If $x=2y$ and $y=3z$ and $z=4$, what is x?

10. If $2m^2+n^2=p$ and if $m=0$ and $n=1$, what is p?

11. If $x=2\frac{1}{2}$ and $y=1\frac{2}{3}$, what is $\frac{1}{x}+\frac{1}{y}$?

12. If $u=\frac{2}{v}$ and $v=\frac{3}{4}$, what is (i) uv, (ii) $\frac{u}{v}$?

13. If $x=\frac{1}{2}$, $y=\frac{2}{3}$, $z=\frac{3}{4}$, what are the values of

(i) x^2; (ii) $2x$; (iii) xy; (iv) xyz; (v) $\frac{1}{x}$;

(vi) $6y$; (vii) x^2y; (viii) yz; (ix) $4z^2$; (x) $\frac{x}{y}$;

(xi) y^2; (xii) $\frac{z}{x}$; (xiii) $\frac{x^2}{z}$; (xiv) $6xz$; (xv) $3y^2$?

14. If $a=1\frac{1}{2}$, $b=\frac{2}{5}$, $c=5$, what are the values of

(i) $4a$; (ii) $c-2a$; (iii) bc; (iv) $a-3b$;

(v) $2ac$; (vi) a^2b^2; (vii) $\frac{6}{a}$; (viii) $10ab$;

(ix) $3b+c$; (x) $3abc$; (xi) $\frac{1}{b}$; (xii) $b-\frac{2}{c}$;

(xiii) $\frac{1}{2}bc$; (xiv) $\frac{c+1}{a}$; (xv) cb^2?

15. If $r=6$, $s=3$, $t=0$, what are the values of

(i) $\frac{st}{r}$; (ii) $2s-rt$; (iii) $\frac{2s}{r}$; (iv) $\frac{12}{rs}$;

(v) r^2-2s^2; (vi) $\frac{s+1}{r}$; (vii) s^3-3r; (viii) $\frac{r-s}{r+s}$?

E.P. 3.

Substitution in Formulae.

Deduce two arithmetical results from each of the following statements.

[*Oral.*]

1. The sum of y and $4y$ is $5y$.

2. If $2x$ is subtracted from $3x$, the remainder is x.

3. If $2p$ is multiplied by $3q$, the result is $6pq$.

4. If $2a+2$ is divided by 2, the quotient is $a+1$.

5. The square of $2c$ is $4c^2$.

6. The sum of N and $N+1$ is $2N+1$.

7. If $b-1$ is subtracted from $b+1$, the remainder is 2.

8. If z is an odd number, $z+3$ is an even number.

9. If N is any whole number, $2N+1$ is an odd number.

10. If $R-r$ is subtracted from $R+r$, the remainder is $2r$.

11. If $\frac{a}{b}$ is multiplied by b, the result is a.

12. If $c-d$ is added to d, the sum is c.

13. If $y-z$ is subtracted from y, the remainder is z.

14. $\frac{x}{2x}$ equals $\frac{1}{2}$ (if x is not zero).

15. $\frac{x}{3}+\frac{x}{6}$ equals $\frac{x}{2}$.

16. If 1 is added to $\frac{a}{b}$, the sum is $\frac{a+b}{b}$.

17. If $\frac{y}{z}$ is multiplied by $\frac{z}{y}$, the product is 1.

18. If $\frac{N}{2}$ is subtracted from $\frac{2N}{3}$, the remainder is $\frac{N}{6}$

19. In n florins there are $4n$ sixpences.

20. In £P there are $8P$ half-crowns.

21. If there are x hours of daylight, the night is $24-x$ hours long.

22. If the nth day of June is a Sunday, the $(n+5)$th day of July is also a Sunday.

23. If the pth day of May is a Saturday, the pth day of September is a Wednesday.

24. From t minutes past eleven to t minutes to twelve there are $60-2t$ minutes.

25. The minute hand of a clock turns through $6m$ degrees in m minutes.

[*Written.*]

26. $N+1$ is a factor of N^2-1. [Take $N=4$ and $N=6$.]

27. The product of $p+1$ and $\frac{1}{p}$ is $\frac{1}{p}+1$. [Take $p=2$ and $p=5$.]

28. The square root of $9x^2$ is $3x$. [Take $x=2$ and $x=10$.]

29. If a man walks v miles an hour, a journey of 24 miles takes $\frac{24}{v}$ hours. [Take $v=3$ and $v=4$.]

30. Lace, costing x farthings per inch, costs £$(66x)$ per mile at the same rate. [Take $x=2$ and $x=5$.]

31. If a stone is dropped from the top of a cliff, it falls $16t^2$ feet in t seconds. [Take $t=2$ and $t=\frac{1}{2}$.]

32. If a, b, c are consecutive even numbers, a being the smallest, $c^2-a^2=8b$. [Take $a=4$ and $a=6$.]

33. Repeat No. 32, if a, b, c are consecutive odd numbers. [Take $a=3$ and $a=7$.]

34. The area of the floor of a rectangular room is 240 sq. ft.; if it is l feet long, it is $\frac{240}{l}$ feet wide. [Take $l=20$ and $l=16$.]

35. F° Fahrenheit is the same temperature as $\frac{5}{9}(F-32)$ degrees Centigrade. [Take $F=50$ and $F=95$.]

36. If all the angles of a polygon with n sides are equal, each angle is $2-\frac{4}{n}$ right angles. [Take $n=3$ and $n=4$ and $n=5$.]

37. From the top of a cliff h feet high, it is possible to see $\sqrt{\left(\frac{3h}{2}\right)}$ miles. [Take $h=54$ and $h=96$.]

38. It is possible to draw n straight lines on a sheet of paper, so that the number of points at which they cut one another is $\frac{1}{2}n^2-\frac{1}{2}n$. [Take $n=3$, $n=4$, $n=5$.]

39. If h is the H.C.F. of two whole numbers a and b, and if l is their L.C.M., then $hl=ab$. [Take $a=12$, $b=30$ and $a=15$, $b=21$.]

40. A bath can be filled by one tap in p minutes or by another tap in q minutes. If both taps are used, the time taken is $\frac{pq}{p+q}$ minutes. [Take $p=4$, $q=6$ and $p=3$, $q=9$.]

E.P. 4.

Like Terms.

Give short-hand expressions for the following :

1. $x + 2x$. **2.** $3a - a$. **3.** $3b + 3b$.
4. $5c + 2c$. **5.** $6y - 2y$. **6.** $8z - 4z$.
7. $6t + t$. **8.** $3R - 3R$. **9.** $10v + 20v$.
10. $xy + xy$. **11.** $4a^2 + 4a^2$. **12.** $6ab - 3ab$.
13. $2t^2 - t^2$. **14.** $ab + ba$. **15.** $5t^2 + 5t^2$.
16. $8cd - cd$. **17.** $10c^2 - 5c^2$. **18.** $4pq + 4qp$.
19. $x^2 + 2x^2 + 3x^2$. **20.** $p + 6p - 3p$. **21.** $10R - 5R - 2R$.
22. $5xy - xy - xy$. **23.** $5xy - xy + xy$. **24.** $16cd - 8dc$.

Like and Unlike Terms.

Give short-hand expressions for the following : *If there is no shorter form, say so.*

25. $2u + v + u$. **26.** $2a + 2 + 2a$. **27.** $3x - y - x$.
28. $b + a + b$. **29.** $cd + c + d$. **30.** $2b - a - b$.
31. $10x - 2x - 5y$. **32.** $4y + x + y$. **33.** $3x + 3y - x + y$.
34. $8x + 0 - x$. **35.** $xy - x - y$. **36.** $7ab - ab - 7$.
37. $2p + 2$. **38.** $5p - q + p$. **39.** $6yz - 2y - 3z$.
40. $2z + z + 3z$. **41.** $x^2 + 4x^2 - 5x^2$. **42.** $2a^2b + ba^2$.
43. $7R - 5r$. **44.** $8st - 4st$. **45.** $4a^2 - 4a$.
46. $6A + 3A - 9A$. **47.** $10v^3 - v^3$. **48.** $c + cd$.
49. $2x + 3y - x + y$. **50.** $4 + 4a + 2f - 2$.
51. $A + B + B + A + 1$. **52.** $A^2 - 3AB - 5AB + A^2$.
53. $x^2y + xy^2 + yx^2$. **54.** $2u - v + u + 3v + 3$.
55. $3ab + 2ba + a + b$. **56.** $ab - ac + bc - ba + ca - cb$.
57. $t^2 - 2t + 4 + 2t^2 - 3t - 1$. **58.** $4yz - 3xz + zy - xy$.
59. $x^3 - 5x + 4 - x^3 + 6x - 3$. **60.** $p^2 + q^2 - p^2 + q^2$.
61. $x^2 - 1 - x - 1$. **62.** $2 + 3c^2 + c^4 - c^2 + 1$.
63. $aba + bab$. **64.** $aabb + abab + baab$.

65. $4 - 6t + 2 - 3t$.

66. $10x + 10x^3 - 10x^2 - 10$.

67. $3x + x^3 - x^2 - x$.

68. $4x^2 - 3x + 5 - 2x - 1$.

69. $a^3 + a^2 + 1 + a^2$.

70. $t^2 + 4t^3 - 4 - t^2$.

71. $5x^4 - 3x^3 - 3x^4 + 3x^3$.

71. $4 - 3y^2 - y - 1 - y^2$.

Arrange the following in (i) descending powers, (ii) ascending powers of x.

73. $3x - x^2 + 2x^3 + 1$.

74. $x + 5x^4 - 2x^2 + 3 - x^3$.

75. $6x^2 - 5x + 3 - 3x^2 - x$.

76. $x + 2x^3 + 4 + 5x$.

Write down the following :

77. The coefficients of x and the constant term in No. 73.

78. The coefficient of x^4 and the constant term in No. 74.

79. The term of highest degree in No. 76.

80. The coefficient of x in No. 74.

81. The coefficient of x^2 in No. 76.

82. The coefficient of c^2 and the constant term in No. 62.

83. The constant term in No. 59.

84. The coefficient of a in No. 69.

Write down expressions to represent the following :

85. Subtract a^2 from a^4.

86. Add 1 to x^3.

87. Add $2x^2$ to $2x^4$.

88. Subtract $2a^2$ from $6a^2$.

89. Add x^2y to yx^2.

90. Subtract $x + x^2$ from $x^2 + x^3$.

E.P. 5.

Products and Quotients.

Give short-hand expressions for the following :

1. $4t \times 4t$.

2. $5a \times 5$.

3. $ab \times 2a$.

4. $6c^2 \div c$.

5. $6c^2 \div 6$.

6. $8d^2e \div 4de$.

7. $10x^2 \times 2x$.

8. $10xy \times 2y$.

9. $3 \times 3x^3$.

10. $ab \times ac$.

11. $2ab \times 2c$.

12. $2a^2b^2 \div 2ab$.

13. $2t^2 \times 3t^2$.

14. $4s^2t \div 2s$.

15. $2pq \times qr$.

16. $x^3 \times x^3$.

17. $x^6 \div x^2$.

18. $3x \times 3y$.

19. $p^2 \times p^3$. 20. $2t \times t^2$. 21. $3ab \times 3ab$.
22. $6y^6 \div 2y^2$. 23. $6a^2b \div 2a$. 24. $3ab \times 2a$.
25. $4t^4 \times 2t^2$. 26. $c \times 2c \times 3c$. 27. $8r^2s^2 \div 2rs^2$.
28. $2x^2 \times 3xy^2$. 29. $10p^2 \div p^2$. 30. $8p^4 \div p$.
31. $3ab^2 \times 3a^2b$. 32. $12x^6y^6 \div 3x^2y^2$. 33. $2p \times 3q \times pq$.
34. $c^3 \times c^3 \div c^3$. 35. $z^2 \times z^4 \div z^6$. 36. $2x^2 \times 3xy \times 4y^2$.
37. The square of $4ab^2$. 38. A square root of $4a^8$.
39. The cube of $3x^2$. 40. The square of $3x^3$.
41. A square root of $16x^2y^4$. 42. The cube of $2y^3z^2$.
43. The cube root of $8a^6$. 44. The square of $5pq^2r^3$.
45. $(2a)^2 + (3a)^2$. 46. $(4b)^3 \div (2b)^2$. 47. $(3c)^2 \times 3c^2$.
48. $(p^4)^2 \div p^2$. 49. $(2pq)^3 \times 3q$. 50. $(xy^2)^2 \times 2x^3$.
51. $(3t)^3 - (2t)^3$. 52. $(2v)^2 \times (3v)^3 \div 6v^2$. 53. $(4z^2)^2 \div 4z$.
54. $(3ab^3)^2 \div 3ab$. 55. $x \times (2x)^2 \times (3x)^3$. 56. $a^2 \times (2\,ab)^2$.
57. $t(2t)^3 \div t^2$. 58. $(2p^2)^3 \div (2p^3)^2$. 59. $4x^3y^3z^3 \div 4xyz$.

E.P. 6.

H.C.F. and L.C.M.

Find the following :

1. H.C.F. of $6x$, $3xy$. 2. H.C.F. of a^2b, ab^2.
3. L.C.M. of $2pq$, $6p^2q$. 4. L.C.M. of $3y^2$, $2yz$.
5. H.C.F. of $5a^2b$, $10ab^3$. 6. H.C.F. of $6c^3x^4$, $9c^2x^3$.
7. L.C.M. of $2xy$, $2xyz$. 8. L.C.M. of $4abxy$, $14byz$.

Find the H.C.F. and L.C.M. of the following :

9. $3x^2$, $2x^3$. 10. $5a^2b$, $10b^2$. 11. $6y^3$, $8y^2z$.
12. $4x^3$, $6y^3$. 13. $4pq$, $5r^2$. 14. $8yx$, $10xy$.
15. 12, $10t$. 16. y^6z^2, y^3z. 17. $9ef^2$, $6fg^2$.
18. x^2y, xy^2. 19. $4b^3$, $9c^3$. 20. $12b^2x$, $8aby$.
21. $10x^2$, $12xy$, $2x^3$. 22. $6a^3b^3$, $9a^2bc$, $12a^4b^2c^2$.
23. $2z^2$, $3z^3$, $4z^4$. 24. $4r^2$, $5rs$, $6s^2$.
25. $6x$, $2xy$, $2y^3$, $4x^2$. 26. $2x^2yz$, $3xy^2z$, $4xyz^3$.
27. 8, $10p$, $4p^3$, $3q^2$. 28. $9x^3y$, $6y^3z$, $12x^2y^2z^2$.
29. $10x^3$, $15abx^2$, $20cxy$. 30. $(2ax)^2$, $(6bx)^2$, $(2cx)^2$.

Fractions.

Simplify the following fractions ; if there is no simpler form, say so.

31. $\frac{a^2}{a}$.
32. $\frac{b^3}{b^2}$.
33. $\frac{3c}{3}$.
34. $\frac{4d}{4}$.
35. $\frac{5e}{5e}$.
36. $\frac{p^3}{3p}$.
37. $\frac{r^6}{r^2}$.
38. $\frac{s}{st}$.
39. $\frac{u}{u^2}$.
40. $\frac{v^3}{v^6}$.
41. $\frac{2xy}{2xz}$.
42. $\frac{2xy}{x^2}$.
43. $\frac{z}{z}$.
44. $\frac{a}{ab}$.
45. $\frac{c^2}{2cd}$.
46. $\frac{3ef}{3fe}$.
47. $\frac{4g^4}{h^4}$.
48. $\frac{kl}{k^3l^3}$.
49. $\frac{4m^4n^4}{mn}$.
50. $\frac{p^3}{3qr}$.
51. $\frac{6t^2u^2}{9u^2v^2}$.
52. $\frac{w^2}{w^2x^2}$.
53. $\frac{yz}{z}$.
54. $\frac{a^6b^6}{a^2b^2}$.
55. $\frac{c^6}{3c^2}$.
56. $\frac{(3d)^2}{(2d)^2}$.
57. $\frac{(2a^3b)^3}{a^3b^3}$.
58. $\frac{x^2}{(2x)^2}$.
59. $\frac{p^2}{(pqr)^2}$.
60. $\frac{x^2}{(yz)^2}$.
61. $\frac{1}{a}+\frac{1}{2a}$.
62. $\frac{1}{b}-\frac{1}{3b}$.
63. $\frac{c}{cd}-\frac{1}{d}$.
64. $e-\frac{1}{e}$.
65. $1+\frac{1}{f}$.
66. $gh+\frac{g}{h}$.
67. $\frac{2}{l}-\frac{l}{2}$.
68. $\frac{1}{2mn}+\frac{3}{n^2}$.
69. $\frac{p}{q}-\frac{p^2}{q^2}$.
70. $r-\frac{s}{t}$.
71. $\frac{1}{u}-\frac{v}{uv}$.
72. $\frac{x^3}{6x^6}-\frac{1}{6x^3}$.
73. $\frac{1}{6yz}+\frac{1}{4z^2}$.
74. $\frac{a}{10bc}-\frac{b}{15ac}$.
75. $\frac{3a}{2x^2}+\frac{2b}{3y^2}$.
76. $a\div\frac{a}{c}$.
77. $\frac{1}{b}\div\frac{1}{d}$.
78. $ef\div\frac{1}{e}$.
79. $1\times\frac{p}{q}$.
80. $\frac{r}{s}\div 1$.
81. $\frac{x^2}{y^2}\div\frac{y}{x}$.
82. $\frac{3}{4}xy\div\frac{6}{xz}$.
83. $\frac{ab}{ab}\div ac$.
84. $b^6\div\frac{1}{b^2}$.
85. $\left(\frac{2c}{d}\right)^2\div cd$.
86. $\frac{1}{e}\times\left(\frac{3e}{f}\right)^2$.
87. $3p^3\div\left(\frac{1}{2p}\right)^3$.
88. $\frac{z}{3yz}+\frac{x}{6xy}$.
89. $\left(p\div\frac{1}{p}\right)^2$.
90. $\frac{uv}{vu}+\frac{u^2v^2}{v^2u^2}$.

E.P. 7.

Use of Brackets.

Express *by means of brackets* the following: do *not* simplify them.

1. Subtract $x+2y$ from z.

2. Subtract the square of $x+2y$ from the square of z.

3. Multiply $a-b$ by $3c$.

4. Add to p twice the result of subtracting q from r.

5. Take the average of x, y, z.

6. By how much does $2l$ exceed $m-n$?

7. Subtract from b twice the result of subtracting c from d.

8. Multiply the number which is one greater than N by the number which is one less than n.

9. From the square of $R+r$ subtract the square of $R-r$.

10. Divide $A-B$ by $a-b$.

11. What is the product of three consecutive whole numbers of which N is the greatest?

12. What is the product of three consecutive even numbers of which $2l$ is the least?

13. How many inches are there in $2a-b$ feet?

14. A cask contains N gallons; how much remains when $x-y$ gallons have been used?

15. A cask contains N gallons; $a+b$ pints are used each week. How many pints remain after t weeks?

16. A man walks 40 miles in 3 days; he walks $x-7$ miles in the first 2 days, how far does he walk the third day?

17. A carpet is $x+4$ feet long and $x-4$ feet wide. What is its area?

18. A box full of sugar weighs W lb. and when empty weighs w lb. How much sugar is there in n boxes?

19. A cellar contains n bottles; six dozen of them hold a pint and the rest a quart each. How many pints are there altogether?

20. A man buys glasses at the rate of 6d. each for the first dozen and 5d. each for every additional glass. What is the cost in pence of N glasses, if N is greater than 12?

21. A journey by car takes $3\frac{1}{2}$ hours; for the first t hours the speed is 20 miles per hour, and for the rest of the time it is 30 miles per hour. What is the distance travelled? [Assume that t is less than $3\frac{1}{2}$.]

22. Express £x ys. in shillings and £x ys. zd. in pence.

23. A man is earning $6a+b$ shillings a week. What are his new wages if this is increased by one-quarter?

24. A path 4 feet wide runs all round a grass plot l feet long, b feet broad. What is the area of the path?

E.P. 8.

Removal of Brackets.

Simplify the following:

1. $a+(b-3c)$.
2. $d+(b-d)$.
3. $2f-(f+g)$.
4. $(k+l)-(k-l)$.
5. $p+2q-(3q-p)$.
6. $(r-2s)-s$.
7. $3t+(t-2s)$.
8. $4y-(x-3y)$.
9. $(x-y)+(y-x)$.
10. $x^2-(x^2-y^2)$.
11. $(a-2z)-(z-2a)$.
12. $1+b-(1-b)$.
13. $2(c-d)-(c+d)$.
14. $3(f+h)-2(h-f)$.
15. $5-2(1-k)$.
16. $p-3(r-p)$.
17. $2q+5(a+2q)$.
18. $4(u-v)-3(u+v)$.
19. $x(x+y)-x(x-y)$.
20. $x(y+x)-y(y+x)$.
21. $yz-z(y-z)$.
22. $(x^2-5x)\div x$.
23. $(4a^2-6ab)\div 2a$.
24. $c-c\left(1-\frac{1}{c}\right)$.
25. $4e(e-2f)-3f(e-2f)$.
26. $a^3(a^3+2b^3)-b^3(a^3-2b^3)$.
27. $3(x+y-2z)-3(x-y+2z)$.
28. $x(y-z)+y(z-x)+z(x-y)$.
29. $(6x^2y-8xy^2)\div 2xy$.
30. $x\left(1+\frac{1}{x}\right)-y\left(1+\frac{1}{y}\right)$.
31. $a-[a-(1+a)]$.
32. $2p-3\{4-\overline{p-1}\}$.
33. $3x+y-\{4x-3(x-y)\}$.
34. $2[x-\overline{y-x}]-3(x+y)$.
35. $b-2\{c-4(b-c)\}$.
36. $a[a(a+b)-b(a-b)]$.

Fill in the gaps in the following:

37. $a-2b-2c=a-2(\qquad)$.
38. $p-q+r=p-(\qquad)$.
39. $x^2-xy=x(\qquad)$.
40. $l+m-n=m-(\qquad)$.

41. $2a-2c-3x-3z=2(\quad)-3(\quad)$.

42. $a^2+ab-p^2+pq=a(\quad)-p(\quad)=a(\quad)+p(\quad)$.

43. $x^2+xy-xz=x(\quad)=xy-x(\quad)$.

44. $3l+m-n=4l-(\quad)=3(l+m)-(\quad)$.

45. $1+x+x^2+x^3=1+x+x^2(\quad)=1+x^2+x(\quad)$.

46. Add $x+2(y-z)$ to $x-2(y+z)$.

47. Subtract $a-(b-c)$ from $c-(b-a)$.

48. Multiply $x-(x-y)$ by $y-(y-x)$.

49. What must be added to $p-(q+r)$ to give $p+(q+r)$?

50. What must be subtracted from $2(r-s)+t$ to leave $r-t$?

E.P. 9.

Easy Problems in Generalised Arithmetic.

1. I walk s miles and then ride 4 miles. How many miles have I travelled ?

2. A box weighs 3 lb.; I put into it two parcels, one weighing W lb., the other w lb. What is the total weight ?

3. A milkman sells m gallons of milk out of 10 gall.; how much has he left ?

4. A boy starts with d pence and then spends n pence; how much has he left ?

5. In a bookcase there are $2x$ books on one shelf, 20 books on the next shelf, and $2y$ books on the remaining shelf. How many books are there altogether ?

6. I buy n apples and then buy 2 more; how many do I buy in all ?

7. I buy a bunch of $3t$ bananas and cut off 3 of them; how many are left ?

8. A basket, weighing W lb. when empty, contains n lb. of apples; if c lb. of apples are now sold, what is the weight of the basket with the remaining apples ?

9. It is now a quarter past ten; in how many minutes' time will it be (i) 10.30, (ii) t minutes past ten, (iii) n minutes to 11 ?

10. I can walk 4 miles an hour. How far can I walk in 3 hrs.; $2\frac{1}{2}$ hrs.; b hrs.; $5b$ hrs. ?

11. What is the cost of the following:

(i) 5 lb. of butter at l pence per lb.
(ii) 2 oz. of pepper at x pence per oz.
(iii) m yd. of silk at 12 shillings per yd.
(iv) x gall. of oil at p shillings per gall.
(v) $2R$ ft. of pipe at n pence per ft.
(vi) N dozen books at P pence each book.

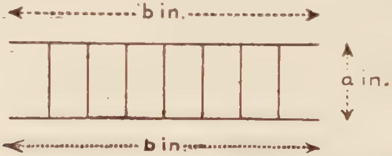
12. How many inches are there in 4 ft., 4 ft. 9 in., l ft., l ft. 9 in., l ft. m in., p ft. q in., v yd. z in.?

13. I spend 5 shillings out of £2. How many shillings are left?

I spend 5 shillings out of £N. How many shillings are left?

I spend f shillings out of £g. How many shillings are left?

14. What is the total length (in inches) of the wire used to make the grid shown here?



Each inch of wire weighs $\frac{1}{5}$ oz.; what is the weight of the grid in ounces?

What is the value of each answer if $a=2$, $b=8$?

15. Eggs are 3 pence each. How much change (in pence) is there out of half a crown, if you buy z eggs?

16. A railway porter is paid b shillings a week and receives in tips n shillings a day. He works 6 days a week, how much does he get each week?

17. A car uses a gallon of petrol every 24 miles. (i) How far will the car run on $2\frac{1}{2}$ gall., $\frac{3}{4}$ gall., t gall.? (ii) How much petrol is used to run 1 mile, 10 miles, s miles?

18. A man buys a horse for £40 and sells it for £P; what is his profit?

19. By selling a cricket bat for z shillings I gain 10 shillings, how much did it cost me?

20. (i) Share v apples among 5 boys. How many does each have? (ii) Share v apples among 5 boys and x girls. How many does each have?

21. A jug holds 2 pints? How many jugs are needed for k pints, n gallons?

22. A pail holds g pints. How many times can it be filled from a tank containing 100 pints, n pints, z gallons?

23. Express in yards, 16 ft., 2 ft., l ft.

24. Express in ounces, 2 lb., $1\frac{1}{2}$ lb., W lb.

25. A family uses 3 loaves of bread a day. How many loaves are needed for n days, t weeks? How long will x loaves last?

E.P. 10.

Harder Problems in Generalised Arithmetic.

1. How many inches are there in $2s$ feet, $\frac{1}{4}t$ yards?

2. What is the length of a train which has n coaches, if each coach is l feet long and the engine is c feet long?

3. A clerk writes n letters an hour, how long does he take to write p letters?

4. A man smokes $\frac{1}{4}$ lb. of tobacco a week; how long does W lb. of tobacco last him?

5. A rug is 6 feet wide and 36 feet long; what is its area in sq. ft., sq. yd.?

6. The length of fence for a square enclosure is $2p$ yd.; what is the area enclosed?

7. A hall is twice as long as it is wide; it is x feet long; what is its floor-area?

8. In No. 7, the hall is 20 ft. high; what is the total area of the four walls?

9. A brick is 3 in. wide, 2 in. deep, t in. long; what is its volume?

10. In No. 9, the material of which the brick is made weighs W oz. per cu. in.; what is the weight of the brick?

11. How many tins measuring 2 in. by 3 in. by 4 in. can be filled from a tank containing V cu. ft. of water?

12. What volume of soil is removed when making a trench $2t$ ft. wide, $3t$ ft. deep, $9t$ ft. long? Answer in (i) cu. ft., (ii) cu. yd.

13. A pencil costs $\frac{3p}{2}$ pence; how many can I buy for half a crown?

14. I bicycle at the rate of $3s$ miles an hour, how far do I go in 40 minutes ?

15. The average speed of a train is v miles an hour, how long does it take to go 50 miles ? How long would it take if the speed was 10 miles an hour less ?

16. Take the number N and from half of it subtract one-third of it.

17. Express in shillings the result of subtracting £x ys. from £y xs., where $y > x$.

18. From a square sheet of cardboard of side $3p$ in., a square of side $2p$ in. is cut away ; what area remains ?

19. A train has $11n$ compartments with 8 seats each and $2n$ compartments with 6 seats each ; how many seats are there in the train ?

20. In No. 19, find the number of such trains required to carry 2000 passengers, if each has a seat ?

21. A brick is t in. broad, $2t$ in. long and h in. high. What is the sum of the lengths of its edges ?

22. A handkerchief costs k pence and a pair of socks costs one shilling more than that. What is the total cost of 8 handkerchiefs and 4 pairs of socks, (i) in pence, (ii) in shillings ?

23. A tank is $3c$ ft. long and $2c$ ft. wide and contains c^3 cu. ft. of water. What is the depth of water, (i) in feet, (ii) in inches ?

24. A tank is $4p$ ft. long and $2p$ ft. wide and contains water to a depth of $3p$ ft. ; what is the area of the wetted surface ?

25. A bicycle wheel revolves N times in 100 yards ; how many times does it revolve in $\frac{1}{8}N$ miles ?

26. It takes x men t days to repair a certain road ; how long would it take $2x$ men ?

27. A photograph x in. long, y in. wide is printed on a sheet of paper $\frac{5x}{4}$ in. long, $2y$ in. wide. What area of the paper is not used ?

28. How many tiles measuring 8 in. by 6 in. are required for the floor of a hall b ft. broad, $2b$ ft. long ?

29. A certain milestone on the Winchester-Southampton road reads x miles to Winchester, $2x$ miles to Southampton. How far is Winchester from Southampton ? What is x if this distance is 12 miles ?

30. A shopkeeper would make a profit of £P by selling a table for £S. For what must he sell it to make a profit of (i) £$(2P)$, (ii) £p?

31. In an examination, a girl, who scores n marks, fails by b marks; a boy passes with c marks in hand. What did he score?

32. The duty on a picture is two-fifths of its value. What is the value if the duty is £$(50P)$?

33. At a shooting gallery, you pay 2d. if you miss and receive 6d. if you hit the target. What is the result if you score x hits and y misses?

34. A, B, C are three parcels; A and B together weigh p lb., B and C together weigh q lb., A and C together weigh r lb.; what is the total weight of A, B, C?

35. A man starts from a boat-house B and rows downstream; he takes t minutes to row 4 miles; he then turns and takes $4t$ minutes to get back to B. How far from B is he at a time $2t$ minutes after leaving B?

E.P. 11.

Positive and Negative Numbers.

1. Subtract
(i) $-a$ from a; (ii) b from $-b$; (iii) $5y$ from $7y$;
(iv) $-3x$ from $5x$; (v) $5p$ from $-2p$; (vi) $-6z$ from $-2z$.

2. Add
(i) $-a$ and a; (ii) $-2b$ and $-3b$; (iii) $-c$ and $5c$;
(iv) $7x$ and $-4x$; (v) 0 and $-2y$; (vi) $-4z$ and $-z$.

3. Multiply
(i) $-a$ by -1; (ii) $2b$ by -2; (iii) $-3x$ by -4;
(iv) $-y$ by y; (v) $-z$ by $-z$; (vi) -5 by $2c$.

4. Divide
(i) $-4a$ by -1; (ii) $-2b$ by -2; (iii) $6y$ by -2;
(iv) $-9x$ by 3; (v) $-x^2$ by $-x$; (vi) $6ab$ by $-3b$.

If $a=-1$, $b=-2$, $c=3$, $d=0$, $x=-4$, $y=1$, write down the values of the following, Nos. 5-47.

5. ab. 6. ac. 7. bd. 8. ax. 9. dy. 10. bc.

11. ay. 12. bx. 13. cd. 14. xy. 15. $\frac{b}{a}$.

16. $\frac{c}{a}$. **17.** $\frac{x}{y}$. 18. $\frac{y}{a}$. 19. $\frac{a}{y}$. **20.** $\frac{d}{y}$.

21. $\frac{d}{a}$. **22.** $\frac{x}{b}$. 23. $\frac{b}{y}$. **24.** $\frac{b}{x}$. 25. b^2.

26. a^3. 27. x^2. 28. a^4. 29. d^2. 30. $a-b$.

31. $b+c$. 32. $d-a$. 33. $x-y$. 34. $c-x$. 35. $y+a$.

36. $x-b$. 37. $a-c$. 38. $b-d$. 39. $y+b$. 40. $y-a$.

41. $x-c$. 42. $2x+y$. 43. $3b-c$. 44. $d-2a$. 45. $3y-b$.

46. $3b-2d$. 47. $b-2a$. 48. $a-2x$. 49. $c+4a$. 50. $2c-6a$.

51. $3y-c$. 52. $2b-x$. 53. $3x-4a$. 54. $a(b-c)$. 55. $b(d-a)$.

56. $x(y-b)$. 57. $c(a-x)$.

Simplify the following:

58. $(-p)(3p)$. 59. $(-2r)(-r)$. 60. $(+3s)(-2t)$.

61. $(-8t)\div(-2)$. 62. $(-6k)\div 3$. 63. $(4l)\div(-1)$.

64. $-(-4a)$. 65. $+(+3b)$. 66. $-(+2c)$.

67. $(-3)(2x)$. 68. $0\times(-3y)$. 69. $0\div(-4z)$.

70. $3p-5p-p$. 71. $a-4a+2a$. 72. $-t-3t-4t$.

73. $x+(-3x)$. 74. $(-y)+(-y)$. 75. $2z-(-z)$.

76. $-(-a)+(+a)$. 77. $-(+c)+(-c)$. 78. $+(-p)-(-p)$.

79. $-(+2b)-(-2c)$. 80. $3r-(-3s)$. 81. $(-2k)+(-l)$.

82. $(-a)\div(ab)$. 83. $c\div(-c^2)$. 84. $(-d)\div(-2d)$.

85. $\left(-\frac{p}{q}\right)\div(-p)$. 86. $\left(\frac{x}{y}\right)(-y)$. 87. $(-x^2)\div(xy)$.

88. $\left(-\frac{2}{r}\right)\times\left(\frac{r}{-2}\right)$. 89. $(-ab)\div(-2b)$. 90. $\left(\frac{a}{-1}\right)\times b$.

91. $(-1)(b-a)+(-a)$. 92. $c+(-2)(c-d)$.

93. $(x^2-2xy)\div(-x)$. 94. $(-3)(p-q)+(-2)(q-p)$.

95. $(r-s)\div(-1)+s$. 96. $-2y+(-3)(x-y)$.

97. $0+(-1)(z-y)$. 98. $(a-b)-(b-a)$.

99. $(-2)(3x)+(-1)(-4x)$. 100. $(-2x)^2-2x^2$.

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101. Subtract $-a-b+c$ from a.

102. Divide $x^2-3xy-x$ by $-x$.

103. Simplify $\frac{x}{-y}+\frac{-x}{y}-\frac{-x}{-y}$.

104. Fill in the blank space in $x-y=(-1)(\quad)$.

105. Simplify (i) $(-a)(-b)(-c)$; (ii) $(-x^3)^2$;
(iii) $(-y)^2(-y^2)$; (iv) $(2p)(-3q)\div(-1)^2$.

106. Subtract $4x^2+7$ from x^2-5x-2.

107. Add $-(1-2x)$ to $3(x+1)$.

108. Divide $x^4y^2-x^2y^4$ by $-xy$.

109. Multiply $1-(2-a)$ by $-a$.

110. Simplify $\frac{b^2}{-b}+\frac{b^2-b^4}{-b^2}+(-b)^2$.

111. If $(x-a)(y-a)=c^2$ and if $a=-1$, $c=-1$, $y=1$, find the value of x.

112. If $r+s+4=0$ and $s=-1$, what is the value of $\frac{r}{s}$?

113. If $x=-\frac{1}{2}$ and $y=-\frac{1}{3}$, what is the value of $\frac{1}{x}-\frac{1}{y}$?

114. Simplify $(a-2a^2)\div(-a)+(2b^2-b)\div(-b)$.

115. If $pq=p+q$ and $q=-1$, what is p?

116. Can you find two numerical values of x such that the square of $x-7$ is 16?

117. Multiply $\frac{1}{-a}+\frac{1}{-b}-\frac{1}{-ab}$ by $-ab$.

118. Divide $x^3y^2z^2-x^2y^3z^2+x^2y^2z^3$ by $-xyz^2$.

119. Fill in the blank space in $2r-3s=(s-r)-(\quad)$.

120. If $y+z=0$, simplify $\frac{y^2+z^2}{yz}$.

E.P. 12.

Simultaneous Equations.

Solve the following pairs of simultaneous equations:

1. $a+b=13$,
$a-b=4$.

2. $u-v=11$,
$u+v=1$.

3. $x-y=9$,
$x+y=0$.

4. $p+q=8$,
$q-p=3$.

5. $2l+m=12$,
$l=m$.

6. $2u-v=7$,
$u+v=8$.

7. $3x - 2y = 6,\quad x + 3y = 13.$

8. $7x - y = 2,\quad 6x = y.$

9. $2x + 5y = 8,\quad 3x + 4y = 5.$

10. $y = 2x + 1,\quad 3y = 5(x + 1).$

11. $3x + 4y = 7,\quad y = 0.$

12. $4x + 3y = 1,\quad 5x + 4y = 2.$

13. $l + m = 2\frac{1}{3},\quad m - l = 1\frac{1}{6}.$

14. $a + 2b = 5\frac{1}{4},\quad a - b = 1\frac{1}{2}.$

15. $4x - 3y = 3,\quad y = 2x.$

16. $t + 2z = 12,\quad z = 4t.$

17. $c = 3 - 2d,\quad d = 12 - 2c.$

18. $3p + 4q - 3 = 0,\quad 2p - 5q + 5 = 0.$

19. $3x + 5y = 21,\quad x + 2y = 7.$

20. $2x + 7y - 22 = 0,\quad 3x + 6y - 15 = 0.$

21. $4x + 3y + 13 = 0,\quad 3x + 2y + 8 = 0.$

22. $2r + s + 10 = 0,\quad 3r - 2s + 1 = 0.$

23. $3p + 2z = 4,\quad 4p + z + 3 = 0.$

24. $2x + 11y = 12,\quad 3x - 2y + 19 = 0.$

25. $c - 7 = d + 2 = 5.$

26. $2p - 1 = 0 = 3 + q.$

27. $x - 3y - 5 = 2x + y - 3 = 0.$

28. $y = 2(x + y - 1) = -2(x + y + 3).$

29. $2x + 3y = x + 1 = 4 - y.$

30. $x + 2y + 3 = 4x + 4y - 1 = 3x + 3y + 2.$

31. $\frac{1}{2}p + \frac{1}{3}q = 1,\quad 2p + 3q + 1 = 0.$

32. $\dfrac{x - 1}{3} + \dfrac{y + 1}{2} = 1,\quad \dfrac{2x + 1}{5} - \dfrac{3y + 1}{4} = 5.$

33. $\dfrac{1}{x} - \dfrac{1}{y} = 2,\quad \dfrac{1}{x} + \dfrac{1}{y} = 8.$

34. $\dfrac{3}{y} - \dfrac{1}{z} = 1,\quad \dfrac{5}{y} + \dfrac{2}{z} = 20.$

35. $\dfrac{2l - m}{3} = \dfrac{l - 3m}{2},\quad \dfrac{l}{3} = 1 - m.$

36. $\dfrac{r - 2s}{r} = \dfrac{3s}{2r},\quad \dfrac{3}{r} - \dfrac{2}{s} = \dfrac{1}{rs}.$

37. $4x - 2y = 4y - 5x = x + y - 3.$

38. $2x - y - 3 = 3y - x + 4 = 5x - 8y - 4.$

39. $\dfrac{3p - 7}{4} = \dfrac{2q + 1}{3} = 5.$

40. $\dfrac{y + 2z}{2y - z} = \dfrac{3y - 2}{z - 7} = 8.$

41. $\dfrac{x - y}{x + y} = \dfrac{3}{4} = \dfrac{x - 1}{y + 2}.$

42. $\dfrac{3x - 1}{2y + 1} = \dfrac{2x + 2}{y + 2} = 4.$

Problems.

43. A knife and fork cost 6s.; the knife costs sixpence more than the fork. What is the cost of each ?

44. A man distributes x oranges among n boys; if each receives 10 oranges, there are 3 left over; but there are four oranges too few for each boy to receive 11 oranges. Find x and n.

45. I am thinking of two numbers which, when added, make 90 and are such that one-third of the smaller is equal to one-seventh of the larger. What are they ?

46. 3 lb. of jam and 2 lb. of butter cost 8s.; also 6 lb. of jam and 3 lb. of butter cost 14s.; find the cost of 1 lb. of jam and of 1 lb. of butter.

47. 3 cows and 4 sheep cost £52; also 4 cows and 6 sheep cost £71; find the cost of one cow and of one sheep.

48. Can you find two numbers such that three times the smaller exceeds twice the larger by 3, and seven times the smaller exceeds five times the larger by 2 ?

49. A boy spends 4s. 6d., partly on the entrance money for an exhibition, and the rest on amusements inside; his brother, who spends twice as much on amusements, spends 7s. altogether. What did each spend on amusements and what was the entrance money ?

50. In 4 years' time a father will be 3 times the age of his son; 4 years ago he was 5 times the age of his son. What are their present ages ?

51. I have to pay for 500 cigarettes two shillings more than for 3 lb. of tobacco; and I have to pay for 4 lb. of tobacco four shillings more than for 600 cigarettes. What do I pay for one lb. of tobacco and for 100 cigarettes ?

52. I wish to give one shilling each to some children, but find I have 1s. 6d. too little to do so. I therefore give them 10d. each and have one shilling over. How many children are there and how much money have I got with me ?

53. A owes £4, B owes £5; A could just pay his debt if he borrowed from B one-eighth of what B has; B could just pay his debt if he borrowed from A two-sevenths of what A has. How much has each ?

54. Find two numbers such that the first is greater than half the second by 4, and three times the first is less than twice the second by 1.

55. If A gives B twopence, B has four times as much as A. If C gives A and B ten pence each, then B has twice as much as A. How much have A and B ?

56. A heap of half-crowns and florins is worth £2 10s.; another heap, which contains only half as many half-crowns but twice as many florins is worth £2 15s. How many coins of each kind are there in the first heap ?

57. Can you discover a fraction such that the result of either adding 1 to the numerator or of subtracting 3 from the denominator is equivalent to $\frac{1}{3}$? Is there more than one answer ?

58. A and B are two jugs; an empty pail is just filled by 5 jug-fulls from A and one from B or by 8 jug-fulls from B. If 3 jug-fulls from both A and B are put into it, there is still room for another 2 pints. How much does each jug hold ?

E.P. 13.

Fractions and Equations.

Simplify the fractions and solve the equations in the following examples :

1. $\frac{1}{2}-\frac{1}{x-2}$.

2. $\frac{1}{2}-\frac{1}{x-2}=\frac{1}{4}$.

3. $\frac{1}{x}+\frac{1}{2x}-\frac{1}{3x}$.

4. $\frac{1}{x}+\frac{1}{2x}-\frac{1}{3x}=\frac{1}{9}$.

5. $\frac{x}{3}-\frac{x+1}{5}$.

6. $\frac{x}{3}-\frac{x+1}{5}=1$.

7. $1-\frac{1}{x+1}$.

8. $2+\frac{1}{2x+1}=1$.

9. $\frac{1}{3x}-\frac{1}{3x+3}$.

10. $\frac{1}{3x}-\frac{1}{3x+3}=\frac{1}{4x}$.

11. $x+3-\frac{x^2}{x-3}$.

12. $\frac{x}{x+2}-\frac{1}{x}=1$.

13. $\frac{x}{x-1}-\frac{1}{x}-1$.

14. $\frac{x}{2x-4}-\frac{x}{3x-6}$.

15. $x+4-\frac{x^2}{x-4}=1$.

16. $\frac{x}{x-1}+\frac{1}{x}=1$.

17. $\frac{x}{x+1}+\frac{x}{x-1}-2.$

18. $\frac{1}{x}-\frac{x+1}{x(x-1)}+\frac{1}{x-1}.$

19. $\frac{5}{x-2}-\frac{3}{x}=\frac{12}{x(x-2)}.$

20. $\frac{x}{2x-4}-\frac{x}{3x-6}=1.$

E.P. 14.

Factors.

Find the factors of the following expressions:

1. $ab-bc-ad+cd.$

2. $t^2-7t+12.$

3. $p^2-3p-4.$

4. $ax-2bx+ay-2by.$

5. $x^2+4x-21.$

6. $y^2+10y+24.$

7. $ab-2ac+2bd-4cd.$

8. $3a^2+5a-2.$

9. $1-11x+24x^2.$

10. $b^2-bc+bd-cd.$

11. $z^2-5z-14.$

12. $2b^2+b-6.$

13. $xy-xz-y^2+yz.$

14. $1-2t-15t^2.$

15. $12a^2-a-6.$

16. $1+a+a^2+a^3.$

17. $10c^2-3c-1.$

18. $10l^2-19lm+6m^2.$

19. $x^2-xy+yz-xz.$

20. $3t^2+9t-30.$

21. $6-p-12p^2.$

22. $1+xy-x-y.$

23. $a^2-16a+48.$

24. $20b^2-49b+30.$

25. $a(a-1)+a-1.$

26. $4t^2+12t-72.$

27. $15+14p-8p^2.$

28. $1-p-q(p-1).$

29. $12y^2-7y+1.$

30. $12x^2-16x-3.$

31. $4ax-6bx+9by-6ay.$

32. $3x^2+xy-2y^2.$

33. $6x^2-2xy+ay-3ax.$

34. $12r^2+32rs+5s^2.$

35. $x^2+(a+b)x+ab.$

36. $28a^2-3ab-18b^2.$

37. $3x^3-5x^2y+6xy^2-10y^3.$

38. $21c^2+34cd+8d^2.$

39. $x^2-(p-q)x-pq.$

40. $20y^2+21yz-54z^2.$

41. $1+x^2+xy+x^3y.$

42. $2a^4+3a^2bc-2b^2c^2.$

43. $ax+4-2x-2a.$

44. $2x^4+11x^2+15.$

45. $(2a+b)(c+d)+c+d.$

46. $3-5t^2-12t^4.$

47. $x-y+(y-x)^2.$

48. $36a^2+27ab-28b^2.$

49. $(c-d)^2+c^2-cd.$

50. $105-17p-12p^2.$

51. $(x+1)^2-(x+1)(2x-3)$.
52. $x(x+4y)-3y(3x-2y)$.
53. $12x^2+5xy-72y^2$.
54. $(x-y)^2-2x(x-y)$.
55. $a(x+y)-xy-a^2$.
56. $x(x+1)+(x+4)(x-3)$.
57. $a(b^2+c^2)-bc(1+a^2)$.
58. $x^2y^2-7xyz-18z^2$.
59. $14a^2-13ab-12b^2$.
60. $6x^2-4xy+10yz-15xz$.

E.P. 15.

Fractions.

Simplify the following expressions :

1. $\dfrac{3x-3y}{3x+3y}$.
2. $\dfrac{2a+2b}{5a+5b}$.
3. $\dfrac{a^2-ax}{ax-x^2}$.
4. $\dfrac{c+c^2}{3+3c}$.
5. $\dfrac{p^2-2p}{p^2+2p}$.
6. $\dfrac{x^2-xy}{x^2-y^2}$.
7. $\dfrac{ab^2+a^2b}{abc}$.
8. $\dfrac{2x-2y}{(x-y)^2}$.
9. $\dfrac{4x-4y}{6y-6x}$.
10. $\dfrac{a^2-b^2}{(a-b)^2}$.
11. $\dfrac{6x^2-6xy}{4xy}$.
12. $\dfrac{3x-3}{(1-x)^2}$.
13. $\dfrac{b^3-b^2c}{b^2c-bc^2}$.
14. $\dfrac{(a-2)^2}{(2a-4)^2}$.
15. $\dfrac{2x-2y}{3x^2y-3xy^2}$.
16. $\dfrac{x-3}{x^2-x-6}$.
17. $\dfrac{a^2+a-6}{2a+6}$.
18. $\dfrac{ab-b^2}{a^2+ab-2b^2}$.
19. $\dfrac{x^2+6x+8}{x^2-4}$.
20. $\dfrac{x^2+xy-2y^2}{x^2-2xy+y^2}$.
21. $\dfrac{x^2-8x+15}{x^2+2x-35}$.
22. $\dfrac{y^2-2y+1}{y^2+3y-4}$.
23. $\dfrac{x^2-x}{(2x-2)^2}$.
24. $\dfrac{x^2+x-2}{x^2-1}$.
25. $\dfrac{x^2-1}{x}\times\dfrac{x^2}{x^2+2x+1}$.
26. $\dfrac{a^2+a-2}{a^2-a-6}\times\dfrac{a-3}{a-2}$.
27. $\dfrac{x^2+xy}{xy-xz}\times\dfrac{y^2-yz}{xz+yz}$.
28. $\dfrac{4x-12}{3x}\times\dfrac{9x^2}{6x-18}$.
29. $\dfrac{2x}{3x-3}\div\dfrac{4y}{x^2-x}$.
30. $\dfrac{5a}{a-3b}\div\dfrac{2a}{3a-b}$.
31. $\dfrac{x^2-x}{ax-a}\div\dfrac{ax}{ax-x}$.
32. $\dfrac{x^2-xy}{xy+y^2}\div\dfrac{xy-y^2}{x^2+xy}$.
33. $\dfrac{x^2-2xy+y^2}{x^2-y^2}\times\dfrac{x^2+2xy+y^2}{x^2+xy}$.
34. $\dfrac{b^2-c^2}{b^2-2bc+c^2}\div\dfrac{c}{b^2-bc}$.

35. $\dfrac{b^2+2bc+c^2}{b^2+2bc} \times \dfrac{b}{b^2-c^2}$.

36. $\dfrac{a^2-4ab+4b^2}{a^2+ab-6b^2} \div \dfrac{a^2+3ab}{a^2+6ab+9b^2}$.

37. $\dfrac{x}{x-2} - \dfrac{2}{x+2}$.

38. $\dfrac{2x-3}{3x-9} - \dfrac{x-2}{2x-6}$.

39. $\dfrac{1}{3a-3b} + \dfrac{1}{6a-6b}$.

40. $\dfrac{a}{a-b} - \dfrac{b}{a+b}$.

41. $\dfrac{4}{x+3} + \dfrac{12}{x^2-9}$.

42. $\dfrac{x}{x-y} + \dfrac{y}{y-x}$.

43. $\dfrac{x+2}{x+1} - \dfrac{x+1}{x+2}$.

44. $\dfrac{ab}{a^2-b^2} - \dfrac{b}{a+b}$.

45. $\dfrac{1}{x^2-4} - \dfrac{1}{x^2+2x-8}$

46. $\dfrac{a}{a-b} + \dfrac{a}{b-a}$.

47. $\dfrac{3}{x-2} - \dfrac{6}{x^2-4}$.

48. $\dfrac{a+2b}{a^2+ab} + \dfrac{a-b}{ab+b^2}$.

49. $\dfrac{x^2+y^2}{x^2-y^2} - \dfrac{x+y}{x-y}$.

50. $\dfrac{x}{x^2-4x+3} - \dfrac{2}{x^2-x-6}$.

51. $\dfrac{2b}{b^2-a^2} + \dfrac{1}{a-b}$.

52. $\dfrac{2b}{b+c} + \dfrac{c}{b-c} - \dfrac{2b^2}{b^2-c^2}$.

E.P. 16.

Quadratics with Rational Roots.

1. If $(x-3)(y-5)=0$, can you find the value of x when (i) $y=7$; (ii) $y=5$; (iii) $y=0$? If so, what is it?

2. If $(a-b)(x+2)=0$, find, when possible, the value of x when (i) $a=3$, $b=1$; (ii) $a=0$, $b=1$; (iii) $a=4$, $b=4$?

3. Do you know anything about the numerical value of x if $xy=10$?

4. What can you say about the values of x and y if

$$(x-3)(y-5)=0\ ?$$

Express the following facts by equations:

5. Either $x=3$ or $x=6$.
6. Either $y=2$ or $y=5$.
7. Either $t=2$ or $t=-2$.
8. Either $z=0$ or $z=4$.
9. Either $x=1$ or $y=1$.
10. Either $x=-2$ or $x=3$.
11. Either $p=-4$ or $p=-7$.
12. Either $y=3$ or $z=-2$.
13. Either $x=\frac{1}{2}$ or $x=3$.
14. Either $t=\frac{2}{3}$ or $t=\frac{4}{7}$.

15. Either $y=0$ or $y=-\frac{3}{4}$.

16. Either $z=-2\frac{1}{2}$ or $z=-1\frac{1}{3}$.

17. $x=\pm 5$.

18. $s=\pm\frac{2}{3}$.

19. $y=0$ or 2 or 5.

20. $z=1$ or -1 or -2.

21. $t=\frac{1}{2}$ or -2 or $1\frac{1}{4}$.

22. $v=0$ or $\pm\frac{3}{4}$

23. $x=-\frac{2}{5}$ or $2\frac{1}{2}$.

24. $y=1$ or $-1\frac{1}{2}$ or $-\frac{1}{3}$.

Solve the following equations :

25. $x(x-3)=0$.

26. $(x-2)(x+3)=0$.

27. $(x-4)^2=0$.

28. $3x(x+2)=0$.

29. $(2x-3)(x-5)=0$.

30. $(3x+1)(x+3)=0$.

31. $5x(5x-2)=0$.

32. $(2x+3)^2=0$.

33. $x(x+1)(x-2)=0$.

34. $(2x-7)(7x+2)=0$.

35. $(x-4)^2(x+4)=0$.

36. $(3-2x)(5+x)=0$.

37. $(4+x)(1-4x)=0$.

38. $(1-x)(2-x)(3+x)=0$.

39. $x^2-8x+15=0$.

40. $x^2-8x=0$.

41. $x^2+11x+28=0$.

42. $x^2+11x=0$.

43. $x^2+5x-6=0$.

44. $x^2-3x-70=0$.

45. $x^2-6x+9=0$.

46. $x^2+12x+36=0$.

47. $x^2+9x=36$.

48. $x^2-11x=60$.

49. $x(x-1)=72$.

50. $x(x+2)=99$.

51. $2x^2-11x+5=0$.

52. $6x^2=x+2$.

53. $x^2=25$.

54. $4x^2=9$.

55. $15x^2+2x=8$.

56. $14x^2=17x+6$.

57. $(2x+3)^2=25$.

58. $20x^2-7x=6$.

59. $4x^2=7x$.

60. $10x^2+33x+20=0$.

61. $(x+1)(x-3)=12$.

62. $(2x-1)(1+3x)=4$.

63. $x(x-1)=3(x-1)$.

64. $(x+2)^2=5(x+2)$.

65. $x-\dfrac{18}{x}=7$.

66. $1+\dfrac{8}{x-1}=\dfrac{18}{x+1}$.

67. $\dfrac{10}{2x-1}=3\left(1-\dfrac{1}{x}\right)$.

68. $x+4=\dfrac{9}{x+4}$.

69. $\dfrac{1}{x-1}-\dfrac{1}{x+2}=\dfrac{1}{6}$.

70. $\dfrac{x-1}{2x+1}=\dfrac{4}{3x}$.

E.P. 17.

The General Quadratic.

Solve the following equations. If the roots are not rational, give each root correct to one place of decimals.

1. $x^2 - 4x = 3.$ **2.** $x^2 + 6x = 8.$ **3.** $x^2 - 5x = 7.$

4. $x^2 + 3x = 5.$ **5.** $x^2 + 8x + 14 = 0.$ **6.** $x^2 - 10x + 12 = 0.$

7. $x^2 - 7x = 18.$ **8.** $x^2 + 5x - 7 = 0.$ **9.** $x^2 + x - 3 = 0.$

10. $2x^2 - 12x = 15.$ **11.** $2x^2 + 3x = 4.$ **12.** $3x^2 - 2x = 1.$

13. $3x^2 - 2x = 2.$ **14.** $5x^2 - 8x + 2 = 0.$ **15.** $4x^2 - 3x = 3.$

16. $x + \frac{1}{x} = 3.$ **17.** $x + 5 = \frac{2}{x}.$ **18.** $2x + \frac{1}{x} = 7.$

19. $\frac{1}{x+1} + \frac{1}{x-1} = \frac{4}{5}.$ **20.** $\frac{x}{x-1} + 1 = \frac{x-2}{2x}.$

21. $\frac{x}{x-2} - \frac{x+1}{x+4} = 1.$ **22.** $\frac{24}{x+4} + \frac{9}{x-5} = 5.$

23. $\frac{2x}{x-1} - \frac{3}{x+2} = 3.$ **24.** $\frac{x}{x+2} + \frac{x+2}{x} = 5.$

25. $\frac{20}{x+1} + \frac{24}{x+2} = 8.$ **26.** $\frac{3}{x+5} - \frac{1}{2x+3} = \frac{1}{4}.$

27. $\frac{1}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} = 0.$ **28.** $\frac{30}{x+1} + \frac{36}{x-1} + 6 = 0.$

29. $\frac{1}{x+1} + \frac{2}{x-1} = \frac{1}{x+3}.$ **30.** $\frac{3}{x+1} + \frac{2}{x-1} = \frac{3(x-1)}{2(x+1)}.$

E.P. 18.

Graphs of Quadratic Functions.

1. A marble rolling in a sloping groove passes at zero hour a marked point B and t seconds later is s feet *down* the groove from B where $s = t^2 - 3t$. Make a table of values of s for values of t from -2 to $+5$ and plot the graph showing the positions of the marble for this interval of time. Why should you include $t = 1{\cdot}5$ in your table ? Use your graph to answer the following questions :

(i) How far above or below B is the marble when $t = 0{\cdot}5$, $2{\cdot}5$, $-1{\cdot}5$, $4{\cdot}5$, $-1{\cdot}7$, $-0{\cdot}8$?

(ii) What is the highest point the marble reaches and at what time ?

(iii) When is the marble (*a*) 4 ft. below B, (*b*) 1 ft. above B, (*c*) 8 ft. below B, (*d*) 1·8 ft. above B ?

(iv) At what time does the marble pass B coming down ?

(v) Find approximate values of t such that (*a*) $t^2 - 3t = 6$, (*b*) $t^2 - 3t = -1$, (*c*) $t^2 - 3t = 3$, (*d*) $t^2 - 3t + 1{\cdot}5 = 0$.

2. Draw the graph of $20x - 5x^2$ for values of x from -2 to $+6$. Use it to find, when possible, approximate solutions of the following equations :

(i) $20x - 5x^2 = 14$; (ii) $20x - 5x^2 = 4$;
(iii) $20x - 5x^2 = -14$; (iv) $5x^2 - 20x = 40$;
(v) $5x^2 - 20x + 8 = 0$; (vi) $5x^2 - 20x + 25 = 0$.

What is the greatest value of $20x - 5x^2$?

Solve graphically, when possible, the following equations :

3. (i) $x^2 + 4x = 1$; (ii) $x^2 + 4x + 2 = 0$;
(iii) $x^2 + 4x + 4 = 0$; (iv) $x^2 + 4x + 5 = 0$.

4. (i) $2x^2 - 3x = 3$; (ii) $2x^2 - 3x + 1 = 0$;
(iii) $2x^2 - 3x + 2 = 0$; (iv) $10x^2 - 15x + 3 = 0$.

5. (i) $3x^2 - 5x = 2$; (ii) $3x^2 - 5x + 1 = 0$;
(iii) $3x^2 - 5x + 2 = 0$; (iv) $3x^2 - 5x + 3 = 0$;
(v) $6x^2 - 10x + 1 = 0$; (vi) $9x^2 - 15x = 4$.

E.P. 19.

Simultaneous Quadratics.

1. $2x + y = 1$, $x^2 - 2xy = 39$.

2. $x - 2y = 1$, $xy = 1$.

3. $x^2 + y^2 = 34$, $x - y = 8$.

4. $2x^2 + y^2 = 3$, $x + y = 2$.

5. $7x - y = 2$, $x^2 + y^2 = 2\frac{1}{2}$.

6. $x^2 + y^2 = -6x$, $y - 2x = 3$.

7. $x^2 + y^2 = x + y = 2$.

8. $2y = 7x - 5xy = 3x - xy$.

9. $xy + 2x - y = 3$, $x + y = 1$.

10. $7xy = x + 2y$, $3x - 4y = 1$.

11. $3x - y = 5$, $(x + y)^2 = 17x - y + 2$.

12. $20y = 9x + 13$, $y(13 - x) = 4(x + 2)$.

13. $x^2 - 3xy = 28$, $3x + 2y = 10$.

14. $x + 2y + xy = -2$. $2x - xy = 9$.

15. $2x - 3y = 1$, $x^2 + y^2 = 2x + 1$.

16. $x^2 + y^2 + 5x = 3$, $5x + 4y = 2$.

17. $9x^2 + 6xy + 16y^2 = 4$. $3x - 2y = 2$.

18. $3x + 4y = 18$, $x^2 + 3xy + 2y^2 = 40$.

19. $3x^2 + 5xy + y^2 = -3$. $2x + 3y = -1$.

20. $3y-5x=4,$
$2x^2+5xy-3y^2=-10.$

21. $\frac{1}{x}-\frac{1}{y}=\frac{1}{x+1}-\frac{1}{y+2}=\frac{1}{12}.$

22. $x^2-15=y^2-10=xy.$

23. $x^2+xy=90,$
$xy+y^2=10.$

24. $2xy-x^2=24,$
$2y^2-xy=30.$

25. $xy+y^2=4,$
$x^2-xy+2y^2=8.$

26. $16(x^2-xy)=18(xy-y^2)=1.$

27. $\frac{1}{x}+\frac{1}{y}=(x-4)(y-4)=2.$

28. $2xy+y^2=16,$
$2x^2-xy=12.$

29. $x^2+2xy=3,$
$y^2-xy=4.$

30. $(x+2y)(2x+y)=20,$
$y^2+4x(x+y)=16.$

E.P. 20.

Literal Equations.

1. $x^2-2x=a^2-1.$

2. $x^2-x\left(\frac{a}{b}+\frac{b}{a}\right)+1=0.$

3. $ax^2+(1+ab)x+b=0.$

4. $ax^2-c^2x=a^2x-ac^2.$

5. $x(x-p)+(p+q)(p-q)=px.$

6. $(x+a)(x-a^3)+4a^4=2a(x+a^3).$

7. $\frac{ax}{b}-\frac{b}{ax}=b-\frac{1}{b}.$

8. $(a+b)x^2+(a+2b+c)x+b+c=0.$

9. $\frac{a^2(x-b)}{a-b}+\frac{b^2(x-a)}{b-a}=x^2.$

10. $x^2-2(ab-c)x=a^2(a^2-2ab+2c).$

11. $\frac{2b}{x-a}=1+\frac{a}{x-b}.$

12. $a(b-c)x^2+b(c-a)x+c(a-b)=0.$

13. $\left(\frac{x-c}{a-b}\right)^2=\frac{cx}{ab}.$

14. $\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}.$

15. $xy=ax+by=bx+ay.$

16. $x(y+1)=y(a+1)=a(x+1).$

17. $x+y=2a,$
$x^2-y^2=4ab.$

18. $ax+by=a^2+ab,$
$x^2-xy=b^2-a^2.$

19. $\frac{a}{x}+\frac{b}{y}=2,$
$ax-by=a^2-b^2.$

20. $ax+by=a^2-b^2,$
$x^2+y^2=a^2+b^2.$

E.P. 21.

Miscellaneous Examples.

1. Multiply $x^2y^2 + 3xy^3 - y^4$ by $2xy - 6y^2$.

2. If $x+2$ is a factor of $2x^2 - x - 5p$, find the value of p.

3. Find an expression which when divided by $p+1$ has $p^2 - p + 1$ for quotient and 3 for remainder.

4. If $a = 10$, $b = -7$, $c = -3$, prove that $a^3 + b^3 + c^3 = 3abc$.

5. Simplify $\dfrac{1}{(r-p)(p-q)} + \dfrac{1}{(q-r)(p-q)} + \dfrac{2}{(q-r)(r-p)}$.

6. Factorise (i) $2x^2 + 5x + 3$; (ii) $2xy - x^2 - y^2$.

7. Simplify $\dfrac{a+b}{b} - \dfrac{a}{a+b} - \dfrac{a^3 - a^2b}{a^2b - b^3}$.

8. Simplify $\dfrac{x^2 + ax - bx - ab}{x^2 - a^2} \div \dfrac{x^2 - ax + bx - ab}{x^2 - b^2}$.

9. Solve $\dfrac{x-2}{x-6} + \dfrac{x-7}{x-5} = 2$.

10. Solve $x = 9 - \dfrac{y}{2}$, $y = 11 + \dfrac{x}{3}$.

11. Solve $(x+4)(2x-5) - 2(x-3)^2 = 5(2x-7)$.

12. Factorise (i) $2 + 5x - 12x^2$; (ii) $a^2 + (p-q)a - pq$.

13. Simplify $\dfrac{1}{2(3x-2y)} - \dfrac{1}{2(3x+2y)} + \dfrac{3x}{9x^2 - 4y^2}$.

14. Solve $\dfrac{x+a}{x+c} + \dfrac{x-c}{x-a} = 2$.

15. Nineteen coins, all shillings or half-crowns, are worth £1 15s. 6d. How many coins are there of each kind ?

16. Find the fraction which becomes equal to $\frac{4}{5}$ when the numerator is increased by 2, and equal to $\frac{5}{8}$ when the denominator is decreased by 3.

17. The function $ax^2 + bx$ has the value 26 if $x = 3$, and has the value 14 if $x = -2$. Find a, b.

18. Solve $\dfrac{1}{2x-3} - \dfrac{3}{2x^2 - 3x} = \dfrac{5}{x}$.

19. Prove that $a^2 + 1$ is a factor of $(a+b)^2 + (ab-1)^2$.

20. Solve $5x^2-6x=10$, giving each root to two places of decimals.

21. Simplify $\dfrac{4a^3-ab^2}{(a+b)^2-a^2}\div\left(\dfrac{2a}{b}-1\right)$.

22. If $\dfrac{1}{u}+\dfrac{1}{v}=\dfrac{2}{a}$, prove that $\dfrac{1}{a-u}+\dfrac{1}{a-v}=\dfrac{2}{a}$.

23. Factorise $1-3(1-2x)+3(1-2x)(1-3x)$.

24. Solve $9x^2-15x=14$.

25. For what numerical values of a, b is $\dfrac{7x+1}{3x^2-7x+2}$ equal to $\dfrac{a}{x-2}+\dfrac{b}{3x-1}$ for all values of x?

26. Simplify $\left(\dfrac{a}{b}+\dfrac{b}{a}+2\right)\div\left(1+\dfrac{b}{a}\right)$.

27. Solve $x^2-2ax=c^2-2ac$.

28. Factorise $(x-y)(x-2y)-(2y-2x)(2x-y)$.

29. A earns 18s. a day. B is paid 6d. an hour more than A and earns 6d. a day less than A, because he works 2 hours a day less than A. How much per hour is paid to A and B.

30. Express as a single fraction $\left(\dfrac{1+x}{1-x}\right)^2-(1+4x)$.

31. If $\dfrac{1}{v}+\dfrac{1}{u}=\dfrac{1}{f}$ and $\dfrac{1}{z}+\dfrac{1}{2u}=\dfrac{1}{f}$, find z in terms of v, f.

32. Simplify $\dfrac{x}{2x+y}+\dfrac{2y}{x-2y}-\dfrac{5xy}{2x^2-3xy-2y^2}$.

33. Make R the subject of the formula

$$V=\pi l[R^2-(R-t)^2].$$

34. Find x if $2x-3=\dfrac{1}{2x-3}$.

35. Simplify $\dfrac{x}{1-x}+\dfrac{2x^2}{(1-x)^2}+\dfrac{x^3}{(1-x)^3}$.

36. Solve $4-xy=5x$, $2xy+9=y$.

SYMBOLS.

$=$ is equal to.

$\equiv$ is identically equal to. See p. 150.

$\bumpeq$ is approximately equal to.

$\neq$ is not equal to.

$\pm$ plus or minus.

$>$ is greater than.

$<$ is less than.

$a \sim b$ means 'the difference between a and b.'

$\overline{a+b}$ means the same as $(a+b)$. See also p. 5.

GLOSSARY AND INDEX.

Ascending powers. See p. 13.

Axes of coordinates. See p. 54.

Axioms of equations. See p. 95.

Binomial. A *binomial* expression contains two terms; *e.g.* $a+b$ or $4x^2-9a^2$.

Brackets. See pp. 22, etc.

Centigrade scale of temperature. The scale in which the freezing point of water is 0° and the boiling point 100°. x degrees Centigrade is written x° C.

Change of subject in a formula. See p. 147.

Checking an equation. See p. 96.

Circle. If the radius of a circle is r units long, (i) the *circumference* is $2\pi r$ units long, (ii) the *area* is πr^2 corresponding units of area.

Coefficient. See p. 12.

Coefficients, detached. See p. 170.

Constants of a formula. If the charge for n lb. of luggage on a particular journey is $an+b$ pence, where n may have any value, but a and b are always the same, then a and b are called the *constants* of this formula. See p. 115.

Constant term. See p. 12.

Coordinates. See p. 54.

Corresponding units. Inch, square inch, cubic inch are *corresponding units* of length, area and volume; so are centimetre, square centimetre, cubic centimetre and so on.

Cross-multiplying. See p. 100.

Cube and cube root. Since $x\times x\times x=x^3$, x^3 is the *cube* of x, and x is a *cube root* of x^3.

Cylinder. If the radius of a cylinder is r, and its height h units, (i) its curved surface is $2\pi rh$ corresponding units of area, (ii) its volume is πr^2h corresponding units of volume.

Degree (i) (of angle). A right angle is divided into 90 equal parts, called *degrees.*

(ii) (of a term). See p. 12.

(iii) (of an equation). For equations of *the first degree*, see p. 108. A quadratic equation is a simple equation of *the second degree.* So an equation in x, in which the highest power of x is x^n, is called *an equation of degree n.*

Descending powers. See p. 12.

Detached coefficients. See p. 170.

Digit. The digits of the number 265 are, 2, 6 and 5. Of these 2 is the hundred digit, 6 the ten digit and 5 the unit digit.

Dividend. If 3 is divided into 17, it goes 5 times, with *remainder* 2; 17 is called the *dividend,* 3 the *divisor* and 5 the *quotient.*

Divisor. See Dividend.

Elimination. See p. 111.

Equation. See p. 91. For the difference between an *equation* and an identity, see p. 149.

Equivalent. Two numbers or quantities are *equivalent* if they have the same value. Thus $\frac{1}{2}$, $\frac{3}{6}$ and $\frac{x}{2x}$ are equivalent fractions.

Factor. If one number or expression divides into a second without remainder, the first is a *factor* of the second.

Fahrenheit scale of temperature. The scale in which the freezing point of water is 32° and the boiling point 212°. x degrees Fahrenheit is written $x°$ F.

Formula. See p. 47.

Function. See p. 59. Also p. 162.

Greater than. In Algebra x is greater than y if x is higher in the scale of number. Thus 2 is greater than -5, and 0 is greater than -100. See Symbols, p. 252.

Highest Common Factor. See p. 16.

Identity. See p. 150.

Index. If any number of factors, each equal to x, are multiplied together, the product is a *power* of x. Thus $x \times x \times x \times x$ is a power of x, and is usually written x^4. The number 4, which shows how many factors are multiplied, is called the *index* of the power of x. x^4 is called the fourth power of x, and x is a fourth root of x^4.

Integer. A whole number is sometimes called an *integer.*

Interest. For the Simple Interest Formula, see p. 47.

Irrational. See Rational.

Less than. In Algebra x is less than y if x is lower in the scale of number. Thus -10 is less than 1. See Symbols, p. 252.

Like terms. Algebraical terms, which can be added or subtracted, are called *like terms.* Thus $3a$ and $5a$, the sum of which is $8a$, are like terms. But $3a^2$ and $5a$ cannot be added in this way, and are called *unlike* terms.

Literal equations. Equations in which the known numbers are represented by letters are called *literal equations.* See p. 144.

Lowest Common Multiple. See p. 16.

Mean proportional. See p. 159.

Minus. (i) Minus, written $-$, is the sign of subtraction; *e.g.* $a-b$ means 'subtract b from a.'

(ii) -1, -2, -3, etc., are the whole numbers (in order) below 0; $-1\frac{1}{2}$ is half-way between -1 and -2, and so on.

Monomial (strictly mononomial). An expression containing only one term.

Multiple. If one number is divided by another and there is no remainder, the first is a multiple of the second; *e.g.* $2a^2b^3$ is a multiple of $2ab$.

Multiplicand, multiplier. If ab is multiplied by x the result is abx; ab is called the *multiplicand*, x the *multiplier* and abx the *product.*

Negative numbers. See p. 65.

Numbers, numbers of things. See p. 38.

Origin. See p. 54.

Plus. Plus, written $+$, is the sign of addition; *e.g.* $a+b$ means 'add b to a.'

Polygon A polygon is a figure which has more than three sides.

Polynomial. An expression which contains more than two terms.

Power. See Index.

Prime number. A number which has no factor (except itself and 1); *e.g.* 3, 7, 19 are prime numbers.

Product. See Multiplicand.

Quadratic Equation. A quadratic equation in x is one in which the highest power of x is x^2.

Quadratic Function. See p. 162.

Quantity. See p. 39.

Quotient. See Dividend.

Rational. Whole numbers and fractions are called *rational* numbers, thus $2\ \ -7$, $\frac{4}{17}$, $0{\cdot}28$ are rational. Numbers such as $\sqrt{2}$, $\sqrt[3]{-5}$, which cannot be represented in this way, are *irrational.* Similarly, algebraical expressions such as $\sqrt{x}$, $2\sqrt[3]{ab}$ are irrational.

Reciprocal. If the product of two numbers is 1, each is the *reciprocal* of the other; *e.g.* x and $\frac{1}{x}$ are reciprocals, so are $\frac{a}{b}$ and $\frac{b}{a}$.

Right-angled triangle. For the relation between the lengths of the sides, see p. 158.

Root. See under Index, Square root, Cube root.

Root of an equation. Since 2 is a value of x which satisfies the equation $x^2+3x-10=0$, 2 is a *root* of this equation.

Rules of signs. See pp. 74, 82.

Scale of numbers. See p. 65.

Simple equation. A simple equation in x contains no power of x above the first.

Simultaneous equations are equations in which there is more than one unknown number.

Square. Since $a \times a = a^2$, a^2 is the *square* of a.

Square root. Since $a \times a = a^2$, a is a square root of a^2; since $-a \times -a = a^2$, $-a$ is another square root of a^2. $\sqrt{x}$, in this book, will mean 'the positive square root of x.' See p. 15.

Subject, change of. See p. 147.

Substitution. If in an algebraical statement or formula we put numbers for the letters, the process is called *substitution*.

Symbols. See pp. 5, 252.

Symmetrical expression. See Example IX. p. 176.

Term. If an algebraical expression is made by adding or subtracting numbers, each of these is called a *term*. Thus in $2x^2+3ax+a^2$, the first term is $2x^2$, the second is $3ax$ and the last is a^2.

Transpose. See p. 100.

Triangle. (1) Right-angled triangle. See p. 158.
(2) Area of triangle. See p. 42.

Unity. The number 1 is sometimes called *unity*.

Unknown (in an equation). See p. 91.

Unlike terms. See Like terms.

Variable. See p. 59.

Verify (an equation). See p. 96.

Zero is the number next after 1, if we count downwards. It is written 0. For multiplication and division by 0, see p. 77.

PART II

CHAPTER XII.

INDICES AND IRRATIONALS.

INDICES.

Definition. If n is a positive integer (not zero), a^n is called the nth *power* of a and represents $a \times a \times a \times \ldots$ to n factors; the number n is called the *index* of a.

Note that the index of a itself is 1, for a really stands for a^1.

Example I. Prove that $a^7 \div a^3 = a^4$.

$$\frac{a^7}{a^3} = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a}$$
$$= a \times a \times a \times a = a^4.$$

Example II. Prove that $(a^3)^4 = a^{12}$.

$$(a^3)^4 = a^3 \times a^3 \times a^3 \times a^3$$
$$= (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a)$$
$$= a^{12}.$$

EXERCISE XII. a.

1. Prove *in full* that

(i) $a^2 \times a^3 = a^5$; (ii) $a^3 \times a^4 = a^7$;
(iii) $a^8 \div a^4 = a^4$; (iv) $a^{12} \div a^3 = a^9$.

2. Write the following in their simplest forms:

(i) $a^3 \times a^5$; (ii) $a^6 \div a^3$; (iii) $a \times a^5$; (iv) $a^{10} \div a^2$;
(v) $a^7 \times a^9$; (vi) $a^8 \times a^4$; (vii) $a^{12} \div a$; (viii) $x^{20} \div x^5$.

3. In what general statement or formula are the following special cases included?

$$a^2 \times a^8 = a^{10}; \quad a^{20} \times a^{41} = a^{61}; \quad a \times a^{80} = a^{81}.$$

4. In what general statement or formula are the following special cases included ?

$$a^7 \div a^4 = a^3\,; \quad a^{100} \div a^{10} = a^{90}\,; \quad a^{13} \div a = a^{12}.$$

5. Prove *in full* that

(i) $(a^2)^5 = a^{10}$; (ii) $(2x^2)^3 = 8x^6$.

6. Simplify

(i) $(a^4)^5$; (ii) $(a^{10})^{10}$; (iii) $(a^8)^6$; (iv) $(3a^9)^3$.

7. In what general statement or formula are the following special cases included ?

$$(a^5)^2 = a^{10}\,; \quad (a^{20})^9 = a^{180}\,; \quad (a^{100})^7 = a^{700}.$$

8. By what must x^7 be multiplied to give x^{21} ?

9. To what power must x^7 be raised to give x^{21} ?

10. By what must x^{50} be divided to give x^{10} ?

11. What is the square of x^4 ?

12. Is $(x^3)^4$ the same as $(x^4)^3$?

13. Is $(x^6)^{10}$ the same as $(x^3)^{20}$?

14. Divide $x^{15} \times x^{10}$ by x^5.

15. Divide $(x^2)^5$ by x^3.

16. Divide $(a^6)^4$ by $(a^4)^6$.

17. Multiply $(a^3)^4$ by $(a^5)^5$.

18. Divide $(a^7)^5$ by a.

19. What is the cube of $4x^4$?

20. Divide the cube of $2x^2$ by the square of $2x^3$.

21. How would the following facts be expressed without using the index notation ?

(i) the velocity of light is $3{\cdot}002 \times 10^{10}$ cm. per sec. ;

(ii) the population of London in 1920 was about $4{\cdot}5 \times 10^6$;

(iii) the value of steel imports in 1913 was about £$12{\cdot}2 \times 10^6$;

(iv) the distance of α Centauri is $2{\cdot}6 \times 10^{13}$ miles ;

(v) $2 \times 10^3 \times 4{\cdot}5 \times 10^6 = 9 \times 10^9$.

22. Write in a shorter form :

(i) 62,800,000 ; (ii) $67{\cdot}1 \times 67{\cdot}1 \times 67{\cdot}1 \times 1{,}000{,}000$.

SUMMARY OF RESULTS.

The examples of the above exercise illustrate the following facts:

If m, n are positive integers,

$$\text{(i)}\quad a^m \times a^n = a^{m+n};$$

$$\text{(ii)}\quad a^m \div a^n = a^{m-n}, \quad \text{for } m > n;$$

$$\text{(iii)}\quad (a^m)^n = a^{mn}.$$

The proofs of these general results follow the methods given in the worked-out examples on page 253 for special cases.

If m is not a positive integer, *e.g.* if $m = \frac{3}{4}$ or $m = -5$ or $m = -\frac{2}{3}$, the symbol a^m has as yet no meaning. We are now going to find out what meaning must be given to it, in order that the relations (i), (ii), (iii) may always hold good.

Note.—The symbol 0^0 is excluded from consideration and has no meaning.

Definition. If n is a positive integer (not zero), a is called *an* nth *root* of a^n.

Note the expression, *an* nth root; a number has more than one nth root; for example, 25 has *two* square roots, viz. $+5$ and -5.

If there is a positive nth root of x it is written $\sqrt[n]{x}$.

Example III. Assign a meaning to x^{-3}.

$$x^5 \div x^8 = x^{5-8} = x^{-3}, \quad \text{from (ii)}.$$

But $$x^5 \div x^8 = \frac{x^5}{x^8} = \frac{1}{x^3}; \qquad \therefore\ x^{-3} = \frac{1}{x^3}.$$

Example IV. Assign a meaning to $x^{\frac{3}{4}}$.

$$x^{\frac{3}{4}} \times x^{\frac{3}{4}} \times x^{\frac{3}{4}} \times x^{\frac{3}{4}} = x^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = x^{\frac{12}{4}}, \quad \text{from (i)},$$
$$= x^3;$$

$$\therefore\ x^{\frac{3}{4}} \text{ is a fourth root of } x^3.$$

This may be written, $x^{\frac{3}{4}} = \sqrt[4]{x^3}$.

Or, we may say $$(x^{\frac{3}{4}})^4 = x^{\frac{3}{4}\times 4}, \quad \text{from (iii)},$$
$$= x^3;$$

$$\therefore\ x^{\frac{3}{4}} \text{ is a fourth root of } x^3.$$

Example V. Assign a meaning to x^0.

$$x^4 \div x^4 = x^{4-4} = x^0, \qquad \text{from (ii).}$$

But $$x^4 \div x^4 = \frac{x^4}{x^4} = 1;$$

$$\therefore\ x^0 = 1.$$

Example VI Prove that $16^{\frac{3}{4}} = \sqrt[4]{16^3} = (\sqrt[4]{16})^3$, and find its value.

Exactly as in Example IV., we see that

$$16^{\frac{3}{4}} \times 16^{\frac{3}{4}} \times 16^{\frac{3}{4}} \times 16^{\frac{3}{4}} = 16^{\frac{12}{4}} = 16^3;$$

$$\therefore\ 16^{\frac{3}{4}} = \sqrt[4]{16^3}.$$

Again, $$16^{\frac{3}{4}} = 16^{\frac{1}{4}} \times 16^{\frac{1}{4}} \times 16^{\frac{1}{4}} = (16^{\frac{1}{4}})^3.$$

But $$16^{\frac{1}{4}} = \sqrt[4]{16};$$

$$\therefore\ 16^{\frac{3}{4}} = (\sqrt[4]{16})^3.$$

It is easier to compute $(\sqrt[4]{16})^3$ than $\sqrt[4]{16^3}$, as it involves smaller numbers.

$$16^{\frac{1}{4}} = \sqrt[4]{16} = 2;$$

$$\therefore\ 16^{\frac{3}{4}} = 2^3 = 8.$$

EXERCISE XII. b.

1. Express as powers of 10 :

(i) $10^3 \div 10$; (ii) $10^3 \div 10^2$; (iii) $10^3 \div 10^3$;

(iv) $10^3 \div 10^4$; (v) $10^3 \div 10^5$.

What are the values of these expressions ?

2. Express as powers of 2 :

(i) $2^4 \div 2$; (ii) $2^4 \div 2^2$; (iii) $2^4 \div 2^3$;

(iv) $2^4 \div 2^4$; (v) $2^4 \div 2^5$; (vi) $2^4 \div 2^6$.

What are the values of these expressions ?

3. Write down 7^5; divide it by 7, and write the quotient in the index form. Repeat this process 8 times. What are the successive quotients expressed in the index form ? What are their values ?

4. Find n if

(i) $x^{10} \div x^3 = x^n$; (ii) $x^{10} \div x^{15} = x^n$;

(iii) $x^{10} \div x^n = x^{-7}$; (iv) $x^{10} \div x^{10} = x^n$.

5. Find n if

(i) $x^4 \times x^7 = x^n$; (ii) $x^5 \times x^{-2} = x^n$; (iii) $x^3 \div x^5 = x^n$;

(iv) $x^n \div x^4 = x^6$; (v) $x^n \times x^3 = x^3$.

6. Express as powers of x, *i.e.* in the form x^n,

(i) $\dfrac{x^6}{x^2}$; (ii) $\dfrac{x^6}{x^9}$; (iii) $\dfrac{x^6}{x^6}$.

7. Express as powers of x

(i) $\dfrac{x^3}{x^8}$; (ii) $\dfrac{1}{x^5}$; (iii) $\dfrac{x^3}{x^3}$; (iv) 1.

8. Use the method of Example III. to assign a meaning to x^{-12}.

9. Simplify

(i) $a^7 \times a^0$; (ii) $a^7 \times 1$; (iii) $a^7 \times a^1$; (iv) $a^7 \times a$:

(v) $a^7 \times 0$; (vi) $a^0 \times 1$; (vii) $a^0 \times a$.

10. Express with positive indices

(i) a^{-2}; (ii) a^{-1}; (iii) $\left(\dfrac{1}{a}\right)^{-3}$; (iv) $a^{-1} \times a^{-1}$.

11. What are the values of

(i) 3^{-2}; (ii) 2^{-3}; (iii) 4^{-1}; (iv) 5^0;

(v) $(\frac{1}{4})^{-2}$; (vi) $(\frac{2}{3})^{-4}$; (vii) $(0{\cdot}1)^{-2}$; (viii) 1^{-3};

(ix) 1^6; (x) $(\frac{2}{3})^{-1}$.

12. If $a=2$, $b=3$, write down the values of

(i) $a^{-1}+b^{-1}$; (ii) $(a+b)^{-1}$; (iii) ab^{-1}; (iv) $(a-b)^{-1}$.

13. Find n if

(i) $x^n \times x^n = x^{16}$; (ii) $x^n \times x^n = x^{14}$; (iii) $x^n \times x^n = x^{15}$.

14. Find n if

(i) $x^6 \times x^6 = x^n$; (ii) $x^7 \times x^7 = x^n$; (iii) $x^{6\frac{1}{2}} \times x^{6\frac{1}{2}} = x^n$.

15. Find n if

(i) $x^n \times x^n = x$; (ii) $x^n \times x^n = 1$.

16. Express as powers of x the square roots of

(i) x^8; (ii) x^7; (iii) x.

17. Simplify

(i) $x^{\frac{2}{3}} \times x^{\frac{2}{3}} \times x^{\frac{2}{3}}$; (ii) $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}}$.

What is the meaning of $x^{\frac{2}{3}}$ and of $x^{\frac{1}{3}}$?

18. Assign a meaning, with reasons, to (i) $x^{\frac{1}{4}}$; (ii) $x^{\frac{3}{4}}$.

19. Find n if

(i) $(x^5)^2 = x^n$; (ii) $(x^{\frac{4}{3}})^3 = x^n$; (iii) $(x^{\frac{1}{2}})^2 = x^n$.

20. Find n if

(i) $(x^n)^3 = x^{27}$; (ii) $(x^n)^5 = x^3$; (iii) $(x^n)^6 = x^4$.

21. What are the values of

(i) $16^{\frac{1}{2}}$; (ii) $9^{\frac{3}{2}}$; (iii) $8^{\frac{1}{3}}$; (iv) $27^{\frac{2}{3}}$.

22. What are the values of

(i) $64^{\frac{4}{3}}$; (ii) $16^{1\cdot 25}$; (iii) $81^{\frac{3}{4}}$; (iv) $8^{\frac{1}{3}}$; (v) $16^{\frac{6}{4}}$.

23. If $a = 16$, $b = 9$, find the values of

(i) $a^{\frac{1}{2}} + b^{\frac{1}{2}}$; (ii) $(a+b)^{\frac{1}{2}}$; (iii) $a^{-\frac{1}{2}} + b^{-\frac{1}{2}}$; (iv) $(a+b)^{-\frac{1}{2}}$.

24. What are the values of

(i) $8^{-\frac{1}{3}}$; (ii) $4^{-\frac{3}{2}}$; (iii) $16^{-\frac{3}{4}}$; (iv) $100^{-1\cdot 5}$; (v) $1^{-\frac{1}{4}}$.

25. Express as powers of x

(i) $\sqrt{x^3}$; (ii) $\sqrt[4]{x^{11}}$; (iii) $\sqrt[6]{x^3}$.

26. Simplify (i) $2x^{\frac{1}{2}} \times 3x^{\frac{1}{3}}$; (ii) $x \times x^{-\frac{1}{2}}$

27. Give reasons for the statement $x^{-\frac{1}{2}} = \dfrac{1}{\sqrt{x}}$.

28. Assign a meaning, with reasons, to

(i) $x^{-\frac{3}{4}}$; (ii) $x^{-2\frac{1}{2}}$.

29. By using tables, write down the values of

(i) $2^{0\cdot 5}$; (ii) $2^{0\cdot 25}$; (iii) $2^{0\cdot 125}$; (iv) $2^{0\cdot 0625}$; (v) $2^{0\cdot 03125}$

What is the effect of repeating this process indefinitely ?

30. By using tables, write down the values of

(i) $10^{0\cdot 5}$; (ii) $10^{0\cdot 25}$; (iii) $10^{0\cdot 125}$; (iv) $10^{0\cdot 0625}$.

What is the effect of repeating this process indefinitely ?

31. Express the following facts without using negative indices :

(i) The charge of an electron is $4\cdot 7 \times 10^{-10}$ units.

(ii) The diameter of an electron is $0\cdot 3 \times 10^{-12}$ cm.

(iii) The mass of an atom of hydrogen is $1\cdot 6 \times 10^{-24}$ gr.

(iv) The diameter of an atom of oxygen is 3×10^{-8} cm.

(v) The time between two collisions of molecules of hydrogen is $1\cdot 06 \times 10^{-10}$ seconds.

(vi) The force of attraction between two grams at a distance 1 cm. apart is $0\cdot 648 \times 10^{-7}$ dynes.

(vii) The wave length of yellow light is 27×10^{-6} inches.

32. Express the following without using fractional or negative indices:

(i) lt^{-2}; mt^{-3}; ml^2t^{-2}.

(ii) $d=kt^{-3}$; $R=\mu^{-\frac{1}{2}}a^{\frac{3}{2}}$; $d=kH^{\frac{1}{3}}N^{-\frac{1}{3}}$.

(iii) $\frac{l}{L}\left(\frac{T}{t}\right)^{-2}$; $\left(\frac{l_1}{l_2}\right)^{-\frac{1}{2}}$; $pv^{0\cdot 25}$.

(iv) $a^{\frac{1}{2}}+b^{\frac{1}{2}}=c^{\frac{1}{2}}$; $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$; $t=2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}$.

(v) $y=e^{-kt}$ when $k=8$, $t=0\cdot 5$.

33. If x is small (compared with 1), $(1+x)^n$ is approximately equal to $1+nx$.

Use this fact to find approximate values for

(i) $(1+\cdot 01)^{\frac{1}{2}}$; (ii) $\sqrt{1\cdot 02}$; (iii) $(1\cdot 01)^{-1}$;

(iv) $\frac{1}{\sqrt{1\cdot 02}}$; (v) $(0\cdot 99)^{\frac{1}{2}}$; (vi) $\frac{1}{0\cdot 99}$;

(vii) $\frac{1}{(1\cdot 001)^2}$; (viii) $\frac{1}{\sqrt{104}}$;

(ix) $\sqrt[3]{1003}$; (x) $\frac{1}{(1003)^2}$.

34. The "derived function" of x^n with respect to x is nx^{n-1} for all values of n. Use this fact to find the "derived functions" with respect to x of

(i) x^3; (ii) $x^{\frac{3}{2}}$; (iii) x^{-2}; (iv) $\sqrt{x}$;

(v) $\frac{1}{x^5}$; (vi) $\frac{1}{\sqrt{x}}$; (vii) x^0; (viii) $\frac{1}{\sqrt[3]{x^5}}$.

35. The "integral" of x^n with respect to x is $\frac{1}{n+1}\cdot x^{n+1}$ for all values of n, except $n=-1$. Use this fact to find the integrals with respect to x of

(i) x^2; (ii) $x^{\frac{1}{2}}$; (iii) x^{-2}; (iv) $\sqrt[3]{x}$; (v) $\frac{1}{x^3}$;

(vi) $\sqrt[2]{x^3}$; (vii) x; (viii) x^0; (ix) $\frac{1}{\sqrt[3]{x}}$.

SUMMARY OF RESULTS.

The examples of the above exercise show what meanings must be given to a^n, when n is zero, fractional or negative, if the laws (i), (ii), (iii) on page 255 hold good.

The necessary interpretations are :

$$\text{(i)}\quad x^0 = 1\,;$$

$$\text{(ii)}\quad x^{-n} = \frac{1}{x^n}\,;$$

$$\text{(iii)}\quad x^{\frac{p}{q}} = \sqrt[q]{x^p} = (\sqrt[q]{x})^p.$$

In particular,

$$x^{\frac{1}{q}} = \sqrt[q]{x}\,;$$

$$x^{-\frac{p}{q}} = \frac{1}{x^{\frac{p}{q}}} = \frac{1}{\sqrt[q]{x^p}}.$$

The proofs of the general results follow the methods given in the worked-out examples on page 255.

EXERCISE XII. c.

1. Write with fractional indices

(i) $\sqrt[3]{x}$; (ii) $(\sqrt[3]{a})^2$; (iii) $(\sqrt[6]{a})^4$; (iv) $\sqrt{ab}$;

(v) $2\sqrt{ab^2}$; (vi) $3\sqrt{ab^3}$; (vii) $\sqrt{ab^2}$.

2. Write the following so that there is no denominator :

(i) $\frac{3}{x^2}$; (ii) $\frac{1}{x}$; (iii) $\frac{1}{xy}$; (iv) $\frac{5}{x\sqrt{y}}$.

3 Find the value of $p^{\frac{r-1}{r}}$ when (i) $r = 1{\cdot}5,\ p = 64$;

(ii) $r = 1{\cdot}25,\ p = 32$.

4. Find n, if (i) $2^n = 8$; (ii) $8^n = 2$; (iii) $3^n = \frac{1}{9}$;

(iv) $5^n = \frac{1}{5\sqrt{5}}$; (v) $(0{\cdot}01)^n = 10$.

5. Simplify (i) $2a^{\frac{1}{2}} \times a$; (ii) $a^{\frac{1}{4}} \times a^{\frac{1}{6}}$; (iii) $ab^{\frac{1}{2}} \times a^{\frac{1}{2}}b$.

6. Simplify (i) $a^{\frac{3}{2}} \div a$; (ii) $4a^{\frac{5}{2}} \div 2a$; (iii) $a \div a^{\frac{1}{3}}$.

7. Simplify (i) $a^2 \div \sqrt{a}$; (ii) $\sqrt[4]{a^3} \div (\sqrt[4]{a})^3$.

8. Multiply $x + x^{-1}$ by x.

9. Divide $\sqrt{x}$ by $\sqrt[3]{x}$.

10. Simplify $\frac{x - x^{-1}}{x - 1}$.

11. Find the value of $\sqrt[5]{32^4}$.

12. Simplify $\frac{1}{x^{-\frac{1}{2}}} \times \sqrt{x^3}$. **13.** If $16^n = 1$, find n.

14. Simplify

(i) $100^{2n} \div 10^n$; (ii) $\left(\frac{25x^4}{2\frac{1}{4}y^6}\right)^{-\frac{1}{2}}$; (iii) $(x^3)^{-2}$;

(iv) $(x^{-3})^{-3}$; (v) $\frac{1}{(x+y)^{-2}}$.

15. Simplify (i) $9^{\frac{1}{2}} + 16^{\frac{1}{2}}$; (ii) $(25 + 144)^{\frac{1}{2}}$;

(iii) $\frac{1}{8^{-\frac{1}{3}}} - \frac{1}{27^{-\frac{1}{3}}}$; (iv) $(4^n + 8^{2n}) \div 2^n$.

16. Solve $9^n = \frac{9}{3^n}$.

IRRATIONAL NUMBERS.

$\sqrt{2}$ is defined as a number whose square is 2. This number cannot be expressed as a fraction in the form $\frac{p}{q}$, where p and q are integers. It is, in fact, a new type of number, and is called an "*Irrational*" number.

Irrational numbers obey the ordinary laws of algebra, *e.g.*

$$\sqrt{a} \times \sqrt{b} = \sqrt{b} \times \sqrt{a} = \sqrt{ab};$$
$$\sqrt{a} + \sqrt{b} = \sqrt{b} + \sqrt{a};$$
$$\sqrt{a}(\sqrt{x} + \sqrt{y}) = \sqrt{ax} + \sqrt{ay}.$$

Example VII. Given that $\sqrt{2} = 1{\cdot}414$ approximately, find $\sqrt{98}$.

$$\begin{aligned}\sqrt{98} &= \sqrt{49} \times \sqrt{2}\\ &= 7\sqrt{2}\\ &= 9{\cdot}898 \text{ approx. or } 9{\cdot}90 \text{ correct to 3 figures.}\end{aligned}$$

Example VIII. Find $\frac{10}{\sqrt{2}}$ correct to 3 figures.

$$\frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2} = 7{\cdot}07 \text{ approx.}$$

It is simpler to "rationalise" the denominator in this way than to divide 10 by 1·414, the approximate value of $\sqrt{2}$.

Example IX. Evaluate $\frac{\sqrt{3}+2}{\sqrt{3}-1}$, given $\sqrt{3}=1{\cdot}732$.

Rationalise the denominator by multiplying both numerator and denominator by $\sqrt{3}+1$.

Then
$$\begin{aligned}\frac{\sqrt{3}+2}{\sqrt{3}-1}&=\frac{(\sqrt{3}+2)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}\\&=\frac{(\sqrt{3})^2+3\sqrt{3}+2}{(\sqrt{3})^2-1}\\&=\frac{5+3\sqrt{3}}{2}=\frac{5+5{\cdot}196}{2}\text{ approx.}\\&=\frac{10{\cdot}196}{2}\\&=5{\cdot}10\text{ correct to 3 figures.}\end{aligned}$$

EXERCISE XII. d.

1. Simplify

(i) $\sqrt{9}+\sqrt{16}$; (ii) $\sqrt{9+16}$; (iii) $\sqrt{9}\times\sqrt{16}$;

(iv) $\sqrt{9\times16}$; (v) $\sqrt{36}\div\sqrt{9}$; (vi) $\sqrt{36\div9}$.

2. Simplify

(i) $\sqrt{6}\times\sqrt{24}$; (ii) $\sqrt{32}\times\sqrt{18}$; (iii) $\sqrt{18}\div\sqrt{200}$.

3. Prove that $\sqrt{8}=2\sqrt{2}$; express in a similar way

(i) $\sqrt{18}$; (ii) $\sqrt{72}$; (iii) $\sqrt{50}$;

(iv) $\sqrt{2000}$; (v) $\sqrt{ab^2}$; (vi) $\sqrt{\left(\frac{a}{x^2}\right)}$.

4. Given $\sqrt{3}=1{\cdot}732$, write down to 3 significant figures the values of

(i) $\sqrt{300}$; (ii) $\sqrt{27}$; (iii) $\sqrt{0{\cdot}75}$.

5. Prove that $\frac{6}{\sqrt{3}}=2\sqrt{3}$; use this idea to obtain to 3 significant figures the values of (i) $\frac{1}{\sqrt{3}}$; (ii) $\frac{5}{\sqrt{12}}$.

6. Express without radical signs in the denominator

(i) $\frac{18}{\sqrt{2}}$; (ii) $\frac{5}{\sqrt{2}}$; (iii) $\frac{100}{\sqrt{10}}$; (iv) $\frac{6}{\sqrt{12}}$; (v) $\frac{b^2}{\sqrt{b}}$.

7. Evaluate to 3 significant figures, given

$$\sqrt{2}=1{\cdot}414,\quad \sqrt{5}=2{\cdot}236.$$

(i) $\sqrt{50}$; (ii) $\sqrt{500}$; (iii) $\sqrt{5000}$; (iv) $\frac{2}{\sqrt{5}}$.

8. (i) Simplify $(\sqrt{2}+1)(\sqrt{2}-1)$.

(ii) Evaluate to 3 figures $\dfrac{1}{\sqrt{2}-1}$.

9. (i) Simplify $(5\sqrt{2}-2\sqrt{3})(5\sqrt{2}+2\sqrt{3})$.

(ii) Express with a rational denominator $\dfrac{6}{5\sqrt{2}+2\sqrt{3}}$.

10. Evaluate to 3 figures

(i) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$; (ii) $\dfrac{1}{\sqrt{3}-\sqrt{2}}$; (iii) $\dfrac{6-2\sqrt{5}}{\sqrt{5}-1}$.

11. Given $\sqrt[3]{10} \simeq 2{\cdot}154$, find to 3 figures $\sqrt[3]{80}$.

12. Evaluate x^2+2x when $x=\sqrt{2}-1$.

13. Simplify

(i) $\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{\sqrt{2}-1}$; (ii) $\dfrac{\sqrt{6}}{(\sqrt{3}+1)(\sqrt{6}-\sqrt{2})}$.

14. Solve $x\sqrt{2}+x=7$: answer to 3 figures.

15. Evaluate $\left(x+\dfrac{1}{x}\right)^2$ if $x=\sqrt{6}-\sqrt{5}$.

16. The perimeter of an isosceles right-angled triangle is 5 inches; find, correct to $\frac{1}{100}$ inch, the length of each side.

17. The area of an equilateral triangle is 6 sq. inches; find correct to $\frac{1}{100}$ inch, the length of a side.

18. Draw a triangle ABC such that $AB=x$ in., $BC=\frac{1}{2}x$ in., $\angle ABC=90°$; from CA cut off CY equal to CB; from AB cut off AX equal to AY; prove that $AB \,.\, BX=AX^2$.

19. AB is the diameter of a circle, centre O; ABC is an equilateral triangle; CO cuts the circle at P; prove that the tangent from C to the circle equals AP.

20. OA, OB are two perpendicular radii of a circle; another circle is drawn touching OA, OB and the arc AB internally; prove that its radius $=OA\,(\sqrt{2}-1)$.

21. (1) What is the square of (i) $\sqrt{a}+\sqrt{b}$, (ii) $\sqrt{a+b}$?

(2) Is it possible for $x+y$ and $\sqrt{x^2+y^2}$ to be equal?

(3) Interpret this result geometrically with reference to a right-angled triangle.

22. (1) What is the square of (i) $\sqrt{x}\times\sqrt{y}$, (ii) $\sqrt{xy}$?

(2) Is $\sqrt{x}\times\sqrt{y}$ equal to $\sqrt{xy}$?

23. Simplify (i) $\dfrac{1}{\sqrt{x}+\sqrt{y}}+\dfrac{1}{\sqrt{x}-\sqrt{y}}$;

(ii) $\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}$.

SUMMARY OF RESULTS.

(i) $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$.

(ii) $\sqrt{a}+\sqrt{b}$ is *not* equal to $\sqrt{a+b}$, unless a or b is zero.

(iii) $\dfrac{a}{\sqrt{b}}=\dfrac{a\sqrt{b}}{\sqrt{b}\times\sqrt{b}}=\dfrac{a\sqrt{b}}{b}$.

(iv) $\dfrac{a}{\sqrt{b}+\sqrt{c}}=\dfrac{a(\sqrt{b}-\sqrt{c})}{(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})}=\dfrac{a(\sqrt{b}-\sqrt{c})}{b-c}$.

Example X. Draw the graph of 2^x from $x=-3$ to $x=+3$

When $x=3$, $2^x=2^3=8$. When $x=0$, $2^x=2^0=1$.

When $x=-3$, $2^x=2^{-3}=\dfrac{1}{2^3}=\dfrac{1}{8}=0{\cdot}125$.

When $x=0{\cdot}5$, $2^x=2^{\frac{1}{2}}=\sqrt{2}=1{\cdot}414$ approx.

When $x=1{\cdot}5$, $2^x=2^{1\frac{1}{2}}=2\times2^{\frac{1}{2}}=2\times\sqrt{2}=2{\cdot}828$ approx.

When $x=2{\cdot}5$, $2^x=2^{2\frac{1}{2}}=2^2\times2^{\frac{1}{2}}=4\times\sqrt{2}=5{\cdot}657$ approx.

We can now write down the following table of values :

x	-3	-2	-1	0	0·5	1	1·5	2	2·5	3
2^x	0·125	0·25	0·5	1	1·41	2	2·83	4	5·66	8

Plotting these values, we obtain the following graph :

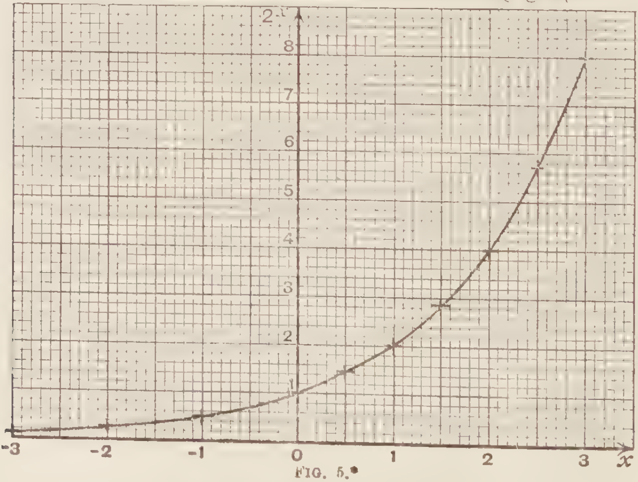

FIG. 5.*

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* Figs. 1-4 appear in the Introduction.

We note that

(i) 2^x increases without limit as x increases ;

(ii) 2^x is positive for all values of x ;

(iii) for x positive $2^x > 1$,
for $x = 0$, $2^x = 1$,
for x negative $2^x < 1$;

(iv) as x is given increasingly large *negative* values, 2^x tends continually closer to the value 0.

EXERCISE XII. e.

1. Read off from the graph of 2^x approximate values of

(i) $2^{0\cdot 2}$; (ii) $2^{-0\cdot 5}$; (iii) $4^{1\cdot 1}$.

2. Use the graph of 2^x to solve $2^x = 5$.

3. What is the greatest integral value of x for which

(i) $2^x < 0\cdot 01$; (ii) $2^x < 0\cdot 001$?

4. Show that $10^{1\cdot 6} < 100$. **5.** Show that $1 > 10^{-0\cdot 8} > 0\cdot 1$.

6. Show that $4^{2\cdot 9} < 64$. **7.** Show that $8^{1\cdot 3} < 16$.

8. Make a rough estimate of the values of

(i) $8^{0\cdot 33}$; (ii) $32^{-0\cdot 39}$; (iii) $10^{0\cdot 00002}$;
(iv) $10^{0\cdot 99}$; (v) $10^{-2\cdot 001}$; (vi) $4^{0\cdot 497}$.

9. (i) By using such facts as $10^{0\cdot 5} = \sqrt{10}$, $10^{0\cdot 25} = \sqrt{(\sqrt{10})}$, $10^{0\cdot 75} = 10^{\frac{3}{4}} = 10^{\frac{1}{2}} \times 10^{\frac{1}{4}}$, etc., compute the values needed for filling in the following table :

$x = 0$	$0\cdot 125$	$0\cdot 25$	$0\cdot 375$	$0\cdot 5$	$0\cdot 625$	$0\cdot 75$	$0\cdot 875$	1
$10^x =$								

(ii) Hence draw the graph of 10^x from $x = 0$ to $x = 1$.

(iii) From your graph, read off the values of $10^{0\cdot 2}$, $10^{0\cdot 4}$, $10^{0\cdot 7}$, $10^{0\cdot 9}$.

(iv) From your graph, read off the values of x for which

(*a*) $10^x = 3$; (*b*) $10^x = 5$; (*c*) $10^x = 8\cdot 1$.

(v) From your graph, read off the values of

(*a*) $\sqrt[5]{10}$; (*b*) $\sqrt[3]{100}$; (*c*) $\sqrt[5]{1000}$.

10. Given that $2 = 10^{0·301}$ approximately, solve the equations

(i) $10^x = 8$; (ii) $5^x = 10$; (iii) $10^x = 200$; (iv) $10^x = 50$.

11. Without making a table of values, sketch roughly the graph of 5^x from $x = 2$ to $x = -1$.

12. Given that $9 = 10^{0·9542}$, express as powers of 10

(i) 3; (ii) 0·9; (iii) 2·7.

CHAPTER XIII.

LOGARITHMS.

We saw in the last chapter (Ex. XII. e. No. **9** (i)) how to construct the graph of 10^x.

Here is the necessary table of values :

$x = 0$	0·125	0·25	0·375	0·5	0·625	0·75	0·875	1
$10^x = 1$	1·33	1·78	2·37	3·16	4·22	5·62	7·50	10

Plotting them, we have the following graph :

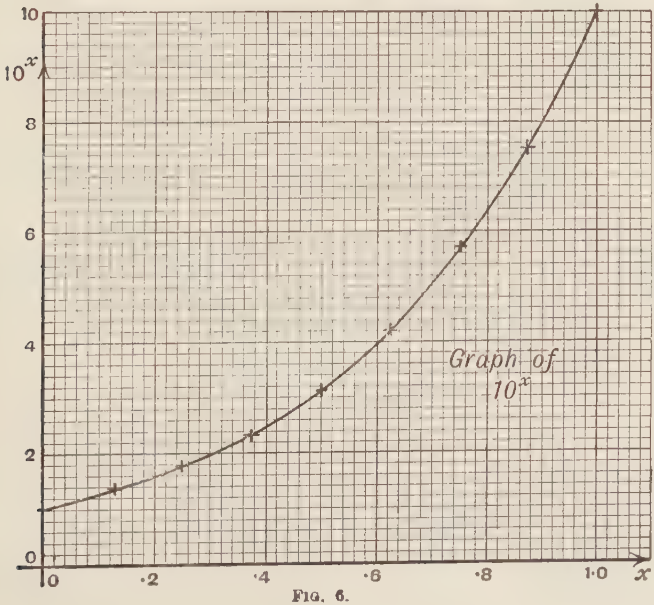

Fig. 6.

From this graph we can express any number between 1 and 10 as a power of 10,

e.g. $4 = 10^{0{\cdot}602}$ approximately.

Definition. If $a = 10^x$, x is called the logarithm of a to base 10.

For example, (i) since $4 = 10^{0{\cdot}602}$,
the logarithm of 4 is 0·602;
(ii) since $1000 = 10^3$,
the logarithm of 1000 is 3;
(iii) since $1 = 10^0$,
the logarithm of 1 is 0.

EXERCISE XIII. a.

1. (i) Find from the graph the value of x for which $10^x = 7$.
(ii) What is the logarithm of 7?

2. (i) Use the graph to express 3·5 as a power of 10.
(ii) What is the logarithm of 3·5?

3. (i) Given that $2 = 10^{0{\cdot}301}$, express as powers of 10
(*a*) 20; (*b*) 200; (*c*) 2000; (*d*) 2,000,000.
(ii) What are the logarithms of
(*a*) 20; (*b*) 200; (*c*) 2000; (*d*) 2,000,000?

4. What are the logarithms of (i) 100; (ii) 1,000,000?

5. Given that $4{\cdot}342 = 10^{0{\cdot}6377}$, find the logarithms of
(i) 43·42; (ii) 434·2; (iii) 43,420; (iv) 4,342,000.

6. Given that $7{\cdot}286 = 10^{0{\cdot}8625}$, find the values of
(i) $10^{1{\cdot}8625}$; (ii) $10^{2{\cdot}8625}$; (iii) $10^{4{\cdot}8625}$; (iv) $10^{6{\cdot}8625}$.

7. What is x if (i) $27{\cdot}2 = 10^x \times 2{\cdot}72$;
(ii) $270{\cdot}2 = 10^x \times 2{\cdot}702$;
(iii) $71{,}000 = 10^x \times 7{\cdot}1$;
(iv) $568{,}123 = 10^x \times 5{\cdot}68$ approximately?

8. What is the whole number in the index when each of the following numbers is expressed as a power of 10?
(i) 73; (ii) 7142·3; (iii) 8·61; (iv) $5{\cdot}6 \times 10^5$;
(v) 100,000; (vi) 1234; (vii) 12,345; (viii) 123,456;
(ix) 12,345,678; (x) 123,456,789.

9. (i) Is $10^{0\cdot78}$ greater or less than 10 ?
  (ii) Is $10^{0\cdot78}$ greater or less than 1 ?
  (iii) Is $10^{5\cdot634}$ greater or less than a million ?
  (iv) Is $10^{4\cdot81}$ greater or less than 10,000 ?

10. (i) Is 103 greater or less than $10^{1\cdot9}$?
  (ii) Is 863·84 greater or less than $10^{3\cdot2}$?
  (iii) Is 7·53 greater or less than $10^{1\cdot1}$?

11. Use the graph to write down approximate values for
(i) $10^{2\cdot85}$; (ii) $10^{4\cdot71}$; (iii) $10^{7\cdot15}$.

12. (i) Use the graph to express (approximately) as powers of 10
(*a*) 29 ; (*b*) 6100 ; (*c*) 8,400,000.

(ii) Use the graph to obtain approximate values for the logarithms of
(*a*) 33 ; (*b*) 54,000,000 ; (*c*) 7400.

SUMMARY OF RESULTS.

The examples in Exercise XIII. a. lead us to the following conclusions :

(1) The logarithm of any number between 1 and 10 has a value between 0 and 1,

e.g. the logarithm of 2 is 0·301 approx.

(2) Moving the decimal point in a number affects the integral part of its logarithm, but has no effect on the figures in the logarithm after the decimal point ;

e.g. the logarithm of 5·64 is 0·7513 ;
∴ the logarithm of 56·4 is 1·7513,
and the logarithm of 56,400 is 4·7513.

(3) The integral part of the logarithm can always be found by inspection ;

e.g. $738{,}100 = 10^5 \times 7\cdot381$;

∴ the integral part of the logarithm of 738,100 is 5.

Note.—The integral part of a logarithm is called its *characteristic.*

The figures in a logarithm after the decimal point are called its *mantissa*.

e.g. the logarithm of 738,100 is 5·8682.

The integer 5 is the characteristic.

The decimal fraction ·8682 is the mantissa.

Since the characteristic can be found by inspection, it is only the mantissa that is printed in logarithm tables.

For a set of logarithm 4-figure tables, see page 552. This table gives logarithms of numbers correct to 4 significant figures. It should be noted that if 4-figure tables are employed for computation, the answer is not necessarily *correct* to 4 figures, because the figures in the fifth place which have been neglected may, as the result of a series of additions or multiplications, affect the figure in the fourth place in the answer.

Example I. Find correct to 3 significant figures the value of

$$3{\cdot}142 \times 41{\cdot}27.$$

From the tables

$$\begin{aligned} 3{\cdot}142 \times 41{\cdot}27 &= 10^{0{\cdot}4972} \times 10^{1{\cdot}6156} \\ &= 10^{0{\cdot}4972+1{\cdot}6156} \\ &= 10^{2{\cdot}1128} \\ &= 129{\cdot}6 \\ &= 130 \text{ to 3 sig. fig.} \end{aligned} \qquad \begin{array}{r} 1{\cdot}6156 \\ {\cdot}4972 \\ \hline 2{\cdot}1128 \\ \hline \end{array}$$

Rough estimate, 3 × 40 = 120.

Example II. Find correct to 3 significant figures the value of

$$407{\cdot}9 \div 14{\cdot}56.$$

From the tables

$$\begin{aligned} 407{\cdot}9 \div 14{\cdot}56 &= 10^{2{\cdot}6106} \div 10^{1{\cdot}1632} \\ &= 10^{2{\cdot}6106-1{\cdot}1632} \\ &= 10^{1{\cdot}4474} \\ &= 28{\cdot}02 \\ &= 28{\cdot}0 \text{ to 3 sig. fig.} \end{aligned} \qquad \begin{array}{r} 2{\cdot}6106 \\ 1{\cdot}1632 \\ \hline 1{\cdot}4474 \\ \hline \end{array}$$

Rough estimate, 400 ÷ 14 = about 30.

Example III. Find correct to 3 significant figures the value of

$$\frac{31{\cdot}7 \times 920000}{415{\cdot}1 \times 1{\cdot}6}.$$

From the tables, we find the expression

$$
\begin{aligned}
&= \frac{10^{1\cdot5011} \times 10^{5\cdot9638}}{10^{2\cdot6181} \times 10^{0\cdot2041}} \\
&= \frac{10^{1\cdot5011+5\cdot9638}}{10^{2\cdot6181+0\cdot2041}} \\
&= \frac{10^{7\cdot4649}}{10^{2\cdot8222}} = 10^{7\cdot4649-2\cdot8222} \\
&= 10^{4\cdot6427} \\
&= 43{,}920 \\
&= 43{,}900 \text{ to 3 sig. fig.}
\end{aligned}
$$

$$
\begin{array}{r|r}
1\cdot5011 & 2\cdot6181 \\
5\cdot9638 & 0\cdot2041 \\
\hline
7\cdot4649 & 2\cdot8222 \\
2\cdot8222 & \\
\hline
4\cdot6427 &
\end{array}
$$

$$
\text{Rough estimate} = \frac{30 \times 900000}{400 \times 2} = \frac{27 \times 10^6}{800} \simeq 3 \times 10^4 \simeq 30{,}000.
$$

Example IV. Find correct to 3 significant figures the value of

$$(4\cdot172)^5.$$

From the tables

$$
\begin{aligned}
(4\cdot172)^5 &= (10^{0\cdot6203})^5 \\
&= 10^{0\cdot6203\times5} \\
&= 10^{3\cdot1015} \\
&= 1263 \\
&= 1260 \text{ to 3 sig. fig.}
\end{aligned}
$$

$$
\begin{array}{r}
0\cdot6203 \\
5 \\
\hline
3\cdot1015
\end{array}
$$

Rough estimate $= 4^5 = 1024$.

Example V. Find correct to 3 significant figures the cube root of 152·7.

From the tables

$$
\begin{aligned}
\sqrt[3]{152\cdot7} &= (152\cdot7)^{\frac{1}{3}} \\
&= 10^{2\cdot1838\times\frac{1}{3}} \\
&= 10^{0\cdot7279} \\
&= 5\cdot344 \\
&= 5\cdot34 \text{ to 3 sig. fig.}
\end{aligned}
$$

$$
\begin{array}{r|l}
3 & 2\cdot1838 \\
\hline
 & 0\cdot7279
\end{array}
$$

Rough estimate, $\sqrt[3]{125} = 5$.

NOTE ON METHOD.

Until the reader is fully experienced, he should follow the methods given above, in which all numbers are written down clearly as powers of ten, in the middle of the page. Any simplification that has to be done should be written in special columns at the side of the page.

In evaluating a complicated expression, such as in Example III., clearness is gained by having two such columns, one for logarithms of factors of the numerator and the other for logarithms of factors of the denominator.

When sufficient skill and experience has been acquired, it will be found that the middle of the page contains little more than the question and the answer, all the subsidiary calculations being done in the side columns; but the work in these side columns must be kept neat, and logarithms must not be allowed to straggle unattended into the middle of the page.

EXERCISE XIII. b.

Evaluate, correct to 3 significant figures,

1. $4{\cdot}83 \times 7{\cdot}24$.
2. $19{\cdot}1 \div 6{\cdot}87$.
3. $(8{\cdot}73)^2$.
4. 407×103.
5. $4{\cdot}723 \times 1{\cdot}246$.
6. $401{\cdot}6 \times 5{\cdot}073$.
7. $26{\cdot}7 \times 81{\cdot}31$.
8. $8{\cdot}138 \times 41{,}900$.
9. $412{\cdot}6 \div 37{\cdot}19$.
10. $46{\cdot}59 \div 5{\cdot}123$.
11. $91{,}923 \div 718{\cdot}6$.
12. $10 \div 4{\cdot}965$.
13. $(1{\cdot}738)^2$.
14. $(2{\cdot}716)^3$.
15. $(114{\cdot}9)^2$.
16. $(94{\cdot}56)^3$.
17. $\dfrac{31{\cdot}2 \times 7{\cdot}84}{9{\cdot}91}$.
18. $\dfrac{863}{4{\cdot}09 \times 11{\cdot}8}$.
19. $\dfrac{639 \times 41{\cdot}3}{83{\cdot}2 \times 55{\cdot}4}$.
20. $\dfrac{(8{\cdot}63)^2}{10{\cdot}08}$.
21. $4{\cdot}16 \times 11{\cdot}3 \times 7{\cdot}28$
22. $\dfrac{(13{\cdot}01)^3}{(4{\cdot}82)^2}$.
23. $6{\cdot}14 \times (8{\cdot}231)^3$.
24. $\dfrac{6{\cdot}172 \times 19{\cdot}41}{(9{\cdot}835)^2}$.
25. $\dfrac{36{\cdot}32 \times 10000}{(21{\cdot}23)^3}$.
26. $\dfrac{100}{(3{\cdot}71)^3}$.
27. $\left(\dfrac{3{\cdot}18 \times 6{\cdot}05}{4{\cdot}45}\right)^2$.
28. $5{\cdot}018 \times (2{\cdot}63)^3$.
29. $2\frac{1}{3} \times 8{\cdot}61 \div 3{\cdot}4$.
30. $(13\frac{1}{4})^2 \div 4{\cdot}931$.

31. $\sqrt{4{\cdot}256}$. **32.** $\sqrt{42{\cdot}56}$.

33. $\sqrt[3]{1{\cdot}523}$. **34.** $\sqrt[3]{15{\cdot}23}$.

35. $\sqrt[3]{152{\cdot}3}$. **36.** $\sqrt[3]{(16{\cdot}35)^2}$.

37. (i) $\sqrt[3]{9}$; (ii) $\sqrt[3]{90}$; (iii) $\sqrt[3]{900}$; (iv) $\sqrt[3]{9000}$.

38. $6{\cdot}14 \times \sqrt{8{\cdot}231}$. **39.** $\sqrt{\dfrac{172{\cdot}8}{9{\cdot}43}}$.

40. $\dfrac{\sqrt{48{\cdot}63}}{(2{\cdot}59)^2}$. **41.** $\dfrac{4{\cdot}13 \times 10^6}{517{\cdot}2} \times \sqrt{39{\cdot}41}$.

42. $(108{\cdot}8)^{\frac{1}{4}}$. **43.** $\sqrt[3]{\dfrac{1{\cdot}752 \times 753{\cdot}6}{1111}}$.

44. $\dfrac{\sqrt{10}}{\sqrt{7}}$. **45.** $\sqrt{2\frac{1}{4} \times 17\frac{3}{11}}$.

EXERCISE XIII. c.

[In the following examples, take $\pi = 3{\cdot}1416 = 10^{0{\cdot}4971}$. Give all results correct to 3 significant figures.]

1. (i) Find the area of a circle of radius 4·37 cm.
(ii) Find the radius of a circle of area 150 sq. cm.

2. The volume of a sphere of radius r cm. is $\frac{4}{3}\pi r^3$ cu. cm.
(i) Find the volume of a sphere of radius 1·083 cm.
(ii) Find the radius of a sphere of volume 5 cu. cm.

3. The distance a body falls in a vacuum in t seconds is $\frac{1}{2} \times 32{\cdot}2 \times t^2$ feet.
(i) How far does it fall in 1·63 seconds?
(ii) How long does it take to fall 83·2 feet?

4. A pendulum of length l feet makes one complete oscillation in $2\pi\sqrt{(l \div 32{\cdot}2)}$ seconds.
(i) Find the time of oscillation for a pendulum of length 50 feet.
(ii) Find the length of a pendulum which takes 2 seconds for a complete oscillation.

5. Find the horse-power of a two-cylinder locomotive from the formula, horse-power $= \dfrac{p\,.\,l\,.\,a\,.\,n}{99000}$, where p = pressure in lb. wt. per sq. in. = 15, l = length of stroke in inches = 17·5, a = area of piston in sq. inches = 254, n = number of revolutions per minute = 250. Answer correct to nearest integer.

6. The following formula is given for the speed of a paddle steamer, V (in knots) $= \sqrt[3]{\dfrac{620P}{S}}$; find V, given that $P =$ horse-power $= 1850$, $S =$ sectional area $= 714$ sq. feet. Find also the horse-power necessary to secure a speed of $15\frac{1}{2}$ knots for the same vessel.

7. Use the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, to find r if $V = 182$, $h = 11{\cdot}4$.

8. The horse-power H of a steam turbine is given by $H = \dfrac{6D^2U^3}{10^8 V}$; find H, if $D =$ mean diameter in feet $= 2{\cdot}4$, $U =$ mean blade speed in ft. per sec. $= 1530$, $V =$ specific volume of steam in cu. ft. $= 285$.

9. The bursting pressure P lb. per sq. inch for a flat circular plate of thickness T inches and radius R inches is given by $P = \dfrac{72000T^2}{R^2}$; find P if $T = 1{\cdot}125$, $R = 8{\cdot}37$.

10. The diameter, D feet, for a 3-bladed propeller is found from Doig's formula $D = \dfrac{52H^{0 \cdot 2}}{R^{0 \cdot 6}}$; find D if the horse-power H is 9000 and the number of revolutions per minute R is 225.

11. The diameter d mm. of a tin wire which is just fused by a current of C amperes is given by the formula $d = \sqrt[3]{\left\{\left(\dfrac{C}{12{\cdot}8}\right)^2\right\}}$; find the value of d if $C = 55$.

12. The force necessary to stop a train of weight 150 tons travelling v miles per hour in d feet is $\dfrac{2{\cdot}23v^2}{d}$ tons; what force will stop it within 87 yards when it is travelling 23 miles per hour?

13. If $M = \dfrac{\pi f d^3}{32{\cdot}2}$, find M when $f = 6{\cdot}15$, $d = 2{\cdot}73$.

14. If $d = k\sqrt[3]{\dfrac{H}{N}}$, find d when $k = 173$, $H = 46$, $N = 3$.

15. If $P = \dfrac{KND^2}{5{\cdot}13}$, find P when $K = 1{\cdot}016$, $N = 6$, $D = 4{\cdot}25$.

16. If $W = \dfrac{Hm}{D^2N^3}$, find W when $N = 160$, $D = 5{\cdot}75$, $H = 15$, $m = 1{\cdot}43 \times 10^7$.

17. If $W = 1880\dfrac{p \cdot A}{\sqrt{T}}$, find W when $p = 14$, $A = 7{\cdot}675$, $T = 306$.

18. Find approximately the values of (i) 2^{20}; (ii) 3^{100}.

Example VI. Express as a power of 10, (i) 0·8634; (ii) 0·08634; (iii) 0·0008634.

$$\begin{aligned} 0{\cdot}8634 &= \frac{8{\cdot}634}{10} = 8{\cdot}634 \times 10^{-1} \\ &= 10^{0\cdot9362} \times 10^{-1} \\ &= 10^{-1+0\cdot9362}; \\ 0{\cdot}08634 &= \frac{8{\cdot}634}{10^{2}} = 8{\cdot}634 \times 10^{-2} \\ &= 10^{-2+0\cdot9362}; \\ 0{\cdot}0008634 &= \frac{8{\cdot}634}{10^{4}} = 8{\cdot}634 \times 10^{-4} \\ &= 10^{-4+0\cdot9362}. \end{aligned}$$

The logarithm of 1 is 0, because $1 = 10^0$.

Therefore the logarithm of any positive number less than 1 is less than 0, *i.e.* is negative: this is illustrated in Example VI.

When the logarithm of a number is negative, the logarithm is always written so that the mantissa (*i.e.* the figures after the decimal point) is positive.

Thus the logarithm of 0·08634 is written as $-2+0{\cdot}9362$, or more shortly as $\bar{2}{\cdot}9362$, the $-$ being placed above the 2 to show that it refers only to the integer **2**.

Example VII. Simplify $\dfrac{18{\cdot}61}{0{\cdot}0451}$.

$$\begin{aligned} \frac{18{\cdot}61}{0{\cdot}0451} &= \frac{10^{1\cdot2697}}{10^{\bar{2}\cdot6542}} \\ &= 10^{2\cdot6155} \\ &= 412{\cdot}6. \end{aligned}$$

$$\begin{array}{r} 1+{\cdot}2697 \\ -2+{\cdot}6542 \\ \hline 2+{\cdot}6155 \\ \hline \end{array}$$

Notes.—(i) At first it is best to write the work out in full as shown in the columns at the side.

(ii) If there is any difficulty in subtracting the integers, use the rule for subtraction: change the sign of the lower line and add.

Example VIII. Simplify $(0{\cdot}0832)^3$.

$$\begin{aligned} (0{\cdot}0832)^3 &= 10^{\bar{2}\cdot9201\times3} \\ &= 10^{\bar{4}\cdot7603} \\ &= 0{\cdot}0005758. \end{aligned}$$

$$\begin{array}{r} -2+\ {\cdot}9201 \\ 3 \\ \hline -6+2{\cdot}7603 \\ -4+\ {\cdot}7603 \end{array}$$

Example IX. Simplify $\sqrt[5]{0{\cdot}0736}$.

$$\begin{aligned}\sqrt[5]{0{\cdot}0736} &= 10^{\bar{2}\cdot8669\times\frac{1}{5}}\\ &= 10^{\bar{1}\cdot7734}\\ &= 0{\cdot}5934.\end{aligned}$$

$$\begin{array}{r} -2+\ {\cdot}8669\\ 5)\underline{-5+3{\cdot}8669}\\ -1+\ {\cdot}7734\\ \hline\end{array}$$

Example X. Simplify $(0{\cdot}0862)^{0\cdot24}$.

$$\begin{aligned}(0{\cdot}0862)^{0\cdot24} &= 10^{\bar{2}\cdot9355\times0\cdot24}\\ &= 10^{-0\cdot48+0\cdot2245}\\ &= 10^{-1+0\cdot52+0\cdot2245}\\ &= 10^{\bar{1}\cdot7445}\\ &= 0{\cdot}5552.\end{aligned}$$

$$\begin{array}{r} 0{\cdot}9355\\ \underline{0{\cdot}24}\\ 18710\\ \underline{37420}\\ 0\ 22452\\ \hline\end{array}$$

EXERCISE XIII. d.

In Examples 1-30, write the answer so that the fractional part of it is positive ; *e.g.* you would write $-1{\cdot}7$ as $2{\cdot}3$.

Simplify

1. $2{\cdot}3+1{\cdot}5$. **2.** $2{\cdot}3+1{\cdot}8$. **3.** $2{\cdot}5+\bar{2}{\cdot}4$.

4. $\bar{2}{\cdot}5+\bar{3}{\cdot}1$. **5.** $\bar{2}{\cdot}5+\bar{4}{\cdot}7$. **6.** $\bar{4}{\cdot}6+2{\cdot}7$.

7. $\bar{2}{\cdot}4+1{\cdot}9+\bar{1}{\cdot}8$. **8.** $2{\cdot}4-\bar{1}{\cdot}3$. **9.** $\bar{3}{\cdot}7-2{\cdot}4$.

10. $3{\cdot}6-\bar{1}{\cdot}8$. **11.** $\bar{3}{\cdot}4-1{\cdot}7$. **12.** $3{\cdot}7-3{\cdot}9$.

13. $2{\cdot}6-3{\cdot}3$. **14.** $2{\cdot}6-3{\cdot}9$. **15.** $\bar{2}{\cdot}6-\bar{3}{\cdot}8$.

16. $\bar{1}{\cdot}4\times2$. **17.** $\bar{1}{\cdot}6\times2$. **18.** $\bar{2}{\cdot}7\times3$.

19. $\bar{3}{\cdot}9\times5{\cdot}8$. **20.** $2{\cdot}4\times(-2)$. **21.** $\bar{2}{\cdot}7\times(-3)$.

22. $\bar{2}{\cdot}6\div2$. **23.** $\bar{1}{\cdot}6\div2$. **24.** $\bar{5}{\cdot}7\div2$.

25. $\bar{3}{\cdot}9\div3$. **26.** $\bar{4}{\cdot}4\div3$. **27.** $\bar{2}{\cdot}8\div5$.

28. $\bar{1}{\cdot}6\div4$. **29.** $\overline{10}{\cdot}1\div9$. **30.** $\bar{3}{\cdot}1\div(-2)$.

EXERCISE XIII. e.

Find correct to 3 significant figures the values of

1. $0{\cdot}456\times7{\cdot}12$. **2.** $0{\cdot}078\times3{\cdot}153$.

3. $0{\cdot}00461\times0{\cdot}912$. **4.** $0{\cdot}000512\times712{\cdot}8$.

5. $47{\cdot}5\div89{\cdot}1$. **6.** $0{\cdot}4523\div9{\cdot}165$.

7. $0{\cdot}2103\div0{\cdot}0172$. **8.** $0{\cdot}3167\div0{\cdot}0715$.

9. $(0{\cdot}156)^2$. **10.** $(0{\cdot}0137)^3$.

11. $\sqrt{0{\cdot}783}$. **12.** $\sqrt{0{\cdot}0783}$.

13. $\sqrt[3]{0{\cdot}07825}$; $\sqrt[3]{0{\cdot}007825}$; $\sqrt[3]{0{\cdot}0007825}$.

14. $\dfrac{1}{7{\cdot}93}$. **15.** $\dfrac{1}{37{\cdot}4}$.

16. $\dfrac{0{\cdot}862 \times 0{\cdot}437}{3{\cdot}14}$. **17.** $\dfrac{415{\cdot}2 \times 0{\cdot}0302}{2468}$.

18. $\dfrac{0{\cdot}00531 \times 416{\cdot}2}{0{\cdot}1018}$. **19.** $\sqrt[3]{\dfrac{1}{915{\cdot}2}}$.

20. $\sqrt[4]{\dfrac{1}{0{\cdot}873}}$. **21.** $\sqrt[3]{\dfrac{4{\cdot}156 \times 0{\cdot}00612}{0{\cdot}0891}}$.

22. $0{\cdot}713 \times \sqrt[5]{0{\cdot}416}$. **23.** $(5{\cdot}16)^{1{\cdot}2}$; $(0{\cdot}516)^{1{\cdot}2}$.

24. $(1{\cdot}472)^{-0{\cdot}8}$. **25.** $(0{\cdot}628)^{-3{\cdot}4}$.

26. $(0{\cdot}372)^{0{\cdot}41}$. **27.** $(0{\cdot}0682)^{0{\cdot}08}$.

28. $(0{\cdot}0072)^{2{\cdot}34}$. **29.** $(0{\cdot}1)^{0{\cdot}1}$.

30. Find approximately the value of (i) $(0{\cdot}2)^{20}$; (ii) $(0{\cdot}9)^{100}$.

EXERCISE XIII. 2.

Evaluate to 3 significant figures

1. $(415{\cdot}7)^2 - (112{\cdot}6)^2$. **2.** $(0{\cdot}718)^2 + (0{\cdot}513)^2$.

3. $\dfrac{0{\cdot}273 + (0{\cdot}864)^2}{10\frac{1}{4}}$. **4.** $\dfrac{5 + \sqrt{8{\cdot}67}}{10 - \sqrt{86{\cdot}7}}$.

5. From the formula $T_1 = T_2\left(\dfrac{p_1}{p_2}\right)^{\frac{\gamma-1}{\gamma}}$, find T_1 when $T_2 = 315$, $p_1 = 7{\cdot}16$, $p_2 = 5{\cdot}95$, $\gamma = 1{\cdot}5$.

6. Find the value of $\pi(R^2 - r^2)h$ when $R = 7{\cdot}16$, $r = 3{\cdot}12$, $h = 2{\cdot}59$.

7. The area of a triangle whose sides are of lengths a, b, c is $\sqrt{\{s(s-a)(s-b)(s-c)\}}$ where $s = \frac{1}{2}(a+b+c)$; find the area of a triangle whose sides are 14·5, 16·2, 18·1 inches.

8. The total surface of a closed circular cylinder is $2\pi r^2 + 2\pi rl$; find the area if $r = 7{\cdot}52$, $l = 5{\cdot}83$. [Use factors.]

9. Find the value of $A \,.\, e^{kt}$ when $A = 4{\cdot}7$, $e = 2{\cdot}718$, $k = 2{\cdot}4$, $t = \frac{1}{3}$.

10. (i) Find the area of a circle of radius 0·47 inch.

(ii) Find the radius of a circle of area 0·76 sq. inch.

11. In the formula $t = 2\pi\sqrt{\dfrac{l}{32{\cdot}2}}$, find t if $l = 1{\cdot}7$.

12. Find R from the formula $R = \dfrac{4l\rho}{\pi d^2}$ where $\rho = 1{\cdot}7 \times 10^{-6}$, $l = 94{,}300$, $d = 0{\cdot}25$.

13. Find n from the formula $n = \dfrac{aH^{\frac{5}{4}}}{P^{\frac{1}{2}}}$ when $a = 22{\cdot}25$, $H = 6$, $P = 180$.

14. The time t seconds for pneumatic transmission through a tube l feet long, d feet in diameter under a pressure of P lb. per sq. inch, is $t = 0{\cdot}000482\sqrt{\dfrac{l^3}{Pd}}$; find t if $l = 42{\cdot}5$, $d = 0{\cdot}24$, $P = 8{\cdot}57$.

15. A circular drain of fall 1 in 100 and diameter d feet can carry off $\dfrac{454{\cdot}3d^3}{1{\cdot}005 + \sqrt{d}}$ cu. ft. of water per minute. Find in tons the weight of water removed in an hour by a drain 1·4 feet in diameter. [1 cu. ft. of water weighs 62·3 lb.]

16. Find f from the formula $f = 1{\cdot}925 \times 10^{-8}q^2d^4$ when $q = 500$, $d = 0{\cdot}3162$.

17. Find the value of $71{\cdot}5e^{-kt}$ where $e = 2{\cdot}718$, if (i) $k = 0{\cdot}25$. $t = 3$; (ii) $k = 0{\cdot}25$, $t = 100$.

18. From the formula for drilling mild steel, $P = 35{,}500D^{0\cdot7}T^{0\cdot6}$, where P is the thrust in lb. required for a drill of diameter D inches to secure a feed of T inches per revolution, calculate P if $D = \frac{7}{8}$, $T = \frac{1}{80}$.

19. If $e = 1 - r\gamma^{-1}$, find e when $\gamma = 1{\cdot}37$, $r = 0{\cdot}4$.

20. Find, to three significant figures, the value of x if

$$x^2 = r^2 - \frac{14{\cdot}64}{r} \text{ and } \pi r^2 = 43{\cdot}7.$$

21. If $\dfrac{p_2}{p_1} = \sqrt{\left\{1 - \dfrac{V^2l}{74{,}300{,}000\,d}\right\}}$, find $\dfrac{p_2}{p_1}$ if $V = 88$, $l = 5280$, $d = 0{\cdot}75$.

22. Find the value of (i) $\left(1 + \dfrac{1}{10^2}\right)^{10^2}$; (ii) $\left(1 + \dfrac{1}{10^6}\right)^{10^6}$, given that $1{\cdot}000001 = 10^{0\cdot000000434294}$.

23. The formula for compound interest states that £P amounts to £$P\left(1 + \dfrac{r}{100}\right)^n$ in n years at r per cent. Find, as accurately as your tables will allow, the amount for

(i) £1000 at 5% for 15 years;

(ii) 15s. 6d. at 6% for 5 years;

(iii) one penny at 4½% for 800 years.

24. What sum invested for a child at birth will amount to £1000 when he comes of age if it is invested at 6% compound interest ?

25. Simplify $\dfrac{1+(1\cdot023)^{-10}}{1+(1\cdot023)^{-1}}$.

26. The present value £P of an annuity of £A per annum to last for n years is given by $P=\dfrac{100A}{r}\left\{1-\left(1+\dfrac{r}{100}\right)^{-n}\right\}$, allowing r per cent. per annum compound interest.

(i) Find the present value of an annuity of £400 to last for 10 years, interest 6%.

(ii) A man is offered a choice between a single payment of £6000 down or a life annuity of £600 ; his expectation of life is 15 years ; which should he choose, reckoning interest at 6 per cent. ?

(iii) What life annuity can a man purchase with £5000 when his expectation of life is 20 years, reckoning interest at 4 per cent. ?

THE SLIDE RULE.

For remarks on the use of the slide rule, the reader is referred to the Introduction.

EXERCISE XIII. g.

With the aid of a slide rule, find to three significant figures the values of

1. $4\cdot62\times1\cdot6$.
2. $4\cdot62\times1\cdot67$.
3. $9\cdot71\times8\cdot5$.
4. $97\cdot1\times0\cdot85$.
5. $312\div20\cdot7$.
6. $3759\div177$.
7. $4156\div112$.
8. $0\cdot0793\div7\cdot12$.
9. $\sqrt{13\cdot9}$.
10. $\sqrt{1438}$.
11. $\dfrac{1\cdot46\times7\cdot52}{7\cdot14}$.
12. $\dfrac{4\cdot56\times6\cdot52}{9\cdot13\times1\cdot25}$.
13. $\dfrac{8\cdot35}{6\cdot91\times5\cdot13}$.
14. $\dfrac{6\cdot31\times5\cdot64\times7\cdot12}{8\cdot15\times3\cdot14}$.
15. $\sqrt{11\cdot7\times1\cdot56}$.
16. $\sqrt{\dfrac{62\cdot7}{21\cdot9}}$.
17. $\left(\dfrac{8\cdot63}{5\cdot78}\right)^2$.
18. $\sqrt{\dfrac{147\times1\cdot19}{27\cdot8}}$.

For additional examples, see Exercise XIII. b.

*LOGARITHMIC THEORY AND NOTATION.

If $x = a^p$, p is defined as the logarithm of x to base a. This may be written in the form

$$p = \log_a x.$$

So far we have only considered logarithms to base 10, which are the ordinary kind of logarithm for numerical work. If there is no doubt as to what base is being used, the symbol is abbreviated and simply written as $\log x$. In numerical work $\log x$ would be used to stand for $\log_{10} x$.

We shall now prove some of the results which have been already used.

I. $\log_a(xy) = \log_a x + \log_a y$.

Let $\log_a x = p$ and $\log_a y = q$;

$$\therefore\ x = a^p \quad\text{and}\quad y = a^q;$$

$$\therefore\ xy = a^p \times a^q = a^{p+q};$$

$$\therefore\ \log_a(xy) = p + q = \log_a x + \log_a y.$$

II. $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.

Proceed as in I.

Then $$\frac{x}{y} = \frac{a^p}{a^q} = a^{p-q};$$

$$\therefore\ \log_a\left(\frac{x}{y}\right) = p - q = \log_a x - \log_a y.$$

III. $\log_a(x^n) = n \log_a x$.

Proceed as in I.

Then $$x^n = (a^p)^n = a^{pn};$$

$$\therefore\ \log_a(x^n) = pn = n \log_a x.$$

Note.—The result, given in III., holds with the same proof for any fractional or negative value of n.

Thus $\log \sqrt[n]{x} = \log\left(x^{\frac{1}{n}}\right) = \dfrac{1}{n}\log x$.

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* This section may be omitted at a first reading: it is however advisable to take it before the chapters on Empirical Formulae and Nomography.

Example XI. If $W = \frac{Kbd^2}{l}$, find W when $K = 16$, $b = 5{\cdot}3$ $d = 7{\cdot}14$, $l = 90$.

$$\begin{aligned} \log W &= \log \frac{Kbd^2}{l} \\ &= \log K + \log b + 2 \log d - \log l \\ &= 1{\cdot}2041 + 0{\cdot}7243 + 2(0{\cdot}8537) - 1{\cdot}9542 \\ &= 1{\cdot}6816; \end{aligned}$$

$$\therefore\ W = 48{\cdot}04.$$

$$\begin{array}{r} 1{\cdot}2041 \\ 0{\cdot}7243 \\ 1{\cdot}7074 \\ \hline 3{\cdot}6358 \\ 1{\cdot}9542 \\ \hline 1{\cdot}6816 \\ \hline \end{array}$$

Example XII. If $H = p\,.\,v^{1{\cdot}13}$, find $\log H$.

$$\begin{aligned} \log H &= \log (p\,.\,v^{1{\cdot}13}) \\ &= \log p + \log (v^{1{\cdot}13}) \\ &= \log p + 1{\cdot}13 \log v. \end{aligned}$$

Example XIII. If $\log 5 = 0{\cdot}699$, find $\log 25$.

$$\begin{aligned} \log 25 &= \log (5^2) \\ &= 2 \log 5 = 2 \times 0{\cdot}699 \\ &= 1\ 398. \end{aligned}$$

Example XIV. Find x, given that $4{\cdot}7^x = 0{\cdot}83$.

$$\begin{aligned} \log (4{\cdot}7)^x &= \log 0{\cdot}83; \\ \therefore\ x \log 4{\cdot}7 &= \log 0{\cdot}83; \\ \therefore\ x &= \frac{\log 0{\cdot}83}{\log 4{\cdot}7} = \frac{\bar{1}{\cdot}9191}{0{\cdot}6721} \\ &= \frac{-1 + {\cdot}9191}{0{\cdot}6721} = -\frac{0{\cdot}0809}{0{\cdot}6721} \\ &= -\frac{10^{\bar{2}{\cdot}9079}}{10^{\bar{1}{\cdot}8275}} \\ &= -10^{\bar{1}{\cdot}0804} \\ &= -0{\cdot}1203; \\ \therefore\ x &= -0{\cdot}120. \quad \textit{Answer.} \end{aligned}$$

$$\begin{array}{r} \bar{2}{\cdot}9079 \\ \bar{1}{\cdot}8275 \\ \hline \bar{1}{\cdot}0804 \\ \hline \end{array}$$

EXERCISE XIII. h.

[Unless otherwise stated, $\log x$ means $\log_{10} x$.]

1. Simplify

(i) $\log 8 - \log 2$; (ii) $\log 8 \div \log 2$;

(iii) $\log 1 \div \log 2$; (iv) $\log (\sqrt{3}) \div \log 3$.

2. Simplify

(i) $\dfrac{\log 25}{\log 5}$; (ii) $\dfrac{\log 100}{\log 10}$; (iii) $\log 2\tfrac{1}{4} \div \log 1\tfrac{1}{2}$.

3. Without using tables, state which is the greater :

(i) $\log 1 + \log 3$ or $\log (1+3)$;

(ii) $\log 2 + \log 4$ or $\log (2+4)$.

4. Without using tables, state which is the greater :

(i) $\log 2 - \log 1$ or $\log (2-1)$;

(ii) $\log 4 - \log 2$ or $\log (4-2)$;

(iii) $\log 6 - \log 3$ or $\log (6-3)$.

5. Find x if
$$\frac{\log x}{\log 2} = \frac{\log 4}{\log 16}.$$

6. Simplify

(i) $\dfrac{\log (x^2)}{\log x}$; (ii) $\dfrac{\log \sqrt{x}}{\log x}$; (iii) $\log xy^2 - \log x$.

7. Simplify

(i) $\log x + \log \dfrac{1}{x}$; (ii) $\log \dfrac{x}{y} + \log y$.

8. If $x^4 = 1000$, find $\log x$.

9. Express the following in a form which does not involve the logarithmic notation :

(i) $\log x + \log y = \log 5$; (ii) $\log x + 3 \log y = 2$;

(iii) $x \log 3 = 3$; (iv) $3 \log x - 2 \log y = 1$;

(v) $x \log 5 = y \log 6$; (vi) $x \log 5 = y \log 6 + 2$.

10. Solve the equations

(i) $2^x = 8192$; (ii) $14{\cdot}5^n = 210{\cdot}3$; (iii) $7{\cdot}81^n = 0{\cdot}128$.

11. If $pv^n = 475$, find n when $p = 3{\cdot}62$, $v = 98{\cdot}5$.

12. If $h = 391Q^n$, find n when $h = 7{\cdot}95$, $Q = 0{\cdot}125$.

13. If $2^x = 3^y$, find the ratio of x to y.

14. (i) What is the error per cent. in taking $\log (1+x)$ equal to $0{\cdot}434x$ when $x = 0{\cdot}1$?

(ii) Hence find an approximate solution of the equation
$$\log (1+x) = \frac{\sqrt{x}}{10}.$$

15. What are the values of

$\dfrac{\log 4 - \log 1}{\log 2 - \log 1}$; (ii) $\dfrac{\log 8 - \log 1}{\log 2 - \log 1}$; (iii) $\dfrac{\log 9 - \log 1}{\log 3 - \log 1}$?

16. Find ϕ from the formula $\phi=\frac{\log t-\log 273}{\log 2{\cdot}718}$ when $t=373$.

17. £P invested at r per cent. compound interest for n years amounts to £$P\left(1+\frac{r}{100}\right)^n$.

In how many years will a sum of money invested at (i) 3%, (ii) 5%, compound interest, double itself ?

Show that the rough formula $n=\frac{70}{r}$ gives a good approximation in each case.

Prove that the correct formula is $n=\frac{\log 2}{\log\left(1+\frac{r}{100}\right)}$.

18. Assuming that the population of a country increases according to a compound interest law, find the annual percentage increase in a country where the population increases by 50 per cent. in ten years.

19. If the barometer reading is h_1 inches at the foot and h_2 inches at the summit of a mountain, the height of the mountain is 60,360 $(\log h_1-\log h_2)$ feet ; find the height of a mountain for which $h_1=30$, $h_2=25$.

20. Evaluate $\log_e 9$.

CHAPTER XIV.

RATIO AND PROPORTION. VARIATION.

THE *ratio* of two numbers a, b is measured by the fraction $\frac{a}{b}$ and is often written $a:b$.

The ratio $na:nb$ equals the ratio $a:b$, because $\frac{na}{nb}=\frac{a}{b}$.

The ratio-relation $x:y=a:b$ may be written either as

$$\frac{x}{y}=\frac{a}{b} \quad \text{or} \quad \frac{x}{a}=\frac{y}{b}.$$

Such ratio-relations as $x:y:z:u:v\ldots=a:b:c:d:e\ldots$ may be written either as

$$\frac{x}{y}=\frac{a}{b} \quad \text{and} \quad \frac{y}{z}=\frac{b}{c} \quad \text{and} \quad \frac{z}{u}=\frac{c}{d}, \text{ etc.,}$$

or as

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{u}{d}=\ldots.$$

Example I. Simplify the ratio $4\frac{1}{5}:3\frac{1}{2}$.

$$\frac{4\frac{1}{5}}{3\frac{1}{2}}=\frac{4\frac{1}{5}\times 10}{3\frac{1}{2}\times 10}=\frac{42}{35}=\frac{6}{5};$$

$\therefore$ the ratio equals 6 : 5.

Example II. If $x:y=4:7$, find the value of the ratio $(x+2y):(4x-y)$.

First Method.

Suppose $x=4a$, then $y=7a$;

$$\therefore \frac{x+2y}{4x-y}=\frac{4a+14a}{16a-7a}=\frac{18a}{9a}=\frac{2}{1}.$$

$\therefore$ the ratio equals 2 : 1 or 2.

Second Method.

$$\frac{x+2y}{4x-y}=\frac{\frac{x}{y}+2}{\frac{4x}{y}-1}=\frac{\frac{4}{7}+2}{\frac{16}{7}-1}=\frac{18}{7}\div\frac{9}{7}=2.$$

Example III. Given

$$x:y=2:3 \quad\text{and}\quad y:z=5:7 \quad\text{and}\quad z:w=3:2,$$

express in its simplest form $x:y:z:w$.

$$\frac{x}{2}=\frac{y}{3} \quad\text{and}\quad \frac{y}{5}=\frac{z}{7} \quad\text{and}\quad \frac{z}{3}=\frac{w}{2};$$

$$\therefore\ \frac{x}{10}=\frac{y}{15} \quad\text{and}\quad \frac{y}{15}=\frac{z}{21} \quad\text{and}\quad \frac{z}{21}=\frac{w}{14};$$

$$\therefore\ \frac{x}{10}=\frac{y}{15}=\frac{z}{21}=\frac{w}{14};$$

$$\therefore\ x:y:z:w=10:15:21:14.$$

EXERCISE XIV. a.

1. Simplify the following ratios :

(i) $3\frac{1}{3}:1\frac{1}{4}$; (ii) $a:(a+\frac{1}{2}a)$;

(iii) $\frac{a}{b}:ab$; (iv) $\left(1+\frac{1}{x}\right):(1+x)$;

(v) $15x^2y:12xy^2$; (vi) $10^2:20^2$;

(vii) $10^6:10^3$; (viii) $(a^2-b^2):(a^2-ab)$;

(ix) a yards : b feet ; (x) x halfcrowns : y shillings.

2. If $3x+5y=7x-13y$, find $x:y$.

3. If $a:b=3:5$, evaluate $(a+b):(a-b)$.

4. If $\frac{x+11}{y+7}=\frac{11}{7}$, prove $\frac{x}{y}=\frac{11}{7}$.

5. Express in ratio form the following equations :

(i) $ab=cd$; (ii) $p^2=qr$; (iii) $(a+b)(c+d)=(p+q)(e+f)$;
(iv) $xyz=pqr$; (v) $abc=de$.

6. Two numbers are in the ratio $a:b$; the first is x; what is the second ?

7. y is the result of increasing x by 10 per cent. ; find $x:y$.

8. By deducting discount at the rate of 3d. in the shilling a bill of £x is reduced to £y ; find $x:y$.

9. A man's salary was £x before the war and has been increased since the war by £y. In what ratio has it been altered ?

10. $ABCD$ is a straight line; $AB = x''$, $BC = 2x''$, $CD = 4x''$; what is the ratio of AC to BD ?

11. The rents of two houses used to be in the ratio $a : b$; but the first has been raised 20 per cent. and the second 50 per cent.; what is their present ratio ?

12. Two partners share their profits in the ratio $a : b$; the profits are £p; how much does each receive ?

13. The ratio of the areas of two triangles is $a : b$ and the ratio of their heights is $p : q$; what is the ratio of their bases ?

14. (i) Find two numbers in the ratio $7 : 5$, whose difference is 8.

(ii) Find two numbers in the ratio $p : q$, whose difference is c.

15. The line AB is divided externally at P in the ratio $5 : 2$; $AB = x''$; find AP.

16. The ratio of the weights of two circular cylinders of the same material is 3 . 2 and the ratio of their diameters is $3 : 4$; find the ratio of their heights.

17. AB is divided externally at P in the ratio $3 : 5$ and externally at Q in the ratio $9 : 7$; find $PQ : AB$.

18. A measuring rod contracts in the ratio $a : b$ where $a < b$; it is used to measure a line whose real length is x inches; what is the length obtained and what is the error per cent. ?

19. If x is positive, is the ratio $(3 + x) : (5 + x)$ greater or less than the ratio $3 : 5$?

20. If x is positive, is the ratio $(8 + x) : (7 + x)$ greater or less than the ratio $8 : 7$?

21. If $\frac{1}{a} + \frac{1}{b} = 3\left(\frac{1}{a} - \frac{1}{b}\right)$, find $a : b$.

22. If $3x - y = 2z$ and $y = 6(z - x)$, find $x : y : z$.

23. A chain of length l feet hangs over a small rough peg; the portions on each side are in the ratio $a : b$; if $a > b$, find the depth of the mid-point of the chain below the peg.

24. $ABCD$ is a straight line; B divides AC in the ratio $p : q$; C divides BD in the ratio $x : y$; find (i) $AB : CD$, (ii) $AB : BD$.

25. If $(8x - 2y)^2 = (3x + 4y)^2$, find $x : y$.

26. If $4x^2 + 9y^2 = 12xy$, find $x : y$.

27. The perimeter of a right-angled triangle is four times the shortest side; find the ratio of the other two sides.

28. The ratio of the radii of two circles is $a:b$ and the ratio of their areas is $(a-x):(b-x)$; express x in terms of a, b.

29. If $\frac{x+y}{x-y}=\frac{7}{4}$, evaluate $\frac{x^2+y^2}{x^2-y^2}$.

30. If x and y are each increased in the ratio $a:b$, in what ratio is $\frac{x+y}{x-y}$ altered ?

31. At a height of h feet it is possible to see a distance of $\sqrt{\frac{3h}{2}}$ miles; an aeroplane climbs from 4000 feet to 9000 feet; in what ratio does the pilot increase his range of view ?

32. The horse-power H required to drive W lb. of water per second through a small circular aperture of area A sq. inches is given by $H=0{\cdot}00015\,\frac{W^3}{A^2}$. In what ratio must the horse-power be altered if the radius of the aperture is doubled and if twice as much water per second is required ?

33. The indicated horse-power H required to produce a speed of V knots in a paddle steamer is given by the approximate formula $H=\frac{A\,.\,V^3}{70}$, where A is its greatest section in sq. yards. (i) In what ratio must the horse-power be increased to raise the speed from 8 to 12 knots ? (ii) The values of A for two steamers are in the ratio 6 : 5 and their speeds are in the ratio 5 : 2; what is the ratio of their indicated horse-powers ?

34. If a carrier is driven through a tube l feet long, d feet in diameter under a pressure of P lb. per sq. inch, the time of transmission is approximately $0{\cdot}0048\sqrt{\frac{l^3}{Pd}}$ seconds; what is the effect on the time of transmission if the length is increased in the ratio 3 : 2, the pressure in the ratio 5 : 2, and the diameter decreased in the ratio 3 : 5 ?

35. If $a > b > x > 0$, arrange the ratios $(a-x):(b-x)$ and $(a+x):(b+x)$ and $a:b$ in ascending order of magnitude.

36. The ratio of a man's expenditure to his income is $a:b$; what is the ratio of his savings to his expenditure ?

37. The ratio of the male to the female voters at an election is $a:b$. If c fewer men and d fewer women had voted, the ratio would have been $p:q$. What was the total number of votes cast in terms of these letters ?

38. If h is small compared with 1000, find an approximate value of the ratio $[(1000+h)^2-1000^2]:h$.

39. If h is small compared with x and y, find an approximate value of the ratio $[(x+h)^2 - x^2] : [(y+h)^2 - y^2]$.

40. The incomes of A and B are in the ratio $5 : 4$; their expenditures are in the ratio $6 : 5$ and their savings are in the ratio $10 : 7$. Find the ratio of A's income to B's expenditure.

If $\frac{a}{b}=\frac{c}{d}$, the numbers a, b, c, d are said to be *in proportion*, and d is called the *fourth proportional* to a, b, c.

If $\frac{a}{b}=\frac{b}{c}$, c is called the *third proportional* to a, b and b is called the *mean proportional* between a, c.

If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=$etc., $a, b, c, d, e, \ldots$ are said to be in *continued proportion.*

The ratio-equality $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{w}{d}=\ldots$ is often written in the form $x:y:z:w:\ldots=a:b:c:d:\ldots$.

Example IV. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a}{b}=\frac{a+c}{b+d}=\frac{a-c}{b-d}$.

Let
$$\frac{a}{b}=\frac{c}{d}=k;$$
$$\therefore\ a=bk;\quad c=dk;$$
$$\therefore\ \frac{a+c}{b+d}=\frac{bk+dk}{b+d}=\frac{k(b+d)}{b+d}=k=\frac{a}{b}$$
and
$$\frac{a-c}{b-d}=\frac{bk-dk}{b-d}=k=\frac{a}{b}.$$

EXERCISE XIV. b.

1. Find the fourth proportional to

(i) 2, 3, 4; (ii) ab, bc, cd; (iii) $\frac{1}{x}$, y, $\frac{1}{y}$.

2. Find the third proportional to

(i) 2, 3; (ii) ab, bc; (iii) $\frac{1}{x}$, y.

3. Find a mean proportional between

(i) 3, 12; (ii) a^3b, ab^3; (iii) a, b.

4. Fill up the gaps in

$$\frac{x}{8}=\frac{y}{3}=\frac{x+y}{\quad}=\frac{x-y}{\quad}=\frac{5x-12y}{\quad}.$$

5. If $x : y : z = 3 : 2 : 7$, find $\frac{x+3y}{y+z}$ and the ratios $\frac{1}{x} : \frac{1}{y} : \frac{1}{z}$.

6. If $\frac{x+y+z}{11} = \frac{x+y-z}{8} = \frac{x-y}{5}$, find $x : y : z$.

7. If $\frac{x}{5} = \frac{y+z}{11} = \frac{x-y+z}{2}$, find $x : y : z$.

8. If $\frac{a}{b} = \frac{c}{d}$, prove $\frac{a}{b} = \frac{\sqrt{a^2+c^2}}{\sqrt{b^2+d^2}}$.

9. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove $\frac{3a-5c}{3b-5d} = \frac{2c+7e}{2d+7f}$.

10. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, fill up the gaps in

$$\frac{a}{b} = \frac{a+c+e}{\qquad} = \frac{\qquad}{b-d+f} = \frac{10a-7c+2e}{\qquad} = \frac{\frac{a}{2}+\frac{e}{3}}{\qquad}.$$

11. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{ac}{bd} = \frac{a^2-3c^2}{b^2-3d^2}$.

12. If $\frac{x+y}{11} = \frac{x-y}{5}$, find $\frac{x^2-y^2}{xy}$.

13. If $6(x^2-y^2) = 5xy$, find $\frac{x+y}{x-y}$.

14. Solve $\frac{x-2y+3}{5} = \frac{3x+2y+1}{7} = \frac{2x+y}{8}$.

15. Solve $\frac{x^2-2x+3}{x^2+2x-3} = \frac{x+5}{x-5}$.

16. X, Y are points on the sides AB, AC of the triangle ABC such that $\frac{AX}{XB} = \frac{AY}{YC}$; prove that (i) $\frac{AX}{AB} = \frac{AY}{AC}$; (ii) $\frac{BX}{AB} = \frac{CY}{AC}$.

17. O is a point inside the triangle ABC; AO is produced to cut BC at D; prove that (i) $\frac{\triangle BAD}{\triangle CAD} = \frac{BD}{CD}$; (ii) $\frac{\triangle BAO}{\triangle CAO} = \frac{BD}{CD}$.

18. If $\frac{a}{b} = \frac{c}{d}$, prove that

$$\text{(i)}\ \frac{a}{b} = \sqrt{\left(\frac{a^2+c^2}{b^2+d^2}\right)}; \qquad \text{(ii)}\ \frac{a-c}{b-d} = \sqrt[5]{\left(\frac{a^5+c^5}{b^5+d^5}\right)}.$$

19. If b is the mean proportional between a, c, prove that $\frac{a^2}{b^2} = \frac{a}{c}$. Interpret this geometrically.

20. z is a function of x, y, and its values are shown in the following double-entry table :

		x			
		0	1	2	3
y	0	0	2	4	6
	1	1	4	9	16
	2	2	6	14	26
	3	3	8	19	36

Find the probable values of z when

(i) $x=1$, $y=1{\cdot}5$; (ii) $x=2{\cdot}5$, $y=2$;
(iii) $x=1{\cdot}5$, $y=1{\cdot}5$; (iv) $x=2{\cdot}2$, $y=2{\cdot}6$.

21. The values of a function z of two variables x, y are shown in the following table :

		x		
		1	2	3
y	1	0·070	0·087	0·104
	2	0·122	0·139	0·156
	3	0·174	0·191	0·208

Find the probable values of z when

(i) $x=1{\cdot}5$, $y=1{\cdot}5$; (ii) $x=2{\cdot}6$, $y=2{\cdot}4$.

SUMMARY OF RESULTS.

If
$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots,$$

then
$$\text{each fraction}=\frac{pa+qc+re+\dots}{pb+qd+rf+\dots}$$
$$=\sqrt[n]{\left\{\frac{pa^n+qc^n+\dots}{pb^n+qd^n+\dots}\right\}}.$$

VARIATION.

If a man is walking at a uniform rate of 4 miles an hour, he walks 1 mile in 15 minutes,

2 miles in 30 minutes,
3 miles in 45 minutes, and so on.

The distance he walks is directly proportional to the time taken.

If he walks x miles in t minutes, we have

$$x = \frac{t}{15} \quad \text{or} \quad \frac{x}{t} = \frac{1}{15}.$$

In this case, x is said to *vary directly* as t.

This is written $x \propto t$.

The result of plotting x against t is a straight line through the origin.

Suppose next there are a number of rectangles each of area 60 sq. in., but of different shapes.

If the breadth is 2″, the length is 30″.

If the breadth is 3″, the length is 20″.

If the breadth is 4″, the length is 15″, and so on.

The breadth is inversely proportional to the length.

If the breadth is b in. and the length is l in.,

$$b = \frac{30}{l} \quad \text{or} \quad \frac{b}{\frac{1}{l}} = 30.$$

In this case, b is said to *vary inversely as* l.

This is the same as saying that b varies directly as $\frac{1}{l}$ or

$$b \propto \frac{1}{l}.$$

And the result of plotting b against $\frac{1}{l}$ is a straight line through the origin.

Conversely, if we are given a table of values for x and y, and if the graph is a straight line through the origin when y is plotted against x, then $y \propto x$.

If the graph is a straight line through the origin when y is plotted against $\frac{1}{x}$, then $y \propto \frac{1}{x}$.

Example V. The weights of a number of circular discs of the same material and thickness but different diameters are given in the following table:

Diameter in cm. $= d =$ 2	3	4	5	6	7	8	9	10
Weight in gr. $= w =$ 20	45	80	125	180	245	320	405	500

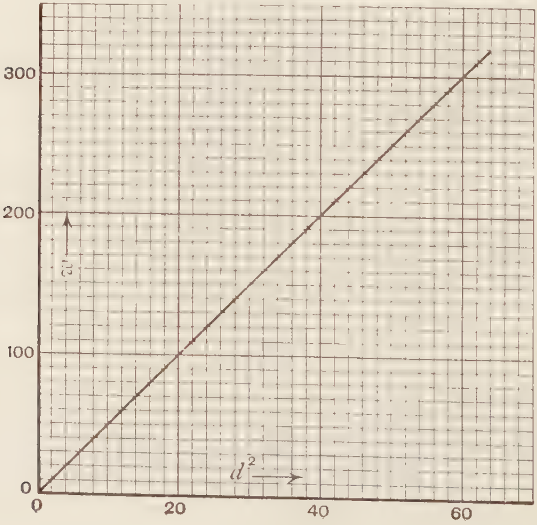
Show that the weight varies as the square of the diameter, and plot w against d^2.

The table connecting d^2 and w is

d^2	4	9	16	25	36	49	64	81	100
w	20	45	80	125	180	245	320	405	500

from which we see that $w = 5d^2$.

$$\therefore\ w \propto d^2.$$



The result of plotting w against d^2 gives a straight-line graph passing through the origin, which is the graphical test that w is proportional to d^2.

Example VI. It is stated that if the velocity of a stream of water is determined by a water-pressure gauge, the velocity varies as the square root of the height in the gauge.

(i) If a speed of 8 feet a second causes a height of 12 inches, what is the speed when the recorded height is 4 inches ?

(ii) If the velocity is v feet per second when the height recorded is h inches, find the equation connecting v and h.

(i) We have $v \propto \sqrt{h}$.

Now $v=8$, $h=12$ is one pair of corresponding values ; and it is required to find the value of v which corresponds to $h=4$.

∴ when $h=4$, the value of v is given by

$$\frac{v}{8}=\frac{\sqrt{4}}{\sqrt{12}};$$

$$\therefore\ v=\frac{8\sqrt{4}}{\sqrt{12}};$$

$$\therefore\ v=4{\cdot}62 \text{ feet per sec.}$$

(ii) Also we have $$\frac{v}{8}=\frac{\sqrt{h}}{\sqrt{12}};$$

$$\therefore\ v=\frac{8}{\sqrt{12}}\cdot\sqrt{h};$$

$$\therefore\ v=2{\cdot}31\,.\,\sqrt{h} \text{ approximately.}$$

EXERCISE XIV. c.

1. A man walking 3 miles an hour travels y yards in x minutes :

(i) Does y vary directly as x ?
(ii) What is the effect on y of doubling x ?
(iii) What is the effect on x of doubling y ?
(iv) What equation connects x and y, and sketch its graph ?

2. x men take y days to level a large field which it would take 12 men 40 days to level :

(i) Does y vary directly as x ?
(ii) What is the effect on y of doubling x ?
(iii) What is the effect on x of halving y ?
(iv) Express y in terms of x, and sketch its graph.

3. If y varies as x, (i) what is the effect on y of trebling x? (ii) what is the effect on x of halving y? (iii) what is the shape of the graph of y?

4. If y varies inversely as x, (i) what is the effect on y of doubling x? (ii) what is the effect on x of multiplying y by 10? (iii) what is the shape of the graph of y? (iv) if x is the length and y the height of a rectangle, what can you say about the area of the rectangle?

5. A rectangle of area 60 sq. cm. is x cm. long and y cm. wide.

(i) What is the variation-relation between y and x?
(ii) Complete the table :

y	10	20	30	40	50
x					
$\frac{1}{x}$					

(iii) Plot $\frac{1}{x}$ against y, taking 3″ as the unit on the $\frac{1}{x}$-axis.

6. If y varies as x^2, what is the effect (i) on x of dividing y by 4; (ii) on y of dividing x by 4? What is the shape of the graph of y?

7. Construct a table of values of x and y to illustrate the cases where

(i) y varies as x;
(ii) y varies inversely as x;
(iii) y varies as x^2.

Sketch the graph of y in each case.

8. To produce a fixed intensity of illumination, the candle-power c of a lamp must be increased four-fold when the distance x feet from the light is doubled; what is the variation-relation between c and x?

9. If multiplying x by 10 always produces the effect of dividing y by 100, what is the variation-relation between x and y?

10. If $y = x + x^2$, in what ratio is y increased when x alters from (i) 1 to 2, (ii) 2 to 4, (iii) 4 to 8? Does y vary as any power of x?

11. In the following tables y varies directly or inversely as some power of x; find the relation between y and x in each case.

What method of plotting must be adopted to give straight-line graphs in the various cases? Carry it out in each case.

(i)	$x =$	-3	-1	0	2	5	
	$y =$	-12	-4	0	8	20	
(ii)	$x =$	1	2	3	4	5	
	$y =$	2	8	18	32	50	
(iii)	$x =$	1	2	5	8	10	12
	$y =$	10	5	2	$1\frac{1}{4}$	1	$\frac{5}{6}$
(iv)	$x =$	1	4	9	25	36	
	$y =$	1	2	3	5	6	

12. A number of triangles each have a base 6 cm. long; if the height of any one of them is h cm. and its area is A sq. cm., complete the following table; what is the variation between A and h?

$h = 3$	4	5	8	10
$A =$				

13. If y varies as x^2, complete the following table:

$x = 0$	1	2	3	5	10
$y =$		12			

14. If y varies as $\frac{1}{x}$, complete the table:

$x = 1$	4	10	20
$y =$		0·8	

15. A cube of aluminium whose edge is x cm. weighs y gr.; complete the table:

$x = 1$	2	4	5
$y =$			325

16. If y varies as $\sqrt{x}$, complete the table:

$x = 0$	1	2	4	25	100
$y =$			4		

17. What is the variation between the variables in the following statements ?

- (i) A circle of radius r cm. is of area A sq. cm.
- (ii) The side of an equilateral triangle is x inches and its area is A sq. in.
- (iii) A train travelling at a uniform rate of x miles an hour takes t hours to go 100 miles.
- (iv) The simple interest on £x at 4 per cent. for 3 years is £y.
- (v) If 5 pints of water are poured into a cylindrical vessel of diameter d inches, the depth is h inches.
- (vi) In a circle of radius 3 cm., an angle of $x°$ at the centre stands on an arc of length y inches.

18. If $y \propto x^2$ and if $y=3$ when $x=2$, find the relation between x and y.

19. If $y \propto \frac{1}{x}$ and if $y=3$ when $x=5$, find the value of y when $x=2$.

20. If $y \propto x^2$, what is the effect on y of doubling x ?

21. If $y \propto \sqrt{x}$, what is the effect on y of increasing x in the ratio 9 : 4 ?

22. If $x \propto \frac{1}{y}$ and if $x \propto z^2$, what is the effect on y of doubling z ?

23. In the given graph, Fig. 8, $y \propto x^2$; find the relation between x and y.

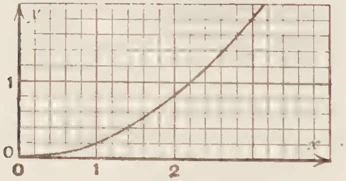


FIG. 8.

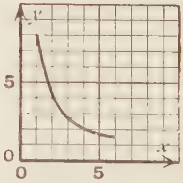


FIG. 9.

24. In the given graph, Fig. 9, y varies inversely as x; find the relation between x and y.

25. If y varies as x and if $y=3$ when $x=2$, represent the relation between y and x by a graph.

26. The area of the surface S sq. in. of a sphere varies as the square of its radius r in. ; its volume V cu. in. varies as the cube of its radius : (i) what is the effect on V of increasing S in the ratio 64 : 1 ? (ii) what is the variation-relation between S and V ?

27. The pressure on the wind-screen of a car travelling v miles an hour on a still day is P lb. per sq. foot. It is found that $P \propto v^2$. Is this consistent with the following measurements?

$v = 5$	20	30
$P = 0{\cdot}123$	1·97	4·43

Express P in terms of v as accurately as the data permit.

28. The range of view at sea varies as the square root of the height of the observer above sea-level. At a height of 20 feet the look-out can see 5·5 nautical miles; what is the range of vision at a height of 40 feet? If a height of h feet commands N nautical miles, express N in terms of h.

29. Two hot-water cans are the same shape: one is 8″ high and holds a quart, the other is 2′ high. What does it hold?

30. Two candlesticks are the same shape; it costs 15 shillings to gild the smaller which is 6″ high; what is the cost of gilding the other which is 15″ high?

31. In what ratio must the radius of a sphere be increased to double (i) its surface, (ii) its volume?

32. A lodger pays ninepence for a scuttle of coal 9 inches wide; what would he expect to pay for a scuttle of the same shape 12 inches wide?

33. It takes x lb. of paint to paint the hull of a battleship 400 ft. long; how much will be required for the hull, similar in shape, of a battleship 530 feet long?

34. The weights of two spheres are in the ratio 2 : 1 and the densities of the materials of which they are made are in the ratio 1 : 2. Compare the radii.

35. The time of a complete oscillation of a pendulum varies as the square root of its length. If a pendulum of length 100 cm. makes one complete oscillation in 2 seconds, find the time of a complete oscillation of a pendulum of length 25 cm. If the time is t sec. when the length is l cm., express t in terms of l.

36. The danger distance from the muzzle of a gun, within which the hearing may be injured by the firing, varies as the fifth root of the weight of the charge. For a 5-lb. charge of cordite the distance is 10 feet. What is the distance for a charge of 187 lb. of cordite?

37. A beam supported at its ends carries a fixed load at its mid-point; for beams of the same cross-section and elasticity the sag in the middle varies as the cube of the length. What is the effect on the sag of trebling the length?

38. (*a*) The weight of a body above ground varies inversely as the square of its distance from the centre of the earth; taking the radius of the earth as 4000 miles, make a table showing the weights of a body which weighs 100 lb. when on the ground at the following heights above the ground: (i) 100 miles, (ii) 1000 miles, (iii) 10,000 miles.

(*b*) Below ground the weight varies directly as its distance from the centre of the earth: make a table showing the weights of the same body at the following depths under ground: (i) 50 miles, (ii) 100 miles, (iii) 1000 miles.

39. If a wire of diameter d mm. fuses when the current in it reaches C ampères, C varies as $d^{\frac{3}{2}}$. The following results were obtained by experiment:

Diameter of copper wire in mm. - - -	0·10	0·51	1·02	1·84
Current fusing wire in ampères - - -	2·55	29·2	82	200

(i) Find the relation between C and d for copper wires as accurately as these data permit.

(ii) What current will fuse a copper wire of diameter 1·5 mm.?

(iii) Fill up the gap in the following table:

Diameter in mm.	Fusing current for copper wire in amp.	Fusing current for tin wire in amp.
0·10	2·55	0·41
1·02	82	

40. If a strong arc light is placed at A at the top of a vertical lamp-post AB, and if P is a point on the level ground, the illumination of the ground at P varies inversely as AP^3.

If $AB = 15'$, $BP = 20'$, the illumination of the ground at P is one unit, *i.e.* is equivalent to the illumination of a surface by a lamp of one candle-power distant one foot from it. What is the illumination of a point Q on the ground such that $BQ = 30'$?

It is said that ordinary small print can be read if the illumination is 0·05 unit. What is the greatest distance from B at which an evening paper on the ground would be legible?

SUMMARY OF RESULTS.

y is said to be a "function" of x, if y can be determined when x is known.

Suppose that y is a function of x, and that the values $y_1, y_2, y_3, y_4, \dots$ of y correspond to the values $x_1, x_2, x_3, x_4, \dots$ respectively of x. If the ratios $y_1 : y_2 : y_3 : y_4$, etc., are respectively equal to the ratios $x_1 : x_2 : x_3 : x_4$, etc., then y is said to *vary directly* as x, or simply *vary* as x.

This is sometimes written $y \propto x$.

If the ratios $y_1 : y_2 : y_3 : y_4$, etc., are equal to the ratios $x_1^n : x_2^n : x_3^n : x_4^n$, etc., then y is said to *vary directly as the nth power of* x, or $y \propto x^n$.

If the ratios $y_1 : y_2 : y_3 : y_4$, etc., are equal to the ratios $\frac{1}{x_1^n} : \frac{1}{x_2^n} : \frac{1}{x_3^n} : \frac{1}{x_4^n}$, etc., then y is said to *vary inversely* as the nth power of x, or $y \propto \frac{1}{x^n}$.

The statement that y varies as x^n may be expressed in any of the following ways:

(i) $y \propto x^n$.

(ii) $\frac{y_p}{y_q} = \frac{x_p^n}{x_q^n}$, where x_p, y_p are any pair of corresponding values of x and y, and x_q, y_q are any other pair.

(iii) $\frac{y_1}{x_1^n} = \frac{y_2}{x_2^n} = \frac{y_3}{x_3^n} = \dots = k$ (say).

(iv) $y = k \,.\, x^n$, where k is a *constant* number independent of the values of x and y.

(v) If x is altered in any ratio $\lambda : 1$, then y is altered in the ratio $\lambda^n : 1$.

(vi) The result of plotting y against x^n gives a straight line through the origin.

JOINT VARIATION.

First suppose we have a number of circular cylinders each of radius 5 in., but of different heights.

If the volume of one of these cylinders of height h in. is V cu. in., then

$$V = 25\pi h.$$

For this set of cylinders, $V \propto h$.

Next suppose we have a number of circular cylinders each of height 7 in., but of different radii.

If the volume of one of these cylinders of radius r in. is V cu. in., then

$$V = 7\pi r^2.$$

For this set of cylinders, $V \propto r^2$.

These facts can be expressed by saying that

$$V \propto h \text{ when } r \text{ is constant}$$

and $$V \propto r^2 \text{ when } h \text{ is constant.}$$

Lastly, if both r and h vary, so that the cylinder may be of any height and radius,

$$V = \pi r^2 h$$

or $$V \propto r^2 h \text{ when both } r \text{ and } h \text{ vary.}$$

This example illustrates the general statement that:

if x varies as y when z is constant

and x varies as z when y is constant,

then x varies as yz when y and z both vary,

or x is said to vary jointly as y and z.

Example VII. The force necessary to stop a train in a given distance varies directly as the weight of the train and the square of its velocity and inversely as the distance. A force of 10 tons will stop a train weighing 200 tons and travelling 30 miles an hour in 200 yards. Find the formula which expresses the force F tons in terms of the velocity, v m.p.h., the distance, d feet, for a train weighing W tons.

$$F \propto \frac{Wv^2}{d}.$$

When $W = 200$, $v = 30$, $d = 600$, we have $F = 10$;

$$\therefore F : \frac{Wv^2}{d} = 10 : \frac{200 \times 30^2}{600} = 10 : 300 = 1 : 30$$

$$\therefore F = \frac{Wv^2}{30d}.$$

Note.—The statement $F \propto \frac{Wv^2}{d}$ can be expressed as follows:

$$F = k \cdot \frac{Wv^2}{d}, \text{ where } k \text{ is a constant.}$$

In the above example, $k = \frac{1}{30}$.

EXERCISE XIV. d.

1. The volume, V cu. in., of a circular cone of height 6 in. and base-radius r in. is given by $V = 2\pi r^2$. The volume, V cu. in., of a circular cone of base-radius 6 in. and height h in. is given by $V = 12\pi h$. For a circular cone base-radius r in., height h, volume V cu. in.,

(i) state how V varies if r is constant;
(ii) state how V varies if h is constant;
(iii) state how V varies if r and h both vary;
(iv) find the formula for V in terms of r and h.

2. One end of a cast-iron beam of rectangular section is built into a wall. For a beam 4″ deep, 10′ long, the greatest weight W tons that can be suspended from the other end is given by $W = 0{\cdot}3b$ where b in. is its breadth. For a beam d″ deep, 10′ long, 8″ broad, the greatest weight W tons is given by $W = 0{\cdot}15d^2$.

For a beam 2″ deep, l feet long, 4″ broad, the greatest weight W tons is given by $W = \frac{3}{l}$.

State how W varies
(i) if d, l are constant;
(ii) if b, l are constant;
(iii) if b, d are constant;
(iv) if b, d, l all vary.

Find the formula for W in terms of b, d, l.

3. A $\frac{1}{4}$-lb. weight is fastened to the end of a string, and is whirled round in a circle. If the speed of the weight is 8 feet a second and the length of the string is l in., the strain in the string is T lb. where $T = \frac{6}{l}$. If the speed of the weight is v feet a second and the length of the string is 6 in., the strain is T lb. where $T = \frac{v^2}{64}$.

State how T varies (i) if v is constant;
(ii) if l is constant;
(iii) if v and l both vary.

Find the formula for T in terms of v and l. [P.T.O.]

If the string breaks under a strain of 5 lb., what is the maximum speed for a string of length 15 in.? How does v vary if T is constant?

4. A mass W lb. is fastened to the end of a string l feet long and is whirled round at the rate of n revolutions per sec.; the strain in the string is T lb.

If $W=1$, $n=4$, then $T=2\pi^2 l$.
If $W=2$, $l=4$, then $T=\pi^2 n^2$.
If $l=2$, $n=6$, then $T=9\pi^2 W$.

(i) How does T vary if l, n, W all vary?
(ii) Find the formula for T in terms of l, n, W.
(iii) How does n vary if T and W are constant?
(iv) How does l vary if W alone is constant?
(v) What is the effect on T of doubling n, halving l and leaving W unchanged?

5. p varies as t if v is constant, and varies inversely as v if t is constant. Complete the given double-entry table, giving the values of p.

		Values of t.		
		300	350	400
Values of v	40			
	50			
	60			240

6. V varies as x if y is constant and varies as y^2 if x is constant; complete the given double-entry table, giving values of V.

		Values of x.		
		5	10	15
Values of y	1			
	2		24	
	3			

7. $S=\frac{6}{x}$ if $y=2$, and $S=\frac{8}{y^2}$ if $x=3$; what is the simplest expression for S in terms of x and y?

8. Write down relations expressing the following facts:

(i) The area of the curved surface of a cone varies as the length of the slant side if the base-radius is constant, and varies as the base-radius if the length of the slant side is constant. [s, l, r.]

(ii) The pressure of a gas varies directly as the absolute temperature if the volume remains constant, and varies inversely as the volume if the temperature remains constant. [p, T, v.]

(iii) The time taken over a journey varies directly as the distance and inversely as the speed (uniform). [t, d, v.]

(iv) The pressure required to drive x cu. ft. of gas per hour through a pipe varies directly as the length of the pipe and as the square of x, and inversely as the fifth power of the diameter. [p, l, d.]

(v) The square of the time for pneumatic transmission through a tube varies directly as the cube of the length of the tube, and inversely as its diameter and the pressure. [t, l, d, p.]

9. On a railway curve the outer rail is raised above the inner by an amount which varies directly as the gauge and the square of the maximum velocity permitted, and inversely as the radius of the curve. If the gauge is 5 feet, the speed 15 m.p.h. and the radius 200 yards, the elevation is 1·5 inches. Express the elevation d inches in terms of the gauge W feet, the speed v m.p.h. and the radius R feet.

10. Lord Kelvin states that the most economical diameter for a copper wire in an electric circuit varies directly as the square root of the normal current and the fourth root of the cost per horse-power per annum, and inversely as the fourth root of the price of copper per lb. Express this by an equation. In what ratio should the diameter be altered if the cost of horse-power rises 50% and the price of copper rises 200%, and the current is reduced by 25%?

11. In steamships of a certain type, if the displacement is d tons and if the indicated horse-power is H for a speed of v knots, the cube of H varies directly as the square of d and the ninth power of v. For a displacement of 1600 tons, the I.H.P. is 690 when the speed is 10 knots. Obtain a general formula.

12. The cost of the coal consumed by a steamer travelling at a steady speed varies directly as the distance and as the square of the speed. What relation connects the cost, the speed and the time taken for the journey? If the coal consumed for a journey of 200 sea miles at a speed of 10 knots cost £80, find the equation connecting the cost, £C, the distance, d sea miles, and the time, T hours.

13. If a mine explodes near a ship, the pressure produced at a point D feet from the charge and at the same level is the sum of two parts, one of which varies directly as the weight of the charge and inversely as the distance, and the other varies directly as the weight of the charge and inversely as the cube of the distance. A charge of 100 lb. No. 1 dynamite produces a pressure of 11,250 lb. per sq. in. at a distance of 10 feet, and a charge of 180 lb. produces a pressure of 12,000 lb. per sq. in. at a distance of 15 feet at the same level. What charge must be used to produce a pressure of 18,000 lb. per sq. in. at a distance of 5 feet at the same level ?

14. If x varies directly as y and inversely as the square root of z, and if $x=8$ when $y=4$ and $z=9$, find the equation connecting x, y, z.

15. If $A \propto BC^2$ and $B \propto xy^2$ and $C \propto \frac{y}{x}$, find the relation between A, x, y.

16. If a stone falls in a vacuum, its speed at any moment varies directly as the time since it started ; and the distance it has fallen varies as the square of the time ; what variation-relation gives the speed in terms of the distance ?

17. The velocity of water issuing in a jet varies directly as the weight of water delivered per sec. and inversely as the area of the jet ; the horse-power necessary to produce it varies directly as the cube of the weight of water delivered per sec. and inversely as the square of the area. What variation-relation gives the speed in terms of (i) the horse-power and the weight, (ii) the horse-power and the area ?

If a 3 H.P. engine can just deliver 20 lb. of water per sec. in a jet of 2 sq. in., what weight of water per sec. is delivered in a jet of area 1 sq. in. by a 6 H.P. engine ? Find also what weight of water per sec. is delivered at a speed of 15 feet per sec. by a 5 H.P. engine. [Take 1 cu. ft. of water to weigh 1000 ounces.]

18. The horse-power of a windmill varies directly as the total sail area and the cube of the velocity of the wind. If the sail area is 1000 sq. ft. and the velocity of the wind 15 m.p.h., the horse-power is 9·7. Find the horse-power if the sail area is 1200 sq. ft. and the velocity of the wind 20 m.p.h. Find also a general formula.

19. The time of pneumatic transmission through a tube varies directly as the square root of the cube of its length, and inversely as the square root of the pressure and the square root of the diameter. If a pressure of 9 lb. per sq. inch drives the carrier through a tube 400 feet long and 3 inches in diameter in 3 secs., find the pressure required to drive it through a tube 100 feet long, $2\frac{1}{2}$ inches in diameter, in 1 sec.

20. x varies inversely as y and z; y varies directly as the square root of x and the square root of w. If x increases in the ratio 3 : 2 and z in the ratio 4 : 3, how does w alter?

SUMMARY OF RESULTS.

If x varies directly as the pth power of y and inversely as the qth power of z, then

$$x = k \cdot \frac{y^p}{z^q},$$

where k is a constant, independent of x, y, z.

If $x = a$, when $y = b$ and $z = c$, k is found from the equation

$$a = k \cdot \frac{b^p}{c^q}$$

or

$$\frac{x}{a} = \frac{y^p}{z^q} \div \frac{b^p}{c^q}.$$

If y is altered in the ratio $\lambda : 1$, and if z is altered in the ratio $\mu : 1$, then

x is altered in the ratio $\frac{\lambda^p}{\mu^q} : 1$.

CHAPTER XV.

FUNCTIONS OF ONE VARIABLE.

A. REPRESENTATION OF FUNCTIONS.

IF the area of a rectangle is fixed, the lengths of its sides cannot be chosen independently of one another ; the choice of a length for one side depends upon the length chosen for the other. In other words, "the length of one side is a *known function* of the length of the other, when the area is given."

For example, if the area is 36 sq. inches, and if one side is of length x inches, then the other side must be of length $\frac{36}{x}$ inches.

Example I. *Sketch* the graph of the function $\frac{36}{x}$.

An accurate drawing is not required and it is not necessary to make a table of values. All that is necessary is to note a few important features of the function as follows :

(i) If x is positive, $\frac{36}{x}$ is positive ;

if x is negative, $\frac{36}{x}$ is negative.

(ii) If x is large and positive, $\frac{36}{x}$ is small, and by making x sufficiently large, we can make $\frac{36}{x}$ as small as we please.

In symbols, when $x \to \infty$, $\frac{36}{x} \to 0$, which reads in words "when x tends to or approaches infinity, $\frac{36}{x}$ tends to or approaches zero."

(iii) If x is small and positive, $\frac{36}{x}$ is large and positive $\left[e.g. \text{ if } x=0{\cdot}1, \frac{36}{x}=360\right]$.

In symbols, when $x \to +0$, $\frac{36}{x} \to +\infty$.

(iv) When x actually $=0$, $\frac{36}{x}$ has no meaning.

(v) For a negative value of x, the value of $\frac{36}{x}$ is equal in magnitude but opposite in sign to its value for the corresponding positive value of x.

In particular, when $x \to -0$, $\frac{36}{x} \to -\infty$,

and when $x \to -\infty$, $\frac{36}{x} \to -0$.

The rough shape of the graph can now be sketched (see Fig. 10).

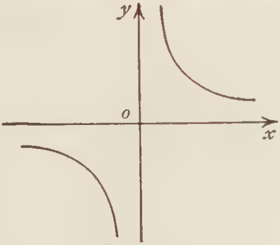


FIG. 10.

The graph is also said to correspond to the equation

$$y=\frac{36}{x} \quad \text{or} \quad xy=36.$$

EXERCISE XV. a.

[In the following examples, squared paper should not be used.]

1. Explain in words how the value of the function $(x-1)(x-3)$ varies as x varies from 1 to 3.

For what values of x does the function $=0$?

For what range of values of x is the function positive ?

2. Can you find a value of x for which the function x^2-4x+4 is negative ?

Construct another function of x which has a similar property.

3. For what values of x is the function x^2+x-6 zero ?

For what range of values of x is the function x^2+x-6 negative ?

4. (i) What is the greatest value of the function $9-(x-1)^2$?
   (ii) For what values of x is this function zero ?
   (iii) To what value does this function tend when $x\to\infty$, and when $x\to-\infty$?
   (iv) Sketch the graph of this function.

5. Sketch the graphs of (i) x^2, (ii) $(x-3)^2$, (iii) $(x+2)^2$.

6. Sketch the graphs of
   (i) $+\sqrt{x}$, (ii) $-\sqrt{x}$, (iii) $\pm\sqrt{x-2}$, (iv) $\pm\sqrt{x+1}$.

7. (i) Can you find (*a*) a positive value of x, (*b*) a negative value of x, such that $\frac{1}{x^2}<0{\cdot}01$?
   (ii) When is this function $\frac{1}{x^2}>100$?
   (iii) Sketch the graph of $\frac{1}{x^2}$.

8. What is the value of the function $\frac{1}{x-3}$
   (i) when x is large and positive, *e.g.* $+1003$?
   (ii) when x is large and negative, *e.g.* -997 ?
   (iii) when x is nearly equal to 3, (*a*) if $x>3$, *e.g.* $3{\cdot}001$, (*b*) if $x<3$, *e.g.* $2{\cdot}999$?
   (iv) Sketch the graph of $\frac{1}{x-3}$.

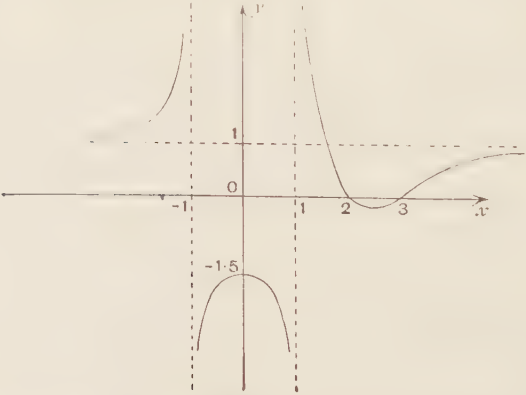


FIG. 11.

9. The graph of a certain function of x is shown in Fig. 11. Describe in words the variation in value of this function of x as x varies from $-\infty$ to $+\infty$

10. *ONP* is a variable triangle with a right angle at N; ON lies along the axis Ox; the length of the hypotenuse OP is 5 and $ON = x$. What is the length of NP?

As x varies, what is the locus of P?

Of what function of x is this locus the graph?

11. The graph of a certain function of x is shown in Fig. 12. Describe in words how the value of the function varies as x varies from -2 to $+4$. If the same series of values of the function recurs again and again every time x is increased by 7, what can you say about the value of the function as $x \to \infty$?

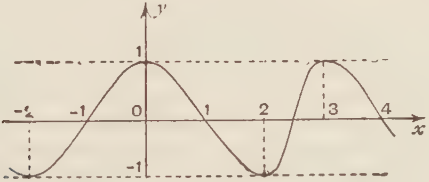


FIG. 12.

12. The values of a certain function of x can be calculated for all values of x, positive and negative. The function never has values greater than 1 or less than -2, and is zero when x equals -1 or 3 or 4. The function is positive if $-1 < x < 3$ or if $4 < x < \infty$, and for other values of x is negative.

Sketch the simplest graph of this function.

13. Sketch the simplest graph of a function which has the following characteristics:

(i) $\to +1$, when $x \to +\infty$.
(ii) $\to +\infty$, when $x \to +1$, provided $x > 1$.
(iii) Is never equal to 1.
(iv) Is not defined if $-1 < x < +1$.
(v) $\to -1$, when $x \to -\infty$.
(vi) $\to -\infty$, when $x \to -1$, provided $x < -1$.

14. (i) Can you find a value of x for which $\dfrac{x-1}{x-2}$ is firstly > 100 and secondly < -100?

(ii) For what value of x is this function zero?
(iii) For what range of values of x is this function negative?
(iv) Can you find a value of x for which the function equals 1?
(v) For what value of x is the function equal to 1·001?
(vi) For what value of x is the function equal to 0·999?
(vii) Sketch the graph of this function.

15. (i) For what values of x is the function $(x-1)(x-3)(x-4)$ zero?

(ii) For what range of values of x is this function positive?

(iii) Find the values of the function when $x=10,\ 0,\ -10$.

(iv) Can you find a value of x for which the function is $>1,000,000$?

(v) Sketch the graph of this function.

16. (i) Can you find a value of x between 0 and 10 for which the function

$$\frac{(x-1)(x-5)}{x-3}$$

is firstly >1000 and secondly <-1000?

(ii) Can you find the value of the function when $x=3$?

(iii) Describe the changes of sign in the function as x increases from -1 to $+8$, stating also where it vanishes.

(iv) What is the approximate error per cent. in taking the function as equal to x when $x=1000$ and when $x=-1000$?

(v) Sketch the graph of this function.

17. Sketch the graph of

(i) $(x-1)(x-2)(x-3)(x-4)(x-5)$;

(ii) $(x-1)(x-2)^2(x-4)(x-5)$;

(iii) $(x-1)(x-2)^3(x-5)$.

18. (i) Find approximately the value of the function

$$\frac{x-1}{(x-2)(x-3)}$$

if $x=1{\cdot}99,\ 2{\cdot}01,\ 2\tfrac{1}{2},\ 2{\cdot}99,\ 3{\cdot}01$.

(ii) What can you say about the value of this function when x is large; about how much, for example, is it if x is a million?

(iii) What is its approximate value if $x=-1,000,000$?

(iv) For what range of values of x is this function negative?

(v) Sketch the graph of the function.

19. Sketch the graph of $\dfrac{x-2}{(x-1)(x-3)}$.

20. Sketch the graph of $\dfrac{x-1}{(x-2)^2}$.

21. Sketch the graph of $\dfrac{(x+1)(x-6)}{(x-1)(x-4)}$.

22. Sketch the graph of $\dfrac{x(x-3)}{(x+2)(x-1)}$.

23. (i) For what range of values of x can the value of the function $\pm\sqrt{(1-x)(x+2)}$ be computed ?

(ii) For what values of x is this function zero ?

(iii) Sketch the graph of this function.

24. Sketch the graph of the function $\pm(x-3)\sqrt{x-2}$.

25. Sketch the graph of the function $+(x-4)\sqrt{(x-1)(x-3)}$.

SUMMARY OF RESULTS.

The function $f(x) \equiv \dfrac{(x-a)(x-b)(x-c)\ldots}{(x-p)(x-q)(x-r)\ldots}$

vanishes when $x=a$ or b or c, etc., and tends to infinity when $x \to p$ or $\to q$ or $\to r$, etc.

The statement that $f(x) \to +\infty$ when $x \to p$ means that $f(x)$ can be made to exceed any number however large, if x is given a value sufficiently close to p or any value closer than this to p.

In order to find the range of values for which a function is positive (or negative), it is usually best to factorise it, when possible.

B. CONSTRUCTION OF FUNCTIONS.

EXERCISE XV. b.

1. In Fig. 13 the coordinates of A, B, P are respectively (2, 3) ; (6, 5) ; (x, y). Express y as a function of x.

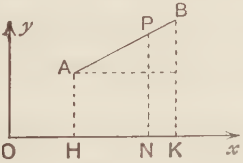


FIG. 13.

2. AB is the diameter of a circle APB ; PN is the perpendicular from P to AB ; if $AB=8$, $AN=x$, $PN=y$, express y as a function of x.

3. $ABCD$ is a rectangle ; P is a point such that the perpendicular PN from P to AB is equal to PC. If $BC=4$, $PN=x$, $NB=y$, express y as a function of x.

4. PN is an altitude of the triangle APB ; O is the mid-point of AB. If $PN=y$, $ON=x$, $AB=6$ and $PA^2+PB^2=30$, express y as a function of x.

5. AB is a diameter of a circle; a line AQR cuts the circle at Q and the tangent at B in R; P is a point on AQ such that $AP=QR$; PN, QK are the perpendiculars from P, Q to AB; $AB=4$, $AN=x$, $PN=y$, $QK=z$; express (i) z as a function of x (use the fact that $BK=AN$), (ii) y as a function of x.

Sketch the graph of y.

6. AOB is a triangle right-angled at O; PN is the perpendicular from a point P on AB to OB; if $AP=3$, $PB=4$, $ON=x$, $PN=y$, express y as a function of x.

7. A point R is taken on the side AB of a triangle ABC of area z sq. inches such that $AR=x \,.\, AB$, where $x > \frac{1}{2}$. RQ, RH are drawn parallel to BC, AC to meet AC, BC at Q, H; QK is drawn parallel to AB to meet BC at K. Express the area of $QRHK$ as a function of x and z. (C.S.C.)

8. AB is a diameter of a circle; CD is a chord parallel to AB and at distance b inches from it; any chord AQ cuts CD at R; RN is drawn perpendicular to AB; QP is drawn parallel to AB and cutting RN at P; if $AB=a$, $AN=x$, $NP=y$ inches, express y as a function of x.

9. AB is a fixed diameter of a given circle; the tangent at B meets a variable chord AP at Q: $AB=d$, $AP=x$, $BQ=y$; express y as a function of x.

10. $ABCD$ is a straight line and AH, BK, CL are three fixed lines perpendicular to it; $AB=a$, $BC=b$, $CD=c$; a variable line cuts AH, BK, CL at P, Q, R and DQ cuts CL at S; if $AP=y$, $CR=z$, $CS=x$, express x as a function of y, z.

11. Assuming the length, breadth and depth of an ordinary match box are in the ratio 10 : 7 : 3, express the volume of the box as a function of the area of match-board used in making it, *i.e.* box and drawer.

12. Taking Ox, Oy as perpendicular axes, sketch the graphs of x^2 and $\frac{1}{x}$ for positive values of x; any line perpendicular to Ox cuts the graphs at P, Q and Ox at N; if $ON=x$, express the area of the triangle OPQ as a function of x.

13. A sector of a circle of unit radius is folded to form a circular cone. If the angle of the sector is x right angles, express the volume of the cone as a function of x, assuming $x < 4$.

14. Take a rectangular sheet of paper of unit area and fold it in half; then fold it again in half with the second crease perpendicular to the first, so that it is now reduced to a quarter of the original size. Repeat this process, taking each time the new crease perpendicular to the last. Suppose the process of folding to be repeated x times in all.

(i) Express the area of the top of the packet as a function of x.

(ii) Supposing a triangular wedge is cut out of the middle of the last crease obtained and the paper is then unfolded, express the number of holes in the original sheet as a function of x.

15. Two metre rules AH, BK, whose zero graduations are at A, B, lie parallel to each other, and in the same sense, on a table; a third metre rule CL, zero graduation at C, touches them at A, B where the graduations on CL are a, b; a straight edge laid across AH, BK, CL meets them where the graduations are x, y, z respectively; express z in terms of x, y.

16. The triangle AOQ is right-angled at O; the bisector of $\angle AOQ$ cuts AQ at P; $OA=a$, $OP=y$, $OQ=x$. Draw PN perpendicular to OQ; (i) express PN and ON in terms of y; (ii) express y as a function of x.

17. AH BK, CL are three lines, graduated in the same way, the graduations being that of a 10″ slide rule; A, B, C are the unit graduations in each case [*e.g.* the graduation for "7" is $10 \log 7$ inches from the unit graduation]. They are placed parallel to each other with C at the mid-point of AB. A straight-edge lies across them and meets AH, BK, CL at the graduations x, y, z respectively; (i) express z in terms of x, y; (ii) how is the expression altered if BK is in the opposite sense both to AH and CL?

18. With the data of Ex. 17, but supposing that C divides AB in the ratio $2:3$, express z in terms of x, y. How is this expression altered if C is the "2" graduation instead of the unit graduation on CL?

19. What is the simplest function of x which is zero when $x=2$ and $x=3$?

20. What is the simplest function of x which is zero when $x=1$ and tends to infinity when $x \to 2$?

21. Find a function of x which is zero for $x=1$ and $x=3$ and is positive if $1<x<3$.

22. Find a function of x which is zero for $x=1$ and $x=-1$ and is never positive.

23. Find a function of x which can be computed for any value of x between 2 and -2, but for no other values.

24. Find a function of x which is equal to 2 when $x=1$, but is never less than 2.

C. GRAPHICAL SOLUTIONS.

Example II. $ABCD$ is a rectangle such that $AB=4''$, $BC=8''$; P, Q are points on BC, CD, such that

$$CQ=\tfrac{1}{2}BP=x \text{ inches.}$$

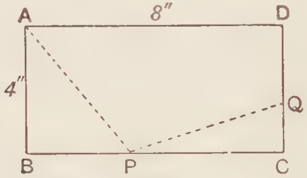


FIG. 14.

Express the area of $APQD$ in terms of x. Represent this function graphically: and find from the graph the length of CQ when the area of $APQD$ is 24 sq. inches.

Check the result by algebra.

(i) The area of $ABCD=4\times 8=32$ sq. inches.

$$CQ=x, \quad BP=2x; \quad \therefore\ PC=8-2x;$$

$\therefore$ area of triangle $ABP=\frac{1}{2}\times 4\times 2x=4x$ sq. inches,
area of triangle $PCQ=\frac{1}{2}(8-2x)\times x=x(4-x)$ sq. inches;

$$\begin{aligned}\therefore \text{ area of } APQD &= 32-4x-x(4-x)\\ &= 32-4x-4x+x^2\\ &= 32-8x+x^2 \text{ sq. inches.}\end{aligned}$$ *Answer.*

(ii) To represent $x^2-8x+32$ by a graph.

Since CQ does not exceed CD, x is not greater than 4.

Construct a table of values for x from 0 to 4.

x	0	1	2	3	4
x^2 $-8x$ 32	0 0 32	1 −8 32	4 −16 32	9 −24 32	16 −32 32
$x^2-8x+32$	32	25	20	17	16

Plotting these, we obtain the required graph.

(iii) From the graph we see that the function equals 24 if $x = 1{\cdot}17$;

$$\therefore\ CQ = 1{\cdot}17'' \text{ when the area of } APQD \text{ is 24 sq. in.}$$

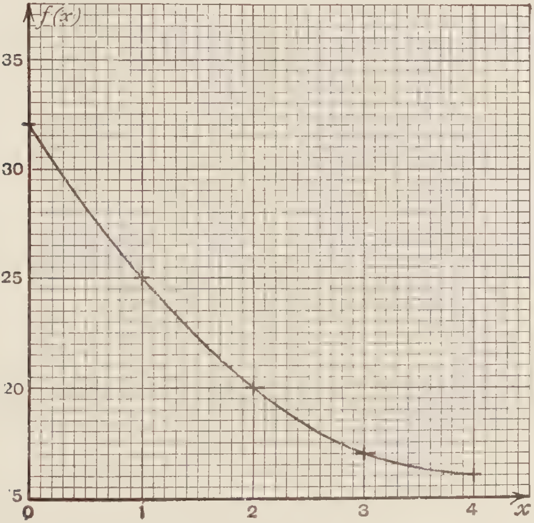


Fig. 15.

(iv) To solve by algebra, we take the equation

$$x^2 - 8x + 32 = 24\,;$$

$$\therefore\ x^2 - 8x + 8 = 0\,;$$

$$\therefore\ x = \frac{8 \pm \sqrt{64 - 32}}{2} = \frac{8 \pm \sqrt{32}}{2}$$

$$= \frac{8 \pm 5{\cdot}657}{2} = \frac{13{\cdot}657}{2} \quad \text{or} \quad \frac{2{\cdot}343}{2}$$

$$= 6{\cdot}828 \quad \text{or} \quad 1{\cdot}171.$$

The value 6·828 is excluded by geometrical considerations;

$$\therefore\ x = 1{\cdot}171.$$

EXERCISE XV. c.

[Examples **1-4** require the use of geometrical instruments.]

1. AB is the diameter of a circle APB; PN is the perpendicular from P to AB; $AB = 10$ cm. By taking different positions for P and making the necessary measurements, construct a table of values connecting the lengths of AP and AN. Represent it graphically, and find the length of AN when $AP = 2AN$.

2. Draw a quadrilateral $AKLB$ having $\angle AKL = 90° = \angle KLB$, $AK = 6$ cm., $KL = 12$ cm., $LB = 3$ cm.; P is any point on KL. Make the necessary measurements for a table of values connecting $AP + PB$ with KP; represent it graphically, and find the length of KP for which $AP + PB$ is least.

3. B is the foot of the perpendicular from A to a line CBD; $AB = 12$ cm., $BC = 4$ cm., $BD = 6$ cm.; P is any point on AB. Make the necessary measurements for a table of values connecting AP with $AP + CP + DP$; represent it graphically, and find the length of AP for which $AP + CP + DP$ is least.

4. With the notation and data of Ex. 1, make a table of values for the relation between AN and the area of the triangle APN; construct the graph, and find the length of AN for which the triangle APN is of maximum area.

If $AN = x$ cm., express the area of the triangle APN as a function of x.

5. A funnel is made in the shape of a pyramid of height a inches standing on a square base of side $2x$ inches. The funnel holds 48 cu. inches of liquid. Prove that (i) $a = \frac{36}{x^2}$; (ii) if the area of the four slant faces is y sq. inches, $y = 4x\sqrt{x^2 + a^2}$.

Tabulate the values of y for x equal to 1, 2, 3, 4, 5, and from the graph find the value of x for which y is least. (C.S.C.)

6. The perimeter of a right-angled triangle is 20 cm., and the shortest side is x cm.; if the area is A sq. cm., express A as a function of x. Find from a graph the value of x for which the area is a maximum. (C.S.C.)

7. The base of an open cistern is a square of side x feet; its volume is 200 cu. ft. The total area of the base and sides is A sq. ft. Express A as a function of x. Represent A graphically for values of x from 4 to 12. Find the depth of the cistern (i) when A is least, (ii) when the total area of the base and sides is 200 sq. ft. (C.S.C.)

8. A sheet of tin is 24 inches square; equal squares are cut out at the four corners, and the sides are then turned up to make a rectangular box; if the side of each square is x inches and the volume of the box is V cu. inches, express V as a function of x, and find graphically the maximum capacity of the box. (C.S.C.)

9. $CDEF$ is a rectangular sheet of paper; $CD = 30$ cm., $DE = 5$ cm.; A, B are the mid-points of CD, EF; P is a point on CA such that $PA = x$ cm. The paper is folded over with PB as crease, and the new position of PD cuts FB at Q; $QB = y$ cm.

Express y as a function of x. Find graphically (i) the least value of y, (ii) the length of AP if QB is 7 cm. Check the answer to (ii) by algebra. [Note that $PQ = QB$, and draw PN perpendicular to FB.] (C.S.C.)

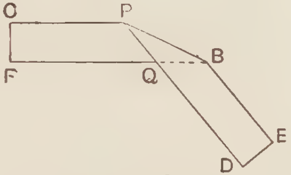


FIG. 16.

10. In the triangle ABC, $AB = AC = 12$ feet, $\angle BAC = 90°$; DHE, FHG are parallels to AC, AB; $AD = AF = x$ feet; express the area of $DHGB$ as a function of x, and find graphically the value of x for which this is a maximum. (C.S.C.)

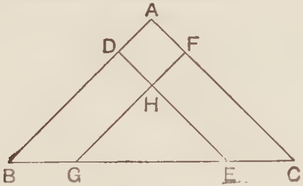


FIG. 17.

11. The base of a box is a square of side x feet, and the length of cord (exclusive of knots) needed for binding it once round each way is 30 feet. Express the volume V cu. ft. of the box as a function of x, and find from a graph the maximum volume of the box.

12. From a circular cone, height 12 cm., base diameter 12 cm., the greatest circular cylinder of radius r cm. is cut. Express its volume V cu. cm. as a function of r, and find from a graph the volume of the greatest cylinder.

13. Fill in the gaps in the following table, and so construct a table of values of the function $4 + 2x - x^2$ for values of x from -2 to $+4$:

x	-2	-1	0	1	2	3	4
4 $+2x$ $-x^2$	4 -4 -4					4 6 -9	
$4 + 2x - x^2$	-4					1	

Draw accurately the graph of this function, and from your graph answer the following questions:

(i) For what values of x does $4+2x-x^2=0$?

(ii) Between what values of x is $4+2x-x^2$ positive?

(iii) What is the maximum value attained by the function $4+2x-x^2$, and for what value of x does the function have its maximum value?

(iv) For what values of x does $4+2x-x^2=1\frac{1}{2}$? Solve the equation $2(4+2x-x^2)=3$.

(v) If $2x-x^2=-2$, what is the value of $4+2x-x^2$? For what values of x does $4+2x-x^2$ have this value? Solve the equation $2x-x^2=-2$.

(vi) Solve the equation $4+2x-x^2=-1$ from the graph, and solve the equation $x^2-2x-5=0$ by formula. Compare the results.

14. Construct a table of values of the function $\frac{1}{5}(2x^2+2x-3)$ for values of x from -4 to $+4$ as follows:

x	-4	-3	-2	-1	0	1	2	3	4
$2x^2$	32						8		
$2x$	-8						4		
-3	-3						-3		
$2x^2+2x-3$	21						9		
$\frac{1}{5}(2x^2+2x-3)$	4·2						1·8		

Draw accurately the graph of this function, and from your graph answer the following questions:

(i) For what values of x is $2x^2+2x-3=0$?

(ii) Between what values of x is $2x^2+2x-3$ negative?

(iii) What is the minimum value attained by the function $\frac{1}{5}(2x^2+2x-3)$, and for what value of x does the function have its minimum value?

(iv) For what values of x does $\frac{1}{5}(2x^2+2x-3)=1$? Solve the equation $2x^2+2x-3=5$.

(v) If $2x^2+2x=7$, what is the value of $2x^2+2x-3$ and of $\frac{1}{5}(2x^2+2x-3)$? For what values of x does $\frac{1}{5}(2x^2+2x-3)$ have this value? Solve the equation $2x^2+2x=7$.

(vi) Solve the equation $x^2+x=1$ from the graph. Check your result by solving the equation by the formula.

15. Construct a table of values of the function

$$\tfrac{1}{5}(x^3 - 3x^2 - 4x - 6)$$

for values of x from -3 to $+5$, and draw the graph of this function.

From your graph answer the following questions :

(i) For what values of x does $x^3 - 3x^2 - 4x - 6 = 0$?

(ii) For what range of values of x is $x^3 - 3x^2 - 4x - 6$ positive ?

(iii) What is the smallest value of the function

$$\tfrac{1}{5}(x^3 - 3x^2 - 4x - 6)$$

for positive values of x ? For what positive value of x does it have the smallest value ?

(iv) What is the largest value of the function for negative values of x, and for what negative value of x does it have its largest value ?

(v) Solve the equation $x^3 - 3x^2 - 4x - 6 = 7$ from the graph.

(vi) Solve the equations : (*a*) $x^3 - 3x^2 - 4x + 4 = 0$;
(*b*) $x^3 - 3x^2 - 4x + 19 = 0$;
(*c*) $x^3 - 3x^2 - 4x - 16 = 0$.

(vii) Solve the equation $x^3 - 3x^2 - 4x + 12 = 0$ from the graph, and also by factorising $x^3 - 3x^2 - 4x + 12$.

16. Draw a graph of the function $\dfrac{1}{x^2}$ for values of x from -3 to $+3$.

(i) With the same axes, and to the same scale, draw the graph of the function $2 - x$. For what values of x are the functions $\dfrac{1}{x^2}$ and $2 - x$ equal ? Hence solve the equation $x^3 - 2x^2 + 1 = 0$.

(ii) Draw the graph of the function $x - 2$ with the same axes, and solve the equation $\dfrac{1}{x^2} = x - 2$.

(iii) Draw the graph of the function $2x + 1$, and solve the equation $x^2(2x + 1) = 1$.

17. Draw graphs of the functions $\dfrac{1}{x}$ and $x^2 - 4$ with the same axes and to the same scale for values of x from -3 to $+3$.

For what values of x are these functions equal ?

Hence solve the equation $x^3 - 4x = 1$.

18. Draw graphs of the functions x^2 and $x - 1$ with the same axes and to the same scale. When are these functions equal ?

What does this tell you about the solutions of the equation $x^2 - x + 1 = 0$?

D. FUNCTIONAL NOTATION.

Any expression whose value can be determined when the value of x is known, can be represented by the symbol $f(x)$.

$f(x)$ is shorthand for the words "a function of x."

In a particular question $f(x)$ might be used to represent the function $5x^4 - 2x + 8$.

Then $f(2)$ would mean the value of this function when $x = 2$.

$$\begin{aligned} \therefore\ f(2) \text{ would mean } & 5(2^4) - 2(2) + 8 \\ &= 80 - 4 + 8 \\ &= 84, \end{aligned}$$

and
$$\begin{aligned} f(-1) \text{ would mean } & 5(-1)^4 - 2(-1) + 8 \\ &= 5 + 2 + 8 \\ &= 15, \end{aligned}$$

and
$$\begin{aligned} f(0) \text{ would mean } & 5(0)^4 - 2(0) + 8 \\ &= 8. \end{aligned}$$

Example III. If $f(x) \equiv x^2 - 3 + \dfrac{1}{x}$, find the value of $f(5)$ and $f(2a)$.

$$f(5) = 5^2 - 3 + \tfrac{1}{5} = 25 - 3 + \tfrac{1}{5} = 22\tfrac{1}{5},$$

$$f(2a) = (2a)^2 - 3 + \frac{1}{2a} = 4a^2 - 3 + \frac{1}{2a}.$$

Example IV. If $f(x) \equiv x^2 + 2x$, find the value of $f(x+h) - f(x)$.

Here
$$\begin{aligned} f(x+h) &= (x+h)^2 + 2(x+h) \\ &= x^2 + 2xh + h^2 + 2x + 2h; \\ \therefore\ f(x+h) - f(x) &= x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x \\ &= 2xh + h^2 + 2h \\ &= h(2x + 2 + h). \end{aligned}$$

EXERCISE XV. d.

1. If $f(x) \equiv x^2 + 2$, find the values of $f(1)$; $f(0)$; $f(-1)$; $f(2a)$; $f(b^3)$.

2. If $f(x) \equiv 10^x$, find the values of $f(1)$; $f(2)$; $f(0)$; $f(-1)$; $f(2x)$.

3. If $f(x) \equiv \log x$, find the values of $f(1000)$; $f(2)$; $f(20)$; $f(1)$; $f(x^3)$.

4. If $f(x) \equiv x^2 - 3x + 5$, find the values of $f(2)$; $f(x+h)$; $f\left(\frac{1}{x}\right)$.

5. If $f(x) \equiv x^2 + 3x$, find the value of $\frac{f(x+h) - f(x)}{h}$. What is the approximate value of this when h is small compared with x ?

6. If $f(x) \equiv \frac{1}{x}$, find the values of $f(1)$ and $\frac{f(1+h) - f(1)}{h}$. What is the approximate value of this last expression when h is small ?

7. If $f(x) \equiv x^2 + 5$, simplify

(i) $f(3x)$;
(ii) $f(x+1) + f(x-1) - 2f(x)$.

8. If $f(x) \equiv x^2$, simplify

(i) $f(x^3)$;
(ii) $f(x+h) - f(x) - \{f(x) - f(x-h)\}$.

9. If $f(x) \equiv (x+7)(7-x)$, show that $f(-x)$ equals $f(x)$. Construct another function which has this property.

10. If $f(x) \equiv \frac{1+x^2}{x}$, show that $f\left(\frac{1}{x}\right)$ is equal to $f(x)$. Construct another function which has this property.

11. For what values of x does $f(x) = 0$ if $f(x)$ is $(x-1)(x-2)$?

12. If $f(x) \equiv 2x^2 - x$, solve the equation $f(x) = f(x-1)$.

13. If $f(x) \equiv \frac{1}{1-x}$ simplify $f(f(x))$.

Find simple functions which have the following properties :

14. $f(3) = 0, f(4) = 0$.
15. $f(3) = 0, f(4) = 8$.

16. $f(3) = 0$, $f(x) \to \infty$ when $x \to 1$, and also $f(1{,}000{,}000)$ is approximately equal to 3.

17. $f(x) = f(-x), f(1) = 3$.
18. $f(x) + f(y) = f(xy)$.

19. $f(x) \times f(y) = f(x+y)$.
20. $f(x) = -f(-x), f(1) = 0$.

21. $f(1) = f(3) = f(5) = -1, f(2) = f(4) = f(6) = +1$.

CHAPTER XVI.

LIMITS AND GRADIENTS.

It has already been noticed, in connection with the graphical representation of functions, that a function may have no meaning for one or more special values of the variable (*e.g.* $\frac{36}{x}$ has no meaning when $x=0$, see p. 307). We shall now examine more closely the behaviour of a function in the neighbourhood of such values. Examples I. and II. illustrate the meaning of a "Limit" of a function: a more formal discussion will be found in the Introduction.

Example I. Plot the values of the functions:

$$\text{(i) } 1-\frac{1}{n}; \qquad \text{(ii) } 1+\frac{1}{n}; \qquad \text{(iii) } 1+(-1)^n\cdot\frac{1}{n}$$

for positive *integral* values of n.

(i) $1-\frac{1}{n}$.

We have the following table of values:

$n=1$	2	3	4	5	6	7	8
$1-\frac{1}{n}=0$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$

which are represented in Fig. 18.

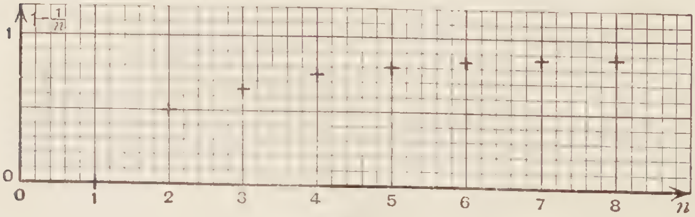


FIG. 18.

(ii) $1+\frac{1}{n}$.

We have the following table of values :

$n=1$	2	3	4	5	6	7	8
$1+\frac{1}{n}=2$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{6}{5}$	$\frac{7}{6}$	$\frac{8}{7}$	$\frac{9}{8}$

which are represented in Fig. 19.

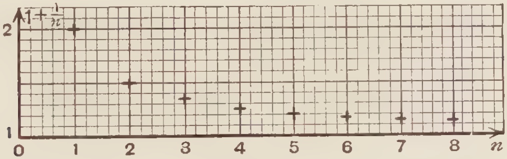


FIG. 19

(iii) $1+(-1)^n\cdot\frac{1}{n}$.

We have the following table of values :

$n=\quad 1$	2	3	4	5	6	7	8
$1+(-1)^n\cdot\frac{1}{n}=0$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{5}{4}$	$\frac{4}{5}$	$\frac{7}{6}$	$\frac{6}{7}$	$\frac{9}{8}$

which are represented in Fig. 20.

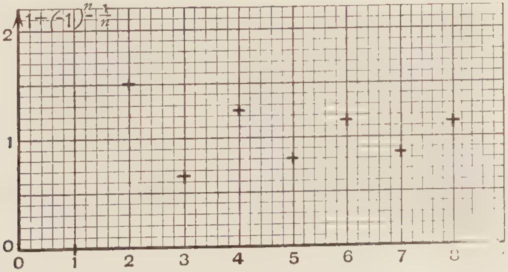


FIG. 20.

Either from the graphs or from examining the functions direct we note that

(i) $1-\frac{1}{n}$ is always less than 1, and that, as n increases, the function steadily increases and approaches the value 1.

(ii) $1+\frac{1}{n}$ is always greater than 1, and that as n increases the function steadily decreases and approaches the value 1.

(iii) $1+(-1)^n\cdot\frac{1}{n}$ is alternately less and greater than 1, and that as n increases the function increases and decreases alternately, but approaches the value 1.

In each case it is possible to find a value of n for which and for all greater values the function differs from 1 by less than a given amount, however small.

E.g. each function differs from 1 by less than 0·001 if $n > 1000$.

But it is impossible to find a value of n for which any of the functions actually equals 1.

Under these conditions we say that:

The Limit of $1+\frac{1}{n}$ as n tends to infinity is 1 [but the limit is not attained in this case]. And we write it as follows:

$$\underset{n\to\infty}{\mathrm{Lt}}\left(1+\frac{1}{n}\right)=1.$$

Similarly $\underset{n\to\infty}{\mathrm{Lt}}\left(1-\frac{1}{n}\right)=1$ and $\underset{n\to\infty}{\mathrm{Lt}}\left[1+(-1)^n\cdot\frac{1}{n}\right]=1$.

Example II. What is the limit of $\frac{(1+h)^2-1}{h}$ as h tends to 0?

$$\frac{(1+h)^2-1}{h}=\frac{1+2h+h^2-1}{h}=\frac{2h+h^2}{h}.$$

If $h=0$, $\frac{2h+h^2}{h}$ becomes $\frac{0}{0}$, which is a meaningless expression.

If $h\neq 0$, $\frac{2h+h^2}{h}=2+h$.

The smaller h becomes, the closer $2+h$ approaches to 2. It never attains the value 2, for h is never actually 0. But we can choose h so that $2+h$ differs from 2 for that and all smaller values of h by less than any given amount, however small;

$\therefore$ its limit is 2;

$$\therefore \underset{h\to 0}{\mathrm{Lt}}\frac{(1+h)^2-1}{h}=2 \text{ (limit not attained).}$$

Note.—The x-coordinate of a point is sometimes called the *abscissa*, and the y-coordinate is called the *ordinate.*

For example, in Figure 21, ON is the *abscissa* of P and PN is the *ordinate* of P.

EXERCISE XVI. a.

1. (i) Plot the values of $\dfrac{2n+1}{n+1}$ for integral values of n from 3 to 10.

(ii) Can you find a value of n for which $\dfrac{2n+1}{n+1}$ differs from 2 by less than $0{\cdot}001$?

(iii) Can you find a value of n for which $\dfrac{2n+1}{n+1}$ equals 2 ?

(iv) What is the limit of $\dfrac{2n+1}{n+1}$ as n tends to infinity?

2. (i) If $s_1 = 1$; $s_2 = 1 + \frac{1}{2}$; $s_3 = 1 + \frac{1}{2} + \frac{1}{4}$; $s_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, etc., plot the values of s_n for values of n from 1 to 5.

(ii) Can you find a value of n for which s_n differs from 2 by less than $0{\cdot}001$?

(iii) Can you find a value of n for which s_n equals 2 ?

(iv) What is the value of $\underset{n \to \infty}{\mathrm{Lt}}\ s_n$?

3. (i) Express as a decimal to 3 figures the value of $\dfrac{n^2}{2n^2+1}$ when $n=5$ and $n=10$.

(ii) What is the limit of $\dfrac{n^2}{2n^2+1}$ when n tends to infinity ?

(iii) Is this limit attained ?

4. (i) Find the values of $\dfrac{(0{\cdot}1)^n}{1+(0{\cdot}1)^n}$ for integral values of n from 1 to 4.

(ii) What is the value of $\underset{n \to \infty}{\mathrm{Lt}} \dfrac{(0{\cdot}1)^n}{1+(0{\cdot}1)^n}$?

5. (i) Plot the values of $\dfrac{2^n}{1+2^n}$ for integral values of n from 1 to 4.

(ii) What is the value of $\underset{n \to \infty}{\mathrm{Lt}} \dfrac{2^n}{1+2^n}$?

(iii) For what integral value of n does $\dfrac{2^n}{1+2^n}$ differ from 1 by less than $\frac{1}{100}$?

6. (i) Find the values of $\frac{x^2-1}{x^2-x}$ for $x=2$, 1·5, 1·1, 1·01.

(ii) What is the value of $\underset{x\to 1}{\mathrm{Lt}}\ \frac{x^2-1}{x^2-x}$?

(iii) Is this limit attained?

(iv) Find a value of x for which $\frac{x^2-1}{x^2-x}$ differs from 2 by less than 0·001.

7. Fig. 21 shows part of the graph of $y=\frac{1}{2}x^3$.

$$ON=1,\quad NM=h.$$

PN, QM are ordinates and PR is perpendicular to QM.

(i) Express $\frac{QR}{PR}$ in terms of h.

(ii) What is the limit of $\frac{QR}{PR}$ as $h\to 0$? Interpret this result geometrically.

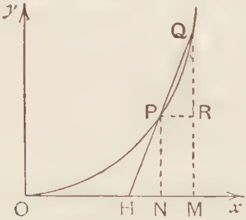


FIG. 21.

8. If a stone is dropped in a vacuum, it falls s feet in t seconds, where $s=16t^2$; the graph of this is represented in Fig. 22.

$$ON=\frac{1}{2},\quad NM=h.$$

PN, QM are ordinates and PR is perpendicular to QM.

(i) Express $\frac{QR}{PR}$ in terms of h.

(ii) What is the limit of $\frac{QR}{PR}$ as $h\to 0$? Interpret this result.

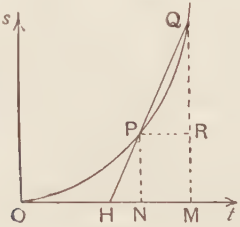


FIG. 22.

9. (i) Draw the graph of $5x - x^2$ from $x = 0$ to 5.

(ii) Draw the ordinate corresponding to $x = 1{\cdot}5$. What is its length ?

(iii) Take any point A on the x-axis and call $OA = a$. Give a geometrical meaning to the ratio

$$\frac{[5(a+h)-(a+h)^2]-[5a-a^2]}{h}.$$

(iv) What is the limit of this function when $h \to 0$?

(v) For what value of a is this limit equal to 0 ? Interpret this result geometrically.

10. (i) Draw the graph of $\frac{1}{x}$ from $x = 4$ to $\frac{1}{4}$.

(ii) Give a geometrical meaning to the ratio $\dfrac{\frac{1}{2+h}-\frac{1}{2}}{h}$.

(iii) What is the limit of this ratio when $h \to 0$? Interpret this result.

(iv) Give a geometrical meaning to the ratio $\dfrac{\frac{1}{a+h}-\frac{1}{a}}{h}$; evaluate its limit when $h \to 0$; and find the value of a for which this limit equals $-\frac{1}{4}$.

11. (i) What is the average of the numbers 1, 2, 3, 4, ... , $(n-1)$?

(ii) Prove that their sum is $\frac{n(n-1)}{2}$.

(iii) Find the value of $\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n^2}[1+2+3+\ldots+(n-1)]$.

12. ABC is an isosceles right-angled triangle; $BC = 10$ cm.; BC is divided into n equal parts, and through each point of division a line is drawn parallel to BA to meet CA, and rectangles are completed as in the figure. By using Ex. 11,

(i) find the sum of the areas of all the rectangles in terms of n;

(ii) find the limit of this sum when n tends to infinity;

(iii) what is the area of the triangle ABC ?

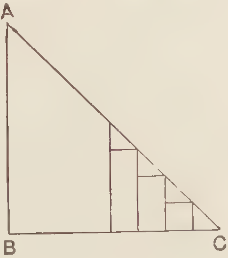


FIG. 23.

13. It can be proved that the sum of the series

$$1^2+2^2+3^2+4^2+\dots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

(i) Find the value of $\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n^3}(1^2+2^2+3^2+\dots+n^2)$.

(ii) Fig. 24 gives the graph of $y=x^2$; $OA=1$; OA is divided into n equal parts, and through each point of division lines are drawn parallel to Oy, and rectangles are completed as in the figure. Express the sum of the areas of all these rectangles in terms of n, and find the limit of this sum when n tends to infinity.

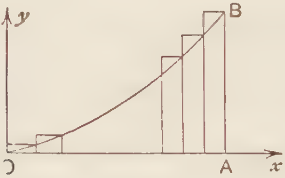


FIG. 24.

14. (i) Use the result given in No. 13 to find the value of

$$\frac{1}{n^3}[1^2+2^2+3^2+\dots+(n-1)^2].$$

(ii) If the rectangles are drawn so that each ends below the graph of $y=x^2$ (the arrangement in Fig. 23), find the sum of the areas of the rectangles in terms of n; and find the limit of this sum when $n\to\infty$.

(iii) By comparing this with the result in Ex. 13, what can you say about the area of the figure bounded by OA, AB and the curve OB in Fig. 24?

CALCULATION OF RATES OF CHANGE FROM STATISTICS.

Example III. The following table gives the height of the mercury barometer at intervals of two hours during a day:

Time	8 a.m.	10 a.m.	Noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.
Height in inches	28·57	28·65	28·85	29·10	29·22	29·12	29·0

Find the average rate at which the barometer was rising

(i) between 10 a.m. and noon ;

(ii) between noon and 2 p.m. ;

(iii) between 4 p.m. and 6 p.m.

Draw the barograph for the day, and find the rate at which the barometer was probably rising at noon.

(i) Between 10 a.m. and noon the barometer rose $0{\cdot}2''$ in 2 hours ;

∴ the average rate of rise was $0{\cdot}1''$ per hour.

(ii) Between noon and 2 p.m. the barometer rose $0{\cdot}25''$ in 2 hours ;

∴ the average rate of rise was $0{\cdot}125''$ per hour.

(iii) Between 4 p.m. and 6 p.m. the barometer fell $0{\cdot}1''$ in 2 hours ;

∴ the average rate of rise was $-0{\cdot}05''$ per hour.

Figure 25 gives the required barograph.

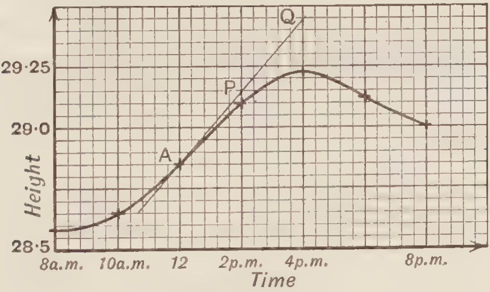


FIG. 25.

It is a curve and not a straight line, because the rate is altering throughout the day. If it continued to rise at the same rate after 12 o'clock as it is rising at 12 o'clock, the barograph would be the straight line formed by drawing the tangent APQ at the point A, which corresponds to 12 o'clock on the graph. In the figure, this tangent is drawn by eye, and cuts the 2 o'clock and 4 o'clock lines (or any other convenient lines) at P and Q. Now the difference in heights registered by P, Q is $0{\cdot}30''$.

∴ the rate of rise at A is $0{\cdot}30''$ in 2 hours,

or $0{\cdot}15''$ per hour.

EXERCISE XVI. b.

1. The following table gives the population of the United Kingdom for various years :

Year	1840	1850	1860	1870	1880	1890	1900
Population (in millions)	27·0	27·7	29·3	31·8	35·2	38·1	42·0

[Give answers in thousands per year.]

(*a*) What was the average rate of increase of population in the period 1840 to 1850 ?

(*b*) What was the average rate of increase of population for the whole period 1840 to 1900 ?

(*c*) During what period of ten years was the population increasing most rapidly, and what was the rate per year then ?

2. At the end of a minutes a car has travelled x miles, and at the end of b minutes it has travelled y miles. What was its average speed

(*a*) for the first a minutes ?

(*b*) for the first b minutes ?

(*c*) during the period a minutes to b minutes ?

If in the above question $a=3$ and $b=4$, what was the average speed of the car during the 4th minute ?

3. The following table gives the distances a car travels, starting from rest :

Time in minutes	5	10	15	20	25	30	35	40	45
Distance in miles	0·7	2·6	5·2	8·3	11·4	14·2	16·7	19	20·5

[Give answers in the form miles per hour.]

(*a*) What was the average speed of the car for the first 5 minutes ?

(*b*) What was the average speed for the first 20 minutes ?

(*c*) What was the average speed for the whole period, 45 minutes ?

(*d*) Draw a graph to illustrate the motion of the car, and by drawing a tangent to the graph at the point determined by $t=35$, find the probable speed of the car 35 minutes after it started.

(*e*) By drawing a tangent to the graph where it appears to be steepest, find the greatest speed attained by the car.

4. A lift ascends 90 ft. in 40 seconds, and its height h feet at intervals of 5 seconds is given by the following table:

t	5	10	15	20	25	30	35	40
h	4	15	30	45	62	77	86	90

Draw a graph to illustrate the motion, and draw tangents to the graph at the points determined by $t=10$, $t=20$, $t=30$.

(i) What is the average speed of the lift for the first 10 seconds ?

(ii) Estimate from your graph the speed of the lift after 10 seconds ?

(iii) What is the average speed of the lift for the first 20 seconds ?

(iv) Estimate its speed after 20 seconds.

(v) What is the average speed of the lift for the last 10 seconds ?

(vi) Estimate its speed after 30 seconds.

5. The weight that can be carried by a certain type of bridge varies with the diameters of the spars used in accordance with the following table:

Diameter of spar in inches -	3	5	7	9	11
Load carried in tons - -	0·1	0·7	2·4	6	12

[Give answers in the form tons per inch.]

What is the average rate at which the load carried increases when the diameter is increased from

(*a*) 3 inches to 5 inches ?

(*b*) 7 inches to 9 inches ?

(*c*) 9 inches to 11 inches ?

(*d*) 7 inches to 11 inches ?

Estimate the 'rate' of increase of the load when the diameter is 7 inches by drawing a graph.

6. Water runs out of a bath, and the volume that runs out is given by the following table:

Time in seconds - -	5	10	15	20	25
Volume in cubic feet - -	6	9	10·5	11·2	11·6

Draw a graph and find the rate of flow at the end of each period of 5 seconds. Show from your results that the rate of flow is proportional to the volume of water remaining in the bath, there being 12 cubic feet of water in the bath originally.

SUMMARY OF RESULTS.

(i) It is possible to have a function of x to which, for one or more special values of x, no meaning can be given.

E.g. $\dfrac{x^2-1}{x-1}$ has no meaning when $x=1$.

(ii) In such cases, the function may tend towards a definite limit as x tends to that value; but the limit is not attained.

E.g. $\underset{x\to 1}{\text{Lt}}\ \dfrac{x^2-1}{x-1}=2$, but no value of x exists for which $\dfrac{x^2-1}{x-1}$ equals 2.

(iii) If a function of x is represented by a graph, the rate at which the function is increasing for any value of x is represented by the slope of the tangent to the graph at the point corresponding to that value of x, provided that the units used in drawing the graph are taken into account.

CALCULATION OF RATES OF CHANGE FROM FORMULAE.

Example IV. Draw the graph of $y=2+3x+x^2$, and find the change of y per unit increase of x, when (i) $x=2$, (ii) $x=a$.

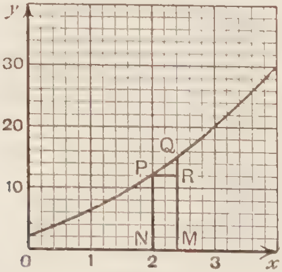


FIG. 26.

We have, by calculation, the table:

$x=0$	1	2	3	4
$y=2$	6	12	20	30

which is represented in Figure 26.

(i) When $ON=2$, $PN=2+3\times 2+2^2=2+6+4=12$.
When $OM=2+h$, $QM=2+3(2+h)+(2+h)^2=12+7h+h^2$;
$\therefore$ when x increases by h, y increases by $QM-PN$ or $QR=7h+h^2$;
$\therefore$ the average increase of y per unit increase of $x=\dfrac{7h+h^2}{h}=7+h$

or
$$\frac{QR}{PR}=7+h;$$
$$\therefore \operatorname*{Lt}_{h\to 0}\frac{QR}{PR}=7;$$

$\therefore$ at $x=2$, the rate of increase of y with respect to x is 7.

(ii) When $ON=a$, $PN=2+3a+a^2$.
When $OM=a+h$,
$$QM=2+3(a+h)+(a+h)^2=2+3a+3h+a^2+2ah+h^2;$$
$$\therefore QR=QM-PN=3h+2ah+h^2;$$
$$\therefore \frac{QR}{PR}=\frac{3h+2ah+h^2}{h}=3+2a+h;$$

$\therefore$ at $x=a$, the rate of increase of y with respect to x,
$$=\operatorname*{Lt}_{h\to 0}(3+2a+h)$$
$$=3+2a.$$

Definition. In Figure 26, $\dfrac{QR}{PR}$ is called the *average gradient* of the graph over the interval NM.

The limit of $\dfrac{QR}{PR}$ when h tends to 0 or $\operatorname*{Lt}_{h\to 0}\dfrac{QR}{PR}$ is called the *gradient* of the graph at P

Figure 27 illustrates the way in which the "average gradient" changes as the point Q is taken nearer and nearer to P.

Q_1, Q_2, Q_3 are successive positions of Q, whilst P remains fixed.

The average gradient over the interval MN is $\dfrac{QR}{PR}$.

The limiting position to which the chord PQ tends as arc $PQ \to 0$ is indicated by the tangent TPT'.

$$\underset{h \to 0}{\text{Lt}} \frac{QR}{PR} = \text{gradient at } P = \text{gradient of the tangent } PT = \frac{PM}{MT}.$$

The gradient measures the "rate of change of y with respect to x," or the change in y per unit increase in x.

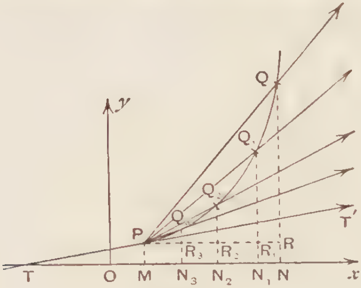


FIG. 27.

EXERCISE XVI. c.

1 What is the gradient of the slopes shown in Figs. 28-30?

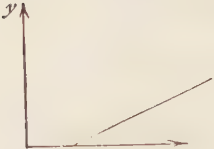


FIG. 28.

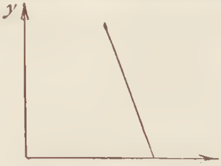


FIG. 29.

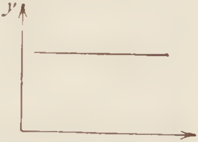


FIG. 30.

2. Figure 31 represents a hill; horizontal scale x-axis is 1 inch : 100 yards; vertical scale y-axis is 1 inch : 10 feet. Find (i) the average gradient from A to B, (ii) the average gradient from P to Q, (iii) the gradients at C, D, Q.

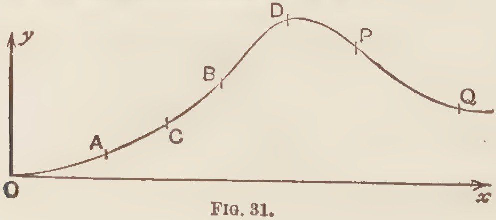


FIG. 31.

3. Draw lines of gradients (i) $\frac{3}{4}$, (ii) $-\frac{1}{2}$.

4. Find the coordinates of a point on the curve in Fig. 31, for the scale given in Ex. 2, where the slope is $\frac{1}{40}$.

5. (i) Draw the graph of $y=2x+3$.
   (ii) Show that the points (1, 5) and (4, 11) lie on it.
   (iii) What is the average gradient of the graph between these two points ?
   (iv) What is the average gradient of the graph over the interval $x=2$ to $x=6$?
   (v) What is the value of y when $x=a$ and when $x=a+h$, and what is the average gradient of the graph over this interval ?
   (vi) Why does the average gradient of the graph not depend on the values of either a or h ?

6. (i) Draw the graph of $y=7-5x$.
   (ii) What is the average gradient of the graph over the interval $x=3$ to $x=7$?
   (iii) What is the gradient of the graph when $x=a$?

7. What is the gradient of the straight line joining the points (2, 5) and (7, 8) ?

8. A marble rolling down an inclined plane travels s feet in t seconds, where $s=3t^2$.
   (i) How far has the marble travelled in 1 second ? What is its average speed for the 1st second ?
   (ii) How far has the marble travelled in 2 seconds ? What is its average speed for the 1st two seconds ? What is its average speed during the 2nd second ?
   (iii) How far has the marble travelled in 2·1 seconds ? What is its average speed in the interval 2 seconds to 2·1 seconds ?
   (iv) How far has it travelled in $(2+h)$ seconds ? What is its average speed in the h seconds between 2 seconds and $(2+h)$ seconds ?
   (v) What does your answer to Question (iv) become when $h=0{\cdot}1$ second, when $h=0{\cdot}01$ second and when $h=0{\cdot}000001$ second ?

   What is its speed exactly 2 seconds after it begins to move ?

   Draw a graph to illustrate the motion from $t=0$ to $t=3$, and find its speed after 2 seconds by drawing a tangent.

9. The distance d ft. that a stone has fallen after t seconds is given by the formula $d = 16t^2$.

(i) How far has the stone fallen after 3 seconds ? What is its average speed for the first 3 seconds ?

(ii) How far has the stone fallen after 2 seconds ? What is its average speed during the third second ?

(iii) How far has the stone fallen after 2·9 seconds ? What is its average speed during the interval from 2·9 seconds to 3 seconds ?

(iv) How far does it fall in $(3 - h)$ seconds ? What is its average speed during the h seconds from $(3 - h)$ seconds to 3 seconds ?

(v) What does your answer to Question (iv) become when $h = 0{\cdot}1$ second, when $h = 0{\cdot}01$ second and when $h = 0{\cdot}000001$ second ?

What is the velocity of the stone 3 seconds after it is dropped ?

10. In the same way as in Ex. 9 work out the speed of the stone 1 second after it is dropped.

11. In the same way as in Ex. 9 work out the speed of the stone a seconds after it is dropped. Evaluate your result for $a = 1$, 2 and 3, and compare with previous results.

12. The distance in feet travelled in t seconds by a body moving in a straight line from a fixed point A is given by the formula $AP = 3t^2 + 5t + 1$. Find AP when $t = 2$ and when $t = 2 + h$. Find the average speed of the body P for the h seconds commencing with the end of the 2nd second. What is this speed when $h = 0{\cdot}1$ sec., when $h = 0{\cdot}01$ sec., when $h = 0{\cdot}0001$ sec. and when $h = -0{\cdot}0000001$ sec. What is the speed of P 2 seconds after the motion starts ?

13. Draw a line AB of length 10 cm. and describe a semicircle with AB as diameter ; P is any point on the semicircle and PN is the perpendicular to AB. Let $AN = x$ and $PN = y$. Find by measurement or calculation the average gradient of the semicircle over the intervals (i) $x = 0$ to 1 ; (ii) $x = 0$ to 2 ; (iii) $x = 3$ to 7 ; (iv) $x = 8$ to 10. Interpret your results geometrically.

14. The area of a circular blot of ink is increasing at a steady rate of 2 sq. cm. per sec. ; if the radius is x cm. after t sec., find the average rate of increase of x over the interval (i) $t = 5$ to $t = 10$; (ii) $t = 5$ to $t = 5{\cdot}1$, assuming that $x = 0$ when $t = 0$.

15. The perimeter of a rectangle is 24 inches: if its area is y sq. in. when one side is of length x in., find

- (i) the average rate of change of y over the interval $x = 3$ to 5;
- (ii) the average rate of change of y over the interval $x = 3$ to $3 + h$;
- (iii) the average rate of change of y over the interval $x = a$ to $a + h$;
- (iv) the rate of change of y when $x = a$;
- (v) the value of x when the rate of change of y is zero; what does this mean?
- (vi) Express y as a function of x and draw its graph.

16. If $y = 3x + x^2$, find (i) the average rate of change of y over the interval $x = 1$ to $1 \cdot 1$; (ii) the gradient at $x = 1$; (iii) the value of x when the gradient is zero.

17. Calculate the gradient of the graph of $\frac{1}{x}$ when (i) $x = 2$, (ii) $x = a$.

18. Calculate the gradient of the graph of $2x^3$ when $x = c$.

19. A marble rolling down a groove travels s feet in t seconds where $s = \frac{1}{4}t^2$; find (i) the average rate of change of s over the interval $t = 1$ to $t = 2$, (ii) the rate of change of s when $t = 1$. What does this mean?

20. The graph of $y = mx + c$ is a straight line, m and c being any constant numbers. Find its gradient.

21. Find the gradient of $y = ax^2 + bx + c$ when $x = 2$.

22. Find the gradient of $y = ax^2 + bx + c$ when $x = x_1$. For what value of x is this gradient equal to zero? What is the geometrical significance of a zero gradient?

23. Give geometrical meanings to the following, taking x to be the x-coordinate of a graph (do not simplify the expressions):

(i) $\dfrac{(x+h)^2 - x^2}{h}$; (ii) $\underset{h \to 0}{\mathrm{Lt}} \dfrac{(x+h)^3 - x^3}{h}$;

(iii) $\dfrac{\sqrt{x+h} - \sqrt{x}}{h}$; (iv) $\underset{h \to 0}{\mathrm{Lt}} \dfrac{\sqrt{x-h} - \sqrt{x}}{h}$;

(v) $\underset{h \to 0}{\mathrm{Lt}} \dfrac{\pi(r+h)^2 - \pi r^2}{h}$; (vi) $\dfrac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h}$.

24. If $f(x) \equiv x^2$, evaluate $\underset{h \to 0}{\mathrm{Lt}} \dfrac{f(x+h) - f(x)}{h}$ and interpret the result.

25. If $f(x) \equiv 5$, what is the gradient of the graph of $f(x)$?

SUMMARY OF RESULTS.

Draw the graph of any function $y=f(x)$; see Figure 32.

Suppose $ON=x$ and $OM=x+h$, so that $PR=NM=h$.

Then $PN=f(x)$ and $QM=f(x+h)$.

$$\therefore\ QR=f(x+h)-f(x).$$

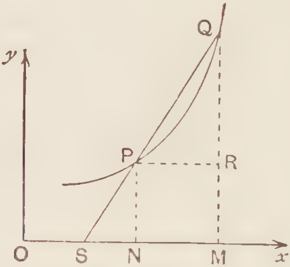


FIG. 32.

The average gradient of PQ is $\dfrac{f(x+h)-f(x)}{h}$.

And the gradient at P is $\underset{h\to 0}{\mathrm{Lt}}\dfrac{f(x+h)-f(x)}{h}$.

The gradient at P is the slope of the tangent at P to the curve.

CHAPTER XVII.

DIFFERENTIATION.

NOTATION.

Figure 33 represents the graph of any function $y = f(x)$.

If $ON = x$, then $PN = y = f(x)$.

Q is any point close to P on the curve and QM its ordinate.

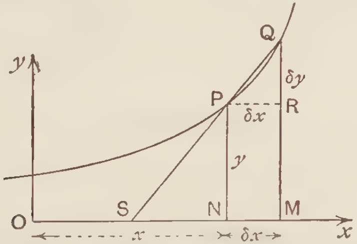


Fig. 33.

The length of NM is represented by the symbol δx, which means "a small increment of the variable x," or colloquially "a little bit of x."

And δy represents the *consequent* change in y; so that $\delta y = QM - PN = QR$.

Thus
$$ON = x; \quad OM = x + \delta x;$$
$$PN = y = f(x); \quad QM = y + \delta y = f(x + \delta x);$$
$$\therefore \ \delta y = f(x + \delta x) - f(x),$$
and
$$\frac{QR}{PR} = \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}.$$

Now $\frac{QR}{PR}$ = the gradient of the chord PQ.

$\therefore$ the gradient of the tangent at P

$$= \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}$$

= rate of change of $f(x)$ with respect to x.

The expression $\underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x}$ is written $\frac{dy}{dx}$ or $\frac{d}{dx}y$, and is called the *differential coefficient* of y with respect to x.

The expression $\underset{\delta x \to 0}{\mathrm{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}$ is written $\frac{df(x)}{dx}$ or $\frac{d}{dx}f(x)$, and is called the *differential coefficient* of $f(x)$ with respect to x.

The process of finding this limit is called "*differentiating with respect to x*," and the result is sometimes called the "*derived function*" of $f(x)$ and written $f'(x)$.

SUMMARY.

(i) $\frac{d}{dx}f(x) = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}$.

(ii) If $y = f(x)$, $\frac{dy}{dx} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{f(x+\delta x)-f(x)}{\delta x}$.

Example I. Differentiate $3x^2+7$ with respect to x

$$\frac{d}{dx}(3x^2+7) = \underset{\delta x \to 0}{\mathrm{Lt}} \frac{[3(x+\delta x)^2+7]-[3x^2+7]}{\delta x}$$

$$= \mathrm{Lt} \frac{[3\{x^2+2x\,\delta x+(\delta x)^2\}+7]-3x^2-7}{\delta x}$$

$$= \mathrm{Lt} \frac{6x\,\delta x+3(\delta x)^2}{\delta x}$$

$$= \mathrm{Lt}\,[6x+3\,\delta x], \text{ provided } \delta x \neq 0,$$

$$= 6x.$$

Example II. Find $\frac{d}{dx}\left(\frac{1}{x}\right)$.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \underset{\delta x \to 0}{\mathrm{Lt}} \left(\frac{\frac{1}{x+\delta x} - \frac{1}{x}}{\delta x} \right)$$

$$= \mathrm{Lt} \frac{x-(x+\delta x)}{x(x+\delta x) \,.\, \delta x}$$

$$= \mathrm{Lt} - \frac{\delta x}{(x^2 + x \,.\, \delta x) \,.\, \delta x}$$

$$= \mathrm{Lt} - \frac{1}{x^2 + x \,.\, \delta x}, \text{ provided } \delta x \neq 0,$$

$$= -\frac{1}{x^2}.$$

EXERCISE XVII. a.

1. (i) Simplify $5(x+\delta x) - 5x$. (ii) Find $\frac{d}{dx}(5x)$.

2. (i) Simplify $(x+\delta x)^2 - x^2$. (ii) Find $\frac{d}{dx}(x^2)$.

3. (i) Simplify $[5(x+\delta x)^2 - 3(x+\delta x) + 7] - [5x^2 - 3x + 7]$.

(ii) Find $\frac{d}{dx}[5x^2 - 3x + 7]$.

4. (i) If $f(x) \equiv (1+x)^2$, what is $f(x+\delta x)$?

(ii) Find $\frac{d}{dx}(1+x)^2$.

5. Differentiate $x(1+x)$.

6. Differentiate (i) x^2; (ii) $3x^2$; (iii) $7x^2$, and write down the value of $\frac{d}{dx}(100x^2)$.

7. Differentiate (i) x^3; (ii) $4x^3$; (iii) $5x^3$; and write down the value of $\frac{d}{dx}(29x^3)$.

8. Given that $\underset{\delta x \to 0}{\mathrm{Lt}} \frac{(x+\delta x)^4 - x^4}{\delta x} = 4x^3$, write down the values of

(i) $\frac{d}{dx}(3x^4)$; (ii) $\frac{d}{dx}(10x^4)$.

9. Given that $\frac{d}{dx}x^5 = 5x^4$, write down the values of

(i) $\frac{d}{dx}(7x^5)$; (ii) $\frac{d}{dx}(x^5 + 7)$.

10. Given that $\dfrac{d}{dx}x^6 = 6x^5$, write down the values of

(i) $\dfrac{d}{dx}(\tfrac{3}{4}x^6)$; (ii) $\dfrac{d}{dx}(5 - x^6)$; (iii) $\dfrac{d}{dx}(x - 2x^6)$.

11. Express in the limit form the fact that $\dfrac{d}{dx}x^n = nx^{n-1}$.

12. Use the facts that

$$\underset{\delta x \to 0}{\text{Lt}} \frac{(x+\delta x)^2 - x^2}{\delta x} = 2x \quad \text{and} \quad \underset{\delta x \to 0}{\text{Lt}} \frac{(x+\delta x)^3 - x^3}{\delta x} = 3x^2$$

to write down the values of

(i) $\underset{\delta x \to 0}{\text{Lt}} \dfrac{[(x+\delta x)^3 + 2(x+\delta x)^2] - [x^3 + 2x^2]}{\delta x}$;

(ii) $\dfrac{d}{dx}(3x^3 + x^2)$; (iii) $\dfrac{d}{dx}(\tfrac{2}{3}x^3 + \tfrac{1}{2}x^2 + 5)$.

13. Express as limits

(i) $\dfrac{d}{dx}f(x)$; (ii) $\dfrac{d}{dx}\phi(x)$; (iii) $\dfrac{d}{dx}[3f(x) - 5\phi(x)]$.

If $\dfrac{d}{dx}f(x) = u$ and $\dfrac{d}{dx}\phi(x) = v$, what is $\dfrac{d}{dx}[3f(x) + 5\phi(x)]$?

14. Find

(i) $\dfrac{d}{dx}(3x)$; (ii) $\dfrac{d}{dx}(5x)$; (iii) $\dfrac{d}{dx}(3x \times 5x)$; (iv) $\dfrac{d}{dx}(3x + 5x)$.

Is $\dfrac{d}{dx}(3x \times 5x)$ equal to $\dfrac{d}{dx}(3x) \times \dfrac{d}{dx}(5x)$?

Is $\dfrac{d}{dx}(3x + 5x)$ equal to $\dfrac{d}{dx}(3x) + \dfrac{d}{dx}(5x)$?

15. (i) Express as limits $\dfrac{d}{dx}(3x^2)$; $\dfrac{d}{dx}(5x^3)$; $\dfrac{d}{dx}(3x^2 + 5x^3)$; $\dfrac{d}{dx}(3x^2 - 5x^3)$; $\dfrac{d}{dx}(3x^2 \times 5x^3)$; $\dfrac{d}{dx}(5x^3 \div 3x^2)$.

(ii) Is $\dfrac{d}{dx}(3x^2) = 3\dfrac{d}{dx}(x^2)$?

(iii) Is $\dfrac{d}{dx}(3x^2 \pm 5x^3) = 3\dfrac{d}{dx}(x^2) \pm 5\dfrac{d}{dx}(x^3)$?

(iv) Is $\dfrac{d}{dx}(3x^2 \times 5x^3) = 3\dfrac{d}{dx}(x^2) \times 5\dfrac{d}{dx}(x^3)$?

(v) Is $\dfrac{d}{dx}(5x^3 \div 3x^2) = 5\dfrac{d}{dx}(x^3) \div 3\dfrac{d}{dx}(x^2)$?

16. What general formula covers the following facts?

$$\frac{d}{dx}x^2 = 2x; \qquad \frac{d}{dx}x^3 = 3x^2; \qquad \frac{d}{dx}x^4 = 4x^3; \qquad \frac{d}{dx}x^5 = 5x^4.$$

17. What general formula covers the following facts?

$$\frac{d}{dx}(7x^2) = 7\frac{d}{dx}(x^2); \quad \frac{d}{dx}(11x^4) = 11\frac{d}{dx}(x^4); \quad \frac{d}{dx}\left(\frac{93}{x}\right) = 93\frac{d}{dx}\left(\frac{1}{x}\right).$$

18. What general formula covers the following facts?

$$\frac{d}{dx}(x^2+x^3) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3); \quad \frac{d}{dx}(7x^2 - 11x^5) = \frac{d}{dx}(7x^2) - \frac{d}{dx}(11x^5).$$

19. Take simple functions for $f(x)$ and $\phi(x)$ to show that $\frac{d}{dx}[f(x) \times \phi(x)]$ is not equal to $\frac{d}{dx}f(x) \times \frac{d}{dx}\phi(x)$.

20. Write down special cases of the general formula

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

when n equals (i) 8; (ii) 50; (iii) -1; (iv) -3; (v) 1; (vi) 0; (vii) $\frac{3}{2}$; (viii) $\frac{1}{2}$; (ix) $-\frac{1}{2}$; (x) $-m$.

21. If $y = x^2$, find δy when $x = 2$, $\delta x = 0 \cdot 1$.

22. If $y = \frac{1}{x}$, find δy when $x = 3$, $\delta x = \frac{1}{3}$.

23. If $y = x^3$, find δy when (i) $x = 1$, $\delta x = 0 \cdot 1$; (ii) $x = 1$, $\delta x = 0 \cdot 01$.

24. If $y = 3x^2$, find δx when $x = 2$, $\delta y = 0 \cdot 1$.

25. If $y = x^2 + x$, find δy when $x = 10$, $\delta x = -1$.

SUMMARY OF RESULTS.

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$, where n is integral or fractional, positive or negative.

(ii) $\frac{d}{dx}(C) = 0$, if C is any constant, *i.e.* a number independent of x.

(iii) $\frac{d}{dx}(Cx^n) = C\frac{d}{dx}(x^n)$
$= Cnx^{n-1}$, where C is any constant.

(iv) $\frac{d}{dx}[f(x) \pm \phi(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[\phi(x)]$.

Example III. Find $\frac{d}{dx}(x)$; $\frac{d}{dx}\left(\frac{1}{x^2}\right)$; $\frac{d}{dx}(\sqrt{x})$.

Since
$$\frac{d}{dx}(x^n) = nx^{n-1}$$

(i) Put $n = 1$, $\therefore \frac{d}{dx}x = 1x^{1-1} = x^0 = 1$,

or, more simply, $\frac{d}{dx}(x) = \underset{\delta x \to 0}{\text{Lt}} \frac{(x + \delta x) - x}{\delta x} = \text{Lt} \frac{\delta x}{\delta x} = 1$.

(ii) Put $n = -2$, $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = -2 \times x^{-}$

$$= -2 \times \frac{1}{x^3} = -\frac{2}{x^3}.$$

iii) Put $n = \frac{1}{2}$, $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

$$= \frac{1}{2} \times \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Example IV. Find $\frac{d}{dx}\left(3x^4 - 7x + 5 - \frac{2}{x}\right)$.

The expression $= \frac{d}{dx}(3x^4) - \frac{d}{dx}(7x) + \frac{d}{dx}(5) - \frac{d}{dx}\left(\frac{2}{x}\right)$

$$= 3\frac{d}{dx}(x^4) - 7\frac{d}{dx}(x) + 0 - 2\frac{d}{dx}(x^{-1})$$
$$= 3(4x^3) - 7(1) + 0 - 2(-1)(x^{-1-1})$$
$$= 12x^3 - 7 + 2x^{-2}$$
$$= 12x^3 - 7 + \frac{2}{x^2}.$$

After a little practice, most of the intermediate steps in the working can be omitted.

SUCCESSIVE DIFFERENTIATION.

If $y = x^4$, $\frac{dy}{dx} = 4x^3$;

$$\therefore \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x^3) = 4\frac{d}{dx}(x^3) = 4 \times 3x^2 = 12x^2.$$

For the sake of brevity, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is written $\frac{d^2y}{dx^2}$: this symbol is called the second differential coefficient of y with respect to x.

Similarly $\dfrac{d^2s}{dt^2}$ is short for $\dfrac{d}{dt}\left(\dfrac{ds}{dt}\right)$.

Example V. Find $\dfrac{d^2}{dx^2}(3x^2+4x-7)$.

$$\frac{d}{dx}(3x^2+4x-7)=\frac{d}{dx}(3x^2)+\frac{d}{dx}(4x)-\frac{d}{dx}(7)=6x+4\,;$$

$$\therefore\ \frac{d^2}{dx^2}(3x^2+4x-7)=\frac{d}{dx}(6x+4)=6.$$

EXERCISE XVII. b.

Differentiate with respect to x the expressions in Examples 1-30.

1. x^7. **2.** $10x^3$. **3.** $x+\dfrac{1}{x}$. **4.** $\dfrac{1}{x^5}$.

5. $\dfrac{3}{x}$. **6.** $\dfrac{2}{x^2}$. **7.** $3x^2-2x$. **8.** $\frac{1}{4}x^4-2$.

9. $x^2\times x^3$. **10.** $(5x)^3$. **11.** $(1-x)^2$. **12.** $\dfrac{1}{x^n}$.

13. ax. **14.** x^a. **15.** $\dfrac{c}{x^3}$. **16.** $b\sqrt{x}$.

17. $\sqrt{x^3}$. **18.** $\dfrac{1}{\sqrt{x}}$. **19.** x^0. **20.** $\sqrt[3]{x}$.

21. $\dfrac{2}{3}x^3-\dfrac{x}{5}$. **22.** $\dfrac{x}{10}+\dfrac{10}{x}$. **23.** $7x^3-2x-5$. **24.** $\dfrac{4}{3}\pi x^3$.

25. $3x^4-\dfrac{1}{2}x+7-\dfrac{6}{x}$. **26.** $5x^4+3x^2+\dfrac{1}{x}-\dfrac{1}{x^2}$.

27. $(x^2+1)(x+2)$. **28.** $\left(x+\dfrac{1}{x}\right)^2$. **29.** x^{3n}. **30.** x^{-k}.

31. If $y=3x^2+5$, find $\dfrac{dy}{dx}$ when $x=1$.

32. If $y=1+2x-x^2$, find $\dfrac{dy}{dx}$ when $x=-2$.

33. If $y=2x^3-9x^2+12x$, find the values of x for which $\dfrac{dy}{dx}=0$.

34. If $y=7x^3-9x$, find $\dfrac{d^2y}{dx^2}$. **35.** If $y^2=x$, find $\dfrac{d^2y}{dx^2}$.

36. If $y=6x^2$, prove that $x\dfrac{dy}{dx}=2y$.

37. If $y=x\,(1-x)$, prove that $1+\dfrac{dy}{dx}=\dfrac{2y}{x}$.

38. If $y=x^3$, prove that $\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=18y$.

39. If $s=100t-16t^2$, find $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$ when $t=1$ and $t=0$.

40. If $y=7x^4$, express (i) $2y\,.\,\frac{dy}{dx}$; (ii) $\frac{d}{dx}(y^2)$ in terms of x.

41. If $y=2z^2$, $z=3+5x$, express in terms of x, (i) $\frac{dy}{dz}$; (ii) $\frac{dz}{dx}$; (iii) y; (iv) $\frac{dy}{dx}$; hence show that $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$.

42. If $y=z+z^3$, $z=1+x^2$, express in terms of x, (i) $\frac{dy}{dz}$; (ii) $\frac{dz}{dx}$; (iii) y; (iv) $\frac{dy}{dx}$; hence show that $\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$.

43. The cubical elasticity of a fluid is equal to $-v\,.\,\frac{dp}{dv}$ where the volume v and the pressure p are connected by the equation $pv=c$ (a constant). Simplify this expression.

44. The radius of the circle which approximates most closely to the shape of the curve $y=\frac{2x^3}{3}$ at any point is

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}\div\frac{d^2y}{dx^2};$$

find its value when $x=1$.

TURNING POINTS.

A function $f(x)$ is called an "*increasing function*" of x for any range of values of x in which, as x increases, the value of $f(x)$ also increases.

It is called a "*decreasing function*" of x if, as x increases, the value of $f(x)$ decreases.

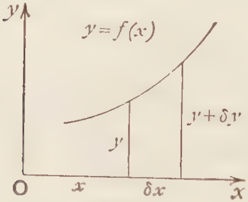


FIG. 34.

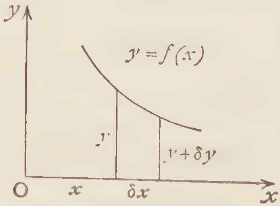


FIG. 35.

Figure 34 represents an increasing function.

Figure 35 represents a decreasing function.

If δx is positive, then δy is positive for an increasing function (Figure 34), and δy is negative for a decreasing function (Figure 35).

Since $\frac{dy}{dx} = \underset{x\to 0}{\mathrm{Lt}} \frac{\delta y}{\delta x}$, we see that $\frac{dy}{dx}$ is positive for an increasing function, and is negative for a decreasing function.

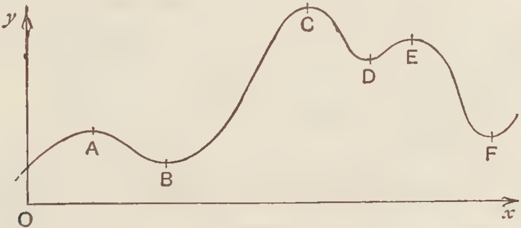


FIG. 36.

Figure 36 represents a function which is an increasing function for some ranges of values of x, and a decreasing function for others.

Thus $f(x)$ is a decreasing function from A to B, and an increasing function from B to C.

The separating point B is called a *turning point*, and the value of $f(x)$ corresponding to the point B is called a *turning value* of the function.

There are two kinds of turning points.

A, C, E, ... correspond to values of x for which the function is greater than at any other point near it: at these points the function is said to be a *maximum*.

B, D, F, ... correspond to values of x for which the function is less than at any other point near it: at these points the function is said to be a *minimum*.

At a turning point, $\frac{dy}{dx}=0$, for the function is neither increasing $\left(i.e.\ \frac{dy}{dx}\text{ positive}\right)$ nor decreasing $\left(i.e.\ \frac{dy}{dx}\text{ negative}\right)$.

EXERCISE XVII. c.

1. Fig. 37 is the graph of $y=x^2$. Is y an increasing or decreasing function (i) from A to O, (ii) from O to B? Is $\frac{dy}{dx}$ positive or negative (α) from A to O, (β) from O to B? Has y a maximum or minimum value anywhere?

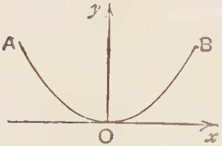


FIG. 37.

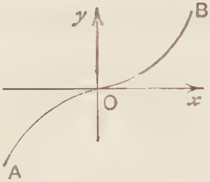


FIG. 38.

2. Fig. 38 is the graph of $y=x^3$. Answer the same questions as in Ex. 1.

3. Fig. 39 is the graph of $y=4+3x-x^2$. Is y an increasing or decreasing function from (i) A to B, (ii) B to C, (iii) C to D, (iv) D to E, (v) E to F? What are the signs of $\frac{dy}{dx}$ for these five portions? Has y a maximum or minimum value anywhere?

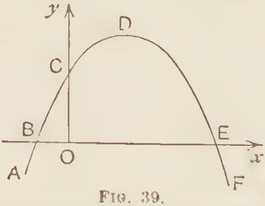


FIG. 39.

4. Draw freehand the graph of a function $y=f(x)$ which starts from the origin O, and is such that

(i) from O to A, y is a decreasing function;

(ii) from A to B, $\frac{dy}{dx}$ is positive and y is negative;

(iii) from B to C, y is a positive increasing function;

(iv) from C to D, $\frac{dy}{dx}$ is negative and y is positive;

(v) from D to E, $\frac{dy}{dx}$ is negative and y is negative;

(vi) from E to F, $\frac{dy}{dx}$ is positive.

Has the function any maximum or minimum values?

5. Draw freehand the graph of a function $y=f(x)$ for which

(i) x is negative, y is negative, $\frac{dy}{dx}$ is positive;

(ii) x is negative, y is negative, $\frac{dy}{dx}$ is negative;

(iii) x is negative, y is positive, $\frac{dy}{dx}$ is positive;

(iv) x is negative, y is positive, $\frac{dy}{dx}$ is negative.

6. (i) Draw freehand the graph $ABCDE$ of a function $y=f(x)$ such that the values of $\frac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively 1, $\frac{1}{2}$, 0, $-\frac{1}{2}$, -1.

(ii) What kind of a point is C?

(iii) Is $\frac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE?

(iv) What is the sign of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2}$ for points on the arc AC and the arc CE?

7. (i) Draw freehand the graph $ABCDE$ of a function $y=f(x)$ such that the values of $\frac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1.

(ii) What kind of a point is C?

(iii) Is $\frac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE?

(iv) What is the sign of $\frac{d^2y}{dx^2}$ for points on the arc AC and the arc CE?

8. (i) Draw freehand the graph $ABCDE$ of a function $y=f(x)$ such that the values of $\frac{dy}{dx}$ (or the gradients of the graph) at A, B, C, D, E are respectively 1, $\frac{1}{2}$, 0, $\frac{1}{2}$, 1.

(ii) Is C a turning point?

(iii) Is $\frac{dy}{dx}$ an increasing or decreasing function for the arc AC and the arc CE?

(iv) What is the sign of $\frac{d^2y}{dx^2}$ for points on the arc AC and the arc CE?

9. Answer the various questions in Ex. 8, taking the values of $\frac{dy}{dx}$ at A, B, C, D, E to be respectively -1, $-\frac{1}{2}$, 0, $-\frac{1}{2}$, -1.

10. AB, CD, EF, GH are portions of the graph of $y = f(x)$.

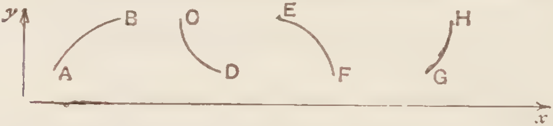


FIG. 40.

What can you say about the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the arcs

(i) AB; (ii) CD; (iii) EF; (iv) GH ?

11. Can one of the minimum values of a function be greater than one of its maximum values ? Illustrate by a figure.

12. Can a function have (i) exactly one minimum and two maximum values, (ii) exactly one minimum and three maximum values ? Illustrate by a figure.

13. Part of the graph of the function $y = f(x)$ is represented by the curve $ACBDPQ$ in Fig. 31, p. 334; copy this freehand, and underneath it draw roughly the corresponding portions of the graphs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx}$.

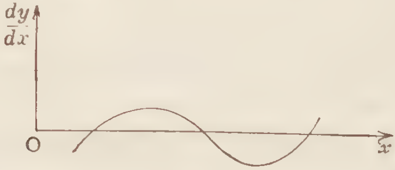


FIG. 41.

14. Fig. 41 represents part of the graph of $\frac{dy}{dx}$; draw roughly the corresponding portions of the graph of y and of $\frac{d^2y}{dx^2}$.

15. Part of the graph of the function $y = f(x)$ is represented by the curve $ABCDEF$ in Fig. 36, p. 347; show in tabular form the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the various portions of the graph, and state at which points either $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ is zero.

16. Fig. 42 shows the graph of $y=x^2$; the tangent at P meets Ox at T; PN is perpendicular to Ox; $ON=x$, $NP=y$; prove that (i) $\frac{PN}{NT}=$ the slope at $P=2x$; (ii) $OT=TN$.

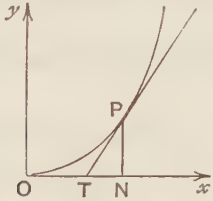


FIG. 42.

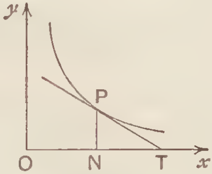


FIG. 43.

17. Fig. 43 shows the graph of $y=\frac{1}{x}$; the tangent at P meets Ox at T; PN is perpendicular to Ox; $ON=x$, $NP=y$; prove that (i) $\frac{PN}{NT}=-$ the slope at $P=\frac{1}{x^2}$; (ii) $ON=NT$.

18. If Fig. 42 represents the graph of $y=3x^2$ and if P is the point (2, 12), find the length of OT.

19. If Fig. 43 represents the graph of $y=\frac{8}{x^2}$ and if P is the point (2, 2), find the length of OT.

20. If Fig. 42 represents the graph of $y=x^3$, prove that

$$OT=\tfrac{2}{3}ON.$$

21. If Fig. 43 represents the graph of $y^2=\frac{1}{x^3}$, prove that

$$ON=\tfrac{3}{5}.OT.$$

SUMMARY OF RESULTS.

For the function $y=f(x)$,

(i) $\frac{dy}{dx}=0$, both at a maximum and a minimum.

(ii) $\frac{d^2y}{dx^2}$ is negative at a maximum;

$\frac{d^2y}{dx^2}$ is positive at a minimum.

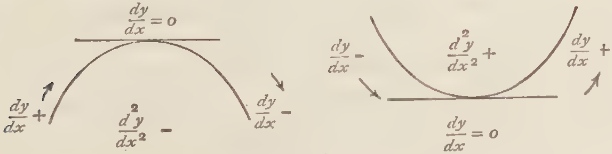


FIG. 44.

(iii) $\frac{dy}{dx}$ changes from + to − in passing through a maximum;

$\frac{dy}{dx}$ changes from − to + in passing through a minimum;

or $\frac{dy}{dx}$ is a decreasing function in passing through a maximum;

$\frac{dy}{dx}$ is an increasing function in passing through a minimum;

(iv) If $\frac{dy}{dx}=0$ but does not change sign, there is neither a maximum nor a minimum.

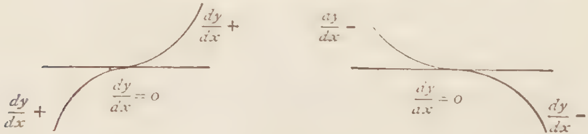


FIG. 45.

MAXIMA AND MINIMA PROBLEMS.

Example VI. The strength of a beam of uniform rectangular section varies as the breadth and the square of the depth. Find the breadth of the strongest rectangular beam that can be cut from a cylindrical tree-trunk of diameter 20 inches.

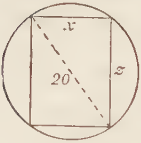


FIG. 46.

Let the breadth be x inches and the depth z inches;

$$\therefore\ x^2+z^2=20^2 \quad \text{(Pythagoras)}$$
$$=400.$$

Now the strength varies as xz^2

$$=kxz^2=kx(400-x^2), \text{ where } k \text{ is a constant,}$$
$$=k(400x-x^3);$$

∴ the function $y=400x-x^3$ is to be a maximum.

$$\frac{dy}{dx}=400-3x^2;$$

$$\therefore\ \frac{dy}{dx}=0 \text{ if } 400-3x^2=0 \quad \text{or} \quad x^2=\frac{400}{3}=133{\cdot}3;$$

$$\therefore\ x=\pm 11{\cdot}54.$$

Also $$\frac{d^2y}{dx^2}=\frac{d}{dx}(400-3x^2)=-6x\,;$$

$$\therefore\ \frac{d^2y}{dx^2} \text{ is negative if } x \text{ is positive};$$

$$\therefore\ x=11{\cdot}54 \text{ gives a } maximum \text{ value for } y\,;$$

$$\therefore\ \text{the required breadth is } 11{\cdot}5 \text{ inches.}$$

Note.—Sometimes it is necessary to find the maximum or minimum values of expressions which are not in a form that can be differentiated by the rules already considered. It may be possible however, either to change the form by substituting or to avoid the difficulty in some other way.

Example VII. Find (i) the value of x for which $\sqrt{x^2-8x+21}$ is a minimum, (ii) the value of x for which $\frac{x-1}{(x+1)^2}$ is a maximum.

(i) $\sqrt{x^2-8x+21}$ has its least value if $x^2-8x+21$ is a minimum;

$$\therefore\ \frac{d}{dx}(x^2-8x+21)=0\,;$$

$$\therefore\ 2x-8=0 \quad\text{or}\quad x=4.$$

It is a minimum because $\frac{d^2}{dx^2}(x^2-8x+21)=2$ and is therefore positive.

(ii) Let $y=\frac{x-1}{(x+1)^2}$ and put $x+1=z$;

$$\therefore\ y=\frac{z-2}{z^2}=\frac{1}{z}-\frac{2}{z^2};$$

$$\therefore\ \frac{dy}{dz}=-\frac{1}{z^2}+\frac{4}{z^3}=\frac{(-z+4)}{z^3}\,;$$

$$\therefore\ \frac{dy}{dz}=0 \text{ if } z=4 \quad\text{or}\quad x+1=4 \quad\text{or}\quad x=3.$$

Also $$\frac{d^2y}{dz^2}=\frac{2}{z^3}-\frac{12}{z^4}=\frac{2z-12}{z^4}=\frac{-4}{4^4} \text{ if } z=4\,;$$

$$\therefore\ \text{for } z=4,\ y \text{ is a maximum, since } \frac{d^2y}{dz^2} \text{ is negative};$$

$$\therefore\ \text{for } x=3,\ \frac{x-1}{(x+1)^2} \text{ is a maximum.}$$

EXERCISE XVII. d.

1. Find the values of x which correspond to turning values of the following functions; determine whether they are maxima or minima; and sketch roughly the graphs of the functions:

(i) x^2-2x; (ii) x^2+4x+3; (iii) $3+8x-10x^2$;

(iv) $x+\dfrac{1}{x}$; (v) $x-\dfrac{1}{x}$; (vi) x^4-4x;

(vii) x^4-x^2; (viii) x^3-x^2-x+1; (ix) x^3-3x^2-9x+7;

(x) x^3-3x^2+3x-1.

2. Find the area of the largest rectangular piece of ground that can be enclosed by 200 hurdles each 4 feet long.

3. The parcel post regulations require that the sum of the length and girth of a parcel shall not exceed 6 feet. Find the volume of the largest box with a square base that can be sent by post.

4. A closed rectangular cistern is to be constructed to contain 80 cu. feet. It is to be 5 feet long. Find the breadth when the *total* area of its surface is a minimum.

5. The strength of a rectangular beam varies as the breadth and the square of the depth. Find the breadth of the strongest rectangular beam which has a perimeter of 4 feet.

6. Find the area of the largest rectangular piece of ground that can be enclosed by 200 hurdles each 4 feet long, if an existing fence is utilised to form one side.

7. A box without a lid is to be made from a sheet of metal of negligible thickness and is to have square ends; it is to hold $4\frac{1}{2}$ cu. feet. What is the least area of metal required?

8. For a steamer travelling v knots, the cost of the coal is £$\dfrac{v^2}{18}$ per hour, and other expenses amount to £8 per hour. What is the most economical speed for a journey of 200 nautical miles?

9. An open gutter of rectangular section is formed out of a long rectangular strip of sheet iron 9 feet wide. Find the maximum area of the cross-section.

10. A is 8 miles north and B is 6 miles east of a point O. Two men, starting at the same time from A and B, walk towards O at 4 miles an hour. If their distance apart is x miles after t hours, prove that $x^2=32t^2-112t+100$, and find when they are nearest together.

11. A rectangular sheet of cardboard is 8″ long and 5″ wide. Equal squares are cut out at each of the corners and the remainder is folded so as to form an open box. Find the maximum volume of the box.

12. Given $\pi r^2 h = 5$, find the value of r for which $2\pi r^2 + 2\pi rh$ is a minimum. What geometrical problem corresponds to this question ?

13. A particle projected in a resisting medium is finally brought to rest : it travels s feet in t seconds, where $s = 6t - \frac{1}{2}t^3$. How far does it go ?

14. Using the data of Ex. 3, find the volume of the largest circular cylinder that can be sent by parcel post.

15. If a very thin rod one foot long swings like a pendulum, the expression $x(1-x)^2$ measures the tendency to break at a place x feet from the point of suspension. Find where the rod is most likely to break.

16. The perimeter of a sheet of metal in the form of a circular sector is 1 foot. For what radius is the area a maximum ?

17. From a circular sheet of paper a sector is removed and the remainder is folded to form a circular cone. What fraction must be removed in order to give the cone of maximum volume ?

18. A rod one foot long is cut into two pieces to form the hypotenuse and one side of a right-angled triangle. How must it be cut to give the triangle of greatest area ?

19. The efficiency of a screw of certain material is $\frac{4h - 3h^2}{3 + 4h}$, where h is the tangent of its pitch. What is the greatest efficiency ? [Put $3 + 4h = x$.]

20. Assuming that the rigidity of a rectangular beam varies as its breadth and the cube of its depth, find the breadth of the most rigid beam that can be cut from a cylindrical trunk of diameter 3 feet.

21. A cylindrical vessel is open at one end and closed at the other ; for a given surface, prove that the volume is greatest if its height equals the radius of the base.

22. A skeleton box with two square ends is formed with 12 pieces of wire, and four other pieces of wire form an equal square round the middle of it. The total amount of wire available is one yard. What is the maximum volume of the box ?

23. A piece of wire 2 feet long is cut into two parts, one of which is bent to form a square and the other a circle. If the sum of the areas is a minimum, find the radius of the circle.

24. If $2x + y = 5$, what is the greatest value of $x^2 + 3xy + y^2$?

25. What is the greatest value of $\frac{x}{x^2 + 4}$?

26. What is the greatest value of $\frac{x}{(x^2 + 1)^2}$?

27. Find the minimum value of $\sqrt{\frac{x}{a}+\frac{a}{x}}$.

28. Find the radius of a circular cylinder which is cut from a sphere of radius 10 inches so that

(i) the volume of the cylinder is a maximum ;
(ii) the curved surface of the cylinder is a maximum.

29. P is a variable point on the line ABC ; $AB=2''$, $BC=3''$. Find the position of P for which $PA^2+PB^2-PC^2$ is a minimum.

30. ABC is a triangular field right-angled at A ; P, Q are points on AB, AC such that a fence from P to Q bisects the field. If $AB=a$, $AC=b$, $AP=x$, $PQ=y$ yards, express y as a function of x. If $a=625$, $b=800$, find the length of AP in order that the fence may be as short as possible; find also the length of the fence. (C.S.C.)

31. $ABCD$ is a rectangular sheet of cardboard (see Fig. 47) from which the shaded portions are removed. The remainder is used to make a closed box, as shown in the figure. If $AB=a$, $BC=b$, $AP=x$ inches, find the volume of the box in terms of a, b, x ; if $a=6$, $b=12$, find the value of x for which the volume is a maximum. (C.S.C.)

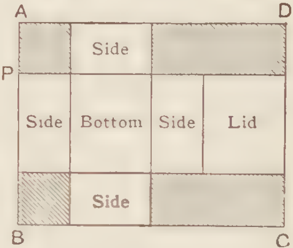


FIG. 47.

APPROXIMATION.

Figure 48 represents the graph of $y=f(x)$.

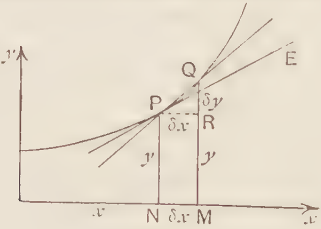


FIG. 48.

With the usual notation,

the slope of the chord PQ is $\frac{\delta y}{\delta x}$,

the slope of the tangent PE is $\frac{dy}{dx}$.

These are obviously not equal to each other.

But $\frac{dy}{dx}$ does equal the limit of $\frac{\delta y}{\delta x}$, when $\delta x \to 0$.

And we may say that

$$\frac{dy}{dx} \bumpeq \frac{\delta y}{\delta x}, \text{ when } \delta x \text{ is small,}$$

or

$$\delta y \bumpeq \frac{dy}{dx} \times \delta x, \text{ when } \delta x \text{ is small.}$$

The smaller δx is, the nearer the slope of the chord PQ is to the slope of the tangent PE.

Example VIII. If $y = x^2$, what is the error in the approximation $\delta y \bumpeq \frac{dy}{dx} \times \delta x$?

Now

$$y + \delta y = (x + \delta x)^2;$$

$$\therefore\ y + \delta y = x^2 + 2x \,.\, \delta x + (\delta x)^2,$$

but

$$y = x^2;$$

$$\therefore\ \delta y = 2x \,.\, \delta x + (\delta x)^2.$$

But

$$\frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x;$$

$$\therefore\ \delta y = \frac{dy}{dx} \times \delta x + (\delta x)^2;$$

$$\therefore\ \delta y \bumpeq \frac{dy}{dx} \times \delta x \text{ with error } (\delta x)^2.$$

The relative size of the error may be seen in a figure.

If $y = x^2$, y sq. inches is the area of a square of side x inches. and $(y + \delta y)$ sq. inches is the area of a square of side $(x + \delta x)$ inches.

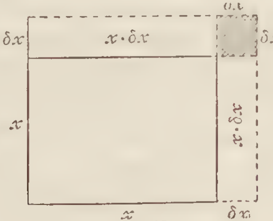


FIG. 49.

In taking δy equal to $\frac{dy}{dx} \times \delta x$ or $2x\,\delta x$, we are supposing its value is the sum of the two rectangles and neglecting the small shaded square in Figure 49.

Suppose x is 1″ and δx is 0·1″, the error is 0·01 sq. in.

EXERCISE XVII. e.

1. Find the error in the relation $\delta y \doteq \frac{dy}{dx} \times \delta x$ if

(i) $y = 5x^2$; (ii) $y = x^3$.

2. The area of an isosceles right-angled triangle of side x inches is A sq. ins.

(i) Find geometrically the value of δA in terms of δx.

(ii) Express A in terms of x, and evaluate $\delta A - \frac{dA}{dx} \times \delta x$.

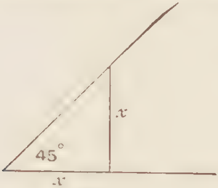


FIG. 50.

3. (i) Find an approximate expression for the increase in area of a circle when the radius increases from r to $r + \delta r$.

(ii) Illustrate the result geometrically.

(iii) Find approximately the difference in area of two circles of radii 10″ and 10·01″.

4. The radius of a spherical soap bubble increases from 1″ to 1·01″; find approximately its change of volume.

5. A body travels s feet in t sec., where $s = 10t - \frac{1}{5}t^3$; find an approximate expression (i) for δs in terms of δt, (ii) for the distance it moves in the interval of time $t = 2$ to $t = 2 \cdot 1$.

6. A gas at constant temperature under a pressure of p lb. per sq. inch occupies v cu. inches, where $pv = c$ (a constant); if the pressure is increased from p to $p + \delta p$, find an approximate expression for the change of volume.

7. Two telegraph posts of equal height are at a distance $2l$ feet apart. When the sag at the mid-point is x feet, the length of wire exceeds the distance between the poles by y feet, where

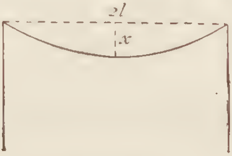


FIG. 51.

$y = \frac{4x^2}{3l}$. Find an approximate value for δy in terms of δx. What does this mean? If the distance between the poles is 30 yards, what is the effect of increasing the sag from 10 inches to 11 inches?

8. At sea-level, water boils at 212° F. At a height h feet above sea-level, the boiling point is lowered t degrees, where $h = 520t + t^2$; find approximately the difference of heights of two places where the boiling points are 200° and 201° F. respectively.

9. The perimeter of a circular pond is measured as 1321 yards; assuming this is correct to the nearest yard, what error may be expected in the area of the pond when calculated from this result?

10. A beam AB 30 feet long supported at each end is just strong enough to carry a load of M tons placed at a point P on the beam such that $M = \frac{1}{10}\left(k + \frac{1}{k} + \frac{1}{2}\right)$, where $\frac{AP}{PB} = k$. Express δM in terms of k and δk. What does this mean? Find approximately what change in the load may be necessary when it is shifted from a point 5 feet from A to a point 6 feet from A?

11. Fig. 52 represents the graph of $y = x^2$, unit 1″ on each axis. If $ON = x$, it can be proved that the area bounded by ON, NP and the arc OP is $A = \frac{1}{3}x^3$ sq. in. Find approximately δA in terms of δx, and interpret the result geometrically.

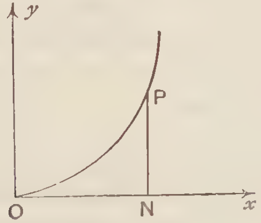


FIG. 52.

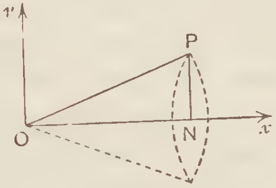


FIG. 53.

12. In Fig. 53 OP is the graph of $y = \frac{1}{2}x$; a circular cone is obtained by revolving OP about Ox. If $ON = x$ and the volume of the cone is V, prove that $V = \frac{1}{12}\pi x^3$. Find approximately δV in terms of δx, and interpret the result geometrically.

13. The bowl of a wine-glass is formed by revolving the graph of $y = x^2$ (see Fig. 52) about Oy. It can be proved that when the depth of wine in the glass is y inches, the volume is $V = \frac{1}{2}\pi y^2$ cu. in. Find approximately δV in terms of δy, and interpret the result geometrically.

14. If the depth of water in a hemispherical bowl of radius a in. is x in., the volume of the water is $V=\pi x^2\left(a-\frac{x}{3}\right)$ cu. in. Find approximately δV in terms of δx, and interpret the result geometrically.

15. If $y=z^5$ and $z=1+3x$, (i) prove that $\delta y \simeq 5z^4 . \delta z$; (ii) express δz in terms of δx; (iii) express δy in terms of x and δx; (iv) hence find $\frac{dy}{dx}$ and $\frac{d}{dx}(1+3x)^5$.

16. If $y=\sqrt{z}$ and $z=1+x^2$, (i) express δy in terms of δz and δz in terms of δx, and so find a relation between δy, δx, x; (ii) hence find $\frac{d}{dx}(\sqrt{1+x^2})$.

17. If $y=z^3$ and $z=x^3-x+7$, (i) find a relation between δy, δx, x; (ii) hence find $\frac{d}{dx}[(x^3-x+7)^3]$.

18. If $y=\frac{1}{z^2}$ and $z=3x-5$, (i) find a relation between δy, δx, x; (ii) hence find $\frac{d}{dx}\left[\frac{1}{(3x-5)^2}\right]$.

19. If $u=y^2$, and if y is a function of x, (i) find a relation between δu, δy, y; (ii) hence express $\frac{du}{dx}$ in terms of $\frac{dy}{dx}$; (iii) prove that $\frac{d}{dx}(y^2)=2y\frac{dy}{dx}$.

20. If V cu. in. is the volume of a sphere of radius r in., then $V=\frac{4}{3}\pi r^3$. The radius of a sphere, which is expanding, is r in. after t sec.; (i) express δV in terms of r and δr; (ii) find a relation between $\frac{dV}{dt}$, $\frac{dr}{dt}$ and r; (iii) interpret this result.

RATE OF CHANGE.

Example IX. A vessel is in the shape of a circular cone of semi-vertical angle 45° Water is poured into it at the rate of 10 cu. in. per sec. At what rate is the level rising after 4 seconds?

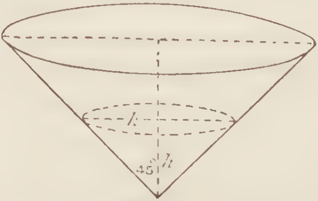


FIG. 54.

When the depth is h in., the surface of the water is a circle of radius h in.;

$\therefore$ the volume of water in the vessel is

$$V = \tfrac{1}{3}\pi h^2 \times h = \tfrac{1}{3}\pi h^3 \text{ cu. in.};$$

$$\therefore \frac{dV}{dh} = \tfrac{1}{3}\pi \times 3h^2 = \pi h^2;$$

$$\therefore \delta V \simeq \pi h^2 . \delta h.$$

Suppose the volume increases from V to $V + \delta V$ in δt secs.

Then $$\delta V = 10 . \delta t;$$

$$\therefore \pi h^2 . \delta h \simeq 10 . \delta t;$$

$$\therefore \frac{\delta h}{\delta t} \simeq \frac{10}{\pi h^2}.$$

Now, when $t = 4$, $V = 40$;

$$\therefore \tfrac{1}{3}\pi h^3 = 40;$$

$$\therefore h^3 = \frac{120}{\pi};$$

$$\therefore h = \sqrt[3]{\frac{120}{\pi}} = 3{\cdot}368;$$

$$\begin{array}{r} 2{\cdot}0792 \\ 0{\cdot}4971 \\ \hline 3\,\big|\,1{\cdot}5821 \\ \hline 0{\cdot}5274 \\ \hline \end{array}$$

$$\therefore \frac{\delta h}{\delta t} \simeq \frac{10}{\pi(3{\cdot}368)^2}$$

$$= 0{\cdot}2806;$$

$$\begin{array}{r} 1{\cdot}0000 \\ 1{\cdot}5519 \\ \hline \bar{1}{\cdot}4481 \\ \hline \end{array} \qquad \begin{array}{r} 0{\cdot}5274 \\ 2 \\ \hline 1{\cdot}0548 \\ 0{\cdot}4971 \\ \hline 1{\cdot}5519 \\ \hline \end{array}$$

$\therefore$ the level is rising at 0·28″ per sec. approximately.

EXERCISE XVII. f.

1. The temperature of a metal cube is being raised steadily so that each edge expands at the rate of 0·01 inch per hour. At what rate is the volume increasing when the edge is 2 inches?

2. The cross-section of a trough is an isosceles right-angled triangle: the trough is 10 feet long. Water is poured into it at the rate of 5 cu. feet per sec. Find the rate at which the level is rising after 8 seconds.

3. The area of a circular ink-blot starts from zero and grows at the rate of 4 sq. inches per sec.; at what rate is the radius increasing (i) when the radius is 1 in., (ii) after 3 seconds?

4. A man 6 ft. high walks towards an electric arc street light at $3\frac{3}{4}$ miles an hour; the light is 24 feet above the level of the street. Find the rate in feet per sec. at which his shadow diminishes in length.

5. A certain quantity of gas is contained in a spherical envelope of volume v cu. in.: the pressure is p lb. per sq. inch, where $pv = 25$. The radius of the envelope increases at the rate of 2 in. per minute; find the rate at which the pressure is altering when the radius of the envelope is 5 inches.

6. A wine-glass is shaped so that when the depth of wine in it is y inches, its volume is $\frac{1}{5}y^4$ cu. in.; wine is poured in at the rate of 2 cu. in. per sec.; at what rate is the level rising when the depth is 2 in.?

7. The candle-power C of an incandescent lamp and its voltage V are connected by the equation $C = \frac{5V^6}{10^{11}}$. Find an expression for the rate of change of candle-power per unit increase of voltage, and evaluate it when $V = 100$.

8. ABC is a triangle right-angled at C; P is a point on AB; $PNCM$ is a rectangle with its corners N, M on CA, CB; $CA = 3$, $CB = 4$, $AP = x$ inches; P moves from A to B at 2 in. per minute; at what rate is the area of the rectangle increasing after (i) 60 sec., (ii) 75 sec., (iii) 90 sec.?

9. A cube is expanding so that its volume after t minutes is $1000 + 0{\cdot}2t + 0{\cdot}01t^2$ cu. inches; at what rate is its edge increasing in length after 10 minutes?

10. Sand is dropped on the ground at a steady rate of 10 cu. in. per sec. and forms a conical pile whose height remains equal to the radius of its base; at what rate is the height increasing after 5 seconds?

11. A bowl 5 inches deep is shaped so that when the depth of water in it is x inches, the amount of water is $8x + x^2$ cu. inches. Water is poured into it at the rate of 4 cu. in. per sec.; at what rate is the level rising when the depth is 3 inches?

12. The volume of a spherical envelope is increasing at the rate of 2 cu. cm. per sec.; at what rate is the area of the surface of the envelope stretching when the volume is 100 cu. cm.?

13. The volume of a circular cylinder is constant and equal to 150 cu. cm.; its height increases at the rate of 1 mm. a sec.; at what rate is its radius altering when its height is 3 cm.?

14. The heat required to raise the temperature of 1 gram of water from 0° C. to t° C. is Q units, where $Q = t + \frac{2}{10^5}t^2 + \frac{3}{10^7}t^3$. The specific heat is the rate of increase of the quantity of heat per degree rise in temperature. Find the specific heat at 60° C.

15. Water runs out of a bath at a rate proportional to the amount in the bath at any moment. Originally the bath contains

30 cu. feet of water; if t sec. later Q cu. feet have run out, express the data of the question by an equation.

SUMMARY OF RESULTS.

(i) An approximate value for

$$f(x+\delta x)-f(x) \quad \text{is} \quad \frac{d}{dx}f(x)\times\delta x.$$

(ii) If in calculating the value of $f(x)$ there is a small error δx in the value substituted for x, the error in the result is approximately $\frac{d}{dx}f(x)\times\delta x$.

(iii) If a distance x feet is described in t seconds, the speed is $\frac{dx}{dt}$ feet per second.

If y is a function of x and x is a function of t, the rate at which y increases is given by

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt},$$

for
$$\delta y \simeq \frac{dy}{dx}\times\delta x \quad \text{and} \quad \delta x \simeq \frac{dx}{dt}\times\delta t.$$

(iv) The formula $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$ is used to differentiate complicated functions;

$$e.g. \text{ if } y=\sqrt{x} \quad \text{and} \quad x=3t^2-5,$$

then
$$y=\sqrt{3t^2-5},$$

and
$$\frac{d}{dt}(\sqrt{3t^2-5})=\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$
$$=\frac{d}{dx}(\sqrt{x})\times\frac{d}{dt}(3t^2-5)$$
$$=\frac{1}{2\sqrt{x}}\times 6t$$
$$=\frac{3t}{\sqrt{3t^2-5}}$$

CHAPTER XVIII.

INTEGRATION.

Example I. If $\frac{dy}{dx}=x^2+7$, express y in terms of x.

We know that $$\frac{d}{dx}x^3=3x^2;$$

$$\therefore\ \frac{d}{dx}(\tfrac{1}{3}x^3)=x^2.$$

Also $$\frac{d}{dx}(7x)=7;$$

$$\therefore\ \frac{d}{dx}(\tfrac{1}{3}x^3+7x)=x^2+7;$$

$\therefore$ *one* value of y is given by $y=\frac{1}{3}x^3+7x$.

But $$\frac{d}{dx}(\tfrac{1}{3}x^3+7x-99) \text{ is also } x^2+7,$$

or $$\frac{d}{dx}(\tfrac{1}{3}x^3+7x+2\tfrac{1}{4}) \text{ is also } x^2+7,$$

because $\frac{d}{dx}$ (any constant) is 0.

$\therefore\ y=\frac{1}{3}x^3+7x+c$, where c is any constant, satisfies the given equation, and is called its *general solution.*

Note.—An equation involving $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$, etc., is called a *differential equation*: and the process of expressing y as a function of x is called *solving* or *integrating* the differential equation.

Whenever a differential equation is integrated, an arbitrary constant enters into the result.

EXERCISE XVIII. a.

1. Prove that $y = x^4 - 5$ is *a* solution of the differential equation $\frac{dy}{dx} = 4x^3$. What is the general solution ?

2. Prove that $y = \frac{1}{5}x^5 + \frac{1}{2}x^2 + 3$ is *a* solution of the differential equation $\frac{dy}{dx} = x^4 + x$. What is the general solution ?

3. The graph of a curve $y = f(x)$ is such that its gradient is everywhere equal to $2x$. (i) What is the differential equation for the curve ? (ii) Prove that $y = x^2$, $y = x^2 + 2$, $y = x^2 - 1$ are each possible solutions, and interpret this result geometrically by drawing the graphs.

4. Prove that $y = \frac{1}{12}x^4 + 97x - 13$ is *a* solution of the differential equation $\frac{d^2y}{dx^2} = x^2$. What is the general solution ?

5. Solve $\frac{dy}{dx} = x$. **6.** Solve $\frac{dy}{dx} = 3x^2 - 2x$.

7. Solve $\frac{dy}{dx} = 4$. **8.** Solve $\frac{d^2y}{dx^2} = 0$.

9. (i) What is $\frac{d}{dx} x^5$? (ii) Solve $\frac{dy}{dx} = x^4$.

(iii) Solve $\frac{dy}{dx} = 3x^4$.

10. (i) What is $\frac{d}{dx} x^8$? (ii) Solve $\frac{dy}{dx} = x^7$.

(iii) Solve $\frac{dy}{dx} = 5x^7 - 2$.

11. (i) What is $\frac{d}{dx}\left(\frac{1}{x}\right)$? (ii) Solve $\frac{dy}{dx} = \frac{1}{x^2}$.

(iii) Solve $\frac{dy}{dx} = \frac{4}{x^2}$.

12. (i) What is $\frac{d}{dx}\left(\frac{1}{x^4}\right)$? (ii) Solve $\frac{dy}{dx} = \frac{1}{x^5}$.

(iii) Solve $\frac{dy}{dx} = 1 - \frac{3}{x^5}$.

13. (i) What is $\frac{d}{dx}(\sqrt{x})$? (ii) Solve $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$.

(iii) Solve $\frac{dy}{dx} = x + \frac{3}{\sqrt{x}}$.

14. Solve $\frac{dy}{dx} = 5x^2 - 3x + 9$.

15. Solve $\frac{dy}{dx} = x^6 - x^2 + 3$.

16. Solve $\frac{dy}{dx} = x^2 + \frac{1}{x^2}$.

17. Solve $\frac{dy}{dx} = 5\sqrt{x}$.

18. Solve $x^3 \frac{dy}{dx} = 3x + 5$.

19. Solve $\frac{dy}{dx} = (x-2)(x-3)$.

20. Solve $\frac{d^2y}{dx^2} = x^2 - x$.

21. Solve $\frac{dy}{dx} = 3x^{100} + \frac{5}{x^{100}}$.

22. If $\frac{dy}{dx} = x - 2$, prove that $y = \frac{1}{2}x^2 - 2x + c$, and find c if $y = 5$ when $x = 6$.

23. If $\frac{ds}{dt} = 50 - 32t$, and if $s = 48$ when $t = 2$, find s in terms of t.

24. If $\frac{d^2s}{dt^2} = 4t$, and if, when $t = 0$, $\frac{ds}{dt} = 5$ and $s = 8$, express s in terms of t.

25. If a body travels s feet in t seconds, its velocity after t seconds is equal to $\frac{ds}{dt}$. Given that $\frac{ds}{dt} = 3 + 4t$, and that it passed a point A after 5 seconds from the start, find its distance from A after another 5 seconds.

26. The gradient of the graph of the function in Fig. 55 is given by $\frac{dy}{dx} = 3x^2 - 1$, and the graph cuts Oy at a distance 3 units from O. Find the function.

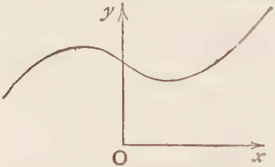


FIG. 55.

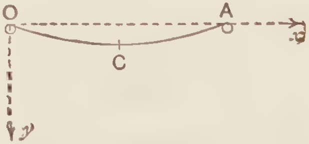


FIG. 56.

27. Fig. 56 represents a uniform lathe OA 20 feet long supported at its ends, which are at the same level. Bending slightly under its own weight, its form obeys the law $\frac{d^2y}{dx^2} = 0{\cdot}0001\,(x^2 - 20x)$; unit 1 foot on each axis.

(i) State from first principles the value of $\frac{dy}{dx}$ at the mid-point C where $x = 10$.
(ii) What is the value of y when $x = 0$?
(iii) Find y in terms of x, and then put $x = 20$ in the result; what should be the value of y?
(iv) What is the slope of the lathe at each of its ends?
(v) What is the sag of the lathe at its mid-point?

28. A chain OA, 100 feet long, hangs vertically from O. Its weight per unit length steadily diminishes from O to A, and at any point x feet from O is $(4 - 0 \cdot 03x)$ lb. per foot. If the work done in winding it up on a drum at O is W foot-lb., then $\frac{dW}{dx} = 4x - 0 \cdot 03x^2$; find the work done in winding it up.

29. A stone is thrown horizontally with a velocity of 12 feet per sec. from the top of a tower O, 100 feet high; Ox is horizontal and Oy vertical (see Fig. 57). If P is its position t seconds after starting, and if $ON = x$ feet, $PN = y$ feet, then x, y obey the laws $\frac{d^2x}{dt^2} = 0$; $\frac{d^2y}{dt^2} = 32$, neglecting air resistance.

(i) Express x and y each in terms of t.

(ii) Express y in terms of x.

(iii) How long does it take to reach the ground?

(iv) How far from the foot of the tower does it strike the ground?

(v) What is the gradient of the path of the stone where it strikes the ground?

(vi) A man is standing on the ground 50 feet from the foot of the tower; where is the stone when it appears to him to be coming straight at him?

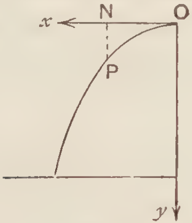


FIG. 57.

30. An elastic spiral spring attached to A, and hanging vertically, supports a body at O. The body is pulled down 3″ and let go. It now oscillates backwards and forwards. If its velocity is v inches per sec. when its distance from O is x in., v and x obey the law $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{16}x$.

(i) Find its velocity when it passes O.

(ii) How far must it be pulled down in order to have a velocity of 2 in. per sec. when passing O?

31. The rate at which a body cools is proportional to the excess of its temperature above that of the atmosphere. The temperature of a body is T° C. at x minutes past one in a room of temperature K° C. Express the law of cooling by a differential equation.

32. In climbing a mountain, the rate at which the atmospheric pressure p lb. per sq. in. decreases per unit increase of the height x feet is proportional to the pressure. Express this law by a differential equation.

Notation. The relation $\frac{dy}{dx}=f(x)$ is usually expressed in the form $y=\int f(x)\,dx$.

In other words, $\int f(x)dx$ represents the function which when differentiated with respect to x gives $f(x)$.

For example
$$\int x^3 dx \equiv \frac{x^4}{4}+c,$$
where c is a constant, because
$$\frac{d}{dx}\left(\frac{x^4}{4}+c\right)\equiv x^3.$$

AREAS AND VOLUMES.

Example II. Figure 58 represents the graph of $y=\frac{x^2}{5}+1$. unit $\frac{1}{2}''$ on each axis.

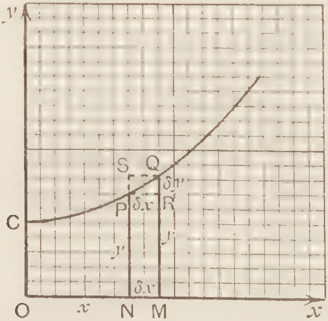


FIG. 58.

If $ON=x$ in., and if the area bounded by CO, ON, NP and arc CP is A sq. in., express A in terms of x.

With the usual notation

δA represents the area bounded by PN, NM, MQ and arc PQ;

$$\therefore\ \delta A > \text{rectangle } PNMR;$$

$$\therefore\ \delta A > y\,\delta x \quad\text{or}\quad \frac{\delta A}{\delta x} > y,$$

and $\delta A <$ rectangle $SNMQ$;

$$\therefore\ \delta A < (y+\delta y)\,\delta x \quad\text{or}\quad \frac{\delta A}{\delta x} < y+\delta y;$$

$$\therefore\ y+\delta y > \frac{\delta A}{\delta x} > y;$$

$$\therefore\ \text{when } \delta y \to 0,\ \frac{dA}{dx} = y = \frac{x^2}{5}+1;$$

$$\therefore\ A = \frac{x^3}{15}+x+c, \text{ where } c \text{ is constant.}$$

But when $x=0$, the area $A=0$;

$$\therefore\ c=0;$$

$$\therefore\ A=\frac{x^3}{15}+x.$$

Would there be any difference in the above argument if the graph sloped downwards instead of upwards?

Example III. Figure 59 represents the graph of $y=\dfrac{x^2}{5}+1$, not drawn to scale; if $OH=2$, $OK=7$, find the area bounded by AH, HK, KB and the arc AB.

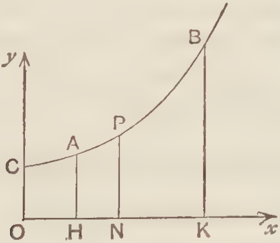


FIG. 59.

It is proved above in Example II. that if $ON=x$, the area of $CONP$ is A, where $\dfrac{dA}{dx}=\dfrac{x^2}{5}+1$;

$$\therefore\ A=\int\left(\frac{x^2}{5}+1\right)dx$$

$$=\frac{x^3}{15}+x+c, \text{ where } c \text{ is a constant;}$$

$$\therefore \text{ the area of } COKB = \frac{7^3}{15} + 7 + c,$$

$$\text{and the area of } COHA = \frac{2^3}{15} + 2 + c\,;$$

$$\therefore \text{ the area of } AHKB = \left[\frac{7^3}{15} + 7\right] - \left[\frac{2^3}{15} + 2\right]$$

$$= \frac{343 - 8}{15} + 5 = \frac{335}{15} + 5 = \frac{67}{3} + 5 = 27\tfrac{1}{3}.$$

Note the disappearance of the arbitrary constant c, owing to the subtraction.

Notation. The symbol $\int_a^b f(x)\,dx$ is used to express the value of $\int f(x)dx$ when $x = b$ *minus* the value of $\int f(x)\,dx$ when $x = a$.

Thus

$$\int_2^7 x^2 dx \equiv \left[\frac{x^3}{3}\right]_{x=7} - \left[\frac{x^3}{3}\right]_{x=2} = \frac{7^3}{3} - \frac{2^3}{3} = \frac{343 - 8}{3} = \frac{335}{3},$$

and we often write $\int_2^7 x^2\,dx \equiv \left[\frac{x^3}{3}\right]_2^7.$

Looking back at Example III. we see that the area of $AHKB$ can be written in any of the following forms:

$$\int_2^7 y\,dx \equiv \int_2^7 \left(\frac{x^2}{5} + 1\right) dx \equiv \left[\frac{x^3}{15} + x\right]_2^7.$$

The expression $\int f(x)\,dx$ is called an *indefinite* integral because its value contains an arbitrary constant.

The expression $\int_a^b f(x)\,dx$ is called a *definite* integral because the arbitrary constant of integration disappears automatically.

Examples II., III. show that $\int_a^b f(x)\,dx$ is the area bounded by two ordinates $x = a$, $x = b$, the x-axis and an arc of the graph of $y = f(x)$.

EXERCISE XVIII. b.

1. Fig. 60 represents the graph of $y=x^2$; $ON=x$; (i) find in terms of x the area bounded by ON, NP, arc OP; (ii) find the area $AHKB$, where $OH=1$, $OK=2$; (iii) express this area as a definite integral.

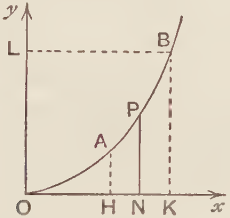


FIG. 60.

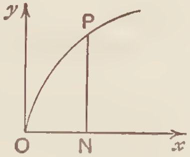


FIG. 61.

2. Fig. 61 represents the graph of $y^2=x$; $ON=x$; find the area ONP.

3. Interpret geometrically $\int_1^3 x\,dx$ and evaluate it (i) by direct calculation, (ii) geometrically.

4. Fig. 62 represents part of the graph of $y=(x-1)(4-x)$; (i) find OA and OB; (ii) find the area between Ox and the portion of the curve above Ox.

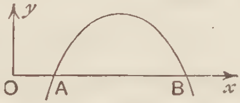


FIG. 62.

5. Draw the graph of $\frac{1}{x^2}$ and find the area (i) between it and the ordinates $x=1$, $x=2$ and the x-axis; (ii) between it and the lines $y=1$, $y=2$ and the y-axis.

6. If Fig. 60 represents the graph of $y=x^3$; prove that the area BLO is three times the area BKO.

7. Find the values of

(i) $\int_1^2 (x^2+5)\,dx$;   (ii) $\int_2^4 \frac{dx}{x^3}$;

(iii) $\int_{-1}^{+1} (x-x^2)\,dx$;   (iv) $\int_0^{\frac{1}{2}} (1+x)^2\,dx$.

8. Fig. 63 represents a solid formed by rotating part of the graph of $y = x^2$ about Ox.

$ON = x$, $NP = y$, $OM = x + \delta x$, $QM = y + \delta y$, $OH = \frac{1}{4}$, $OK = \frac{3}{4}$.

The figure shows cross-sections perpendicular to Ox.

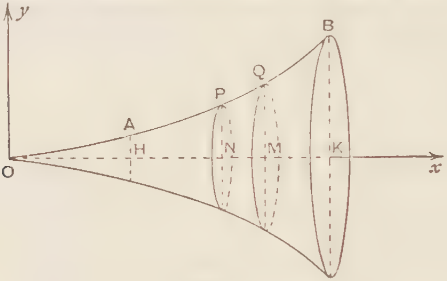


FIG. 63.

V is the volume of the solid between O and the cross-section through P.

(i) What is the area of the cross-section through P?

(ii) What is the area of the cross-section through Q?

(iii) Interpret $\pi y^2 \, \delta x$ geometrically.

(iv) Prove that $\pi (y + \delta y)^2 . \delta x > \delta V > \pi y^2 . \delta x$.

(v) Prove that $V = \pi \int_0^x y^2 \, dx$.

(vi) Express V in terms of x.

(vii) Find the volume between the cross-sections at A and B, and express it also as a definite integral.

(viii) If the graph of $y = f(x)$ is drawn, interpret geometrically $\int_0^x \pi y^2 \, dx$.

(ix) If the graph of $y = f(x)$ is drawn, interpret geometrically $\int_0^y \pi x^2 \, dy$.

9. (i) Draw the graph of $y = \frac{1}{3}x$; take a point P on it and draw PN perpendicular to Ox.

(ii) What solid is formed by revolving OP about Ox?

(iii) Find, by the method of Ex. 8, the volume of the solid between O and the section through P perpendicular to Ox, in terms of x.

10. The graph of $y = \frac{1}{x}$ is rotated about Oy; find the volume of the solid between two planes perpendicular to Oy and at distances 1 and 2 from O.

11. Fig. 64 represents a section of a hemisphere of radius of 10 in., centre O; $ON = x$ in.

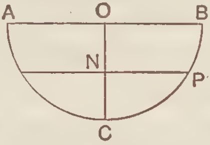


FIG. 64.

(i) Find PN in terms of x.

(ii) Find the area of the cross-section of the hemisphere at N perpendicular to ON.

(iii) Find an approximate expression for the volume of the hemisphere contained between two parallel planes at distances x and $x + \delta x$ from O.

(iv) If the volume between the sections through O and through N, perpendicular to ON, is V cu. in., express V in terms of x.

(v) Express as a definite integral the volume of the spherical segment between C and the section through N; and simplify it.

(vi) Find the volume of the hemisphere.

12. A basin is formed by revolving the graph of $y = \frac{1}{8}x^4$ about the vertical axis Oy; how much water is in the basin when the greatest depth is 4 inches? [Unit for each axis, one inch.]

13. (i) Sketch the graph of $y = \frac{1}{27}(2x + 3)(9 - 2x)$ from $x = 0$ to $x = 3$.

(ii) A beer barrel is formed by rotating this portion of the graph about Ox. [Unit for each axis, one foot.] Find the capacity of the barrel in cu. feet.

14. (i) Sketch the graphs of $y^2 = x$ and $x^2 = 8y$ for positive values of x.

(ii) Show that they cross each other at the point (4, 2).

(iii) Show that the area of the portion common to both is $\int_0^4 \left(\sqrt{x} - \frac{x^2}{8}\right) dx$, and evaluate it.

15. If $f(x) = \int_0^x \left(1 - \frac{y^2}{4}\right) dy$, find the value of $\int_0^1 f(x)\, dx$. [This may be stated more shortly as follows: find the value of

$$\int_0^1 \int_0^x \left(1 - \frac{y^2}{4}\right) dy\, dx.]$$

16. Using the notation explained in Ex. 15. find the value of

$$\int_{-1}^{1}\int_{1}^{x} y\,(1-y)\,dy\,dx.$$

17. Show that $\int_0^1 (x+2)(x+3)\,dx = \int_1^2 (x+1)(x+2)\,dx$. Interpret this result geometrically.

18. A lathe of length 10 feet is built into a wall horizontally at one end and is deflected by a weight fastened at the other end. The deflection at x feet from the wall is $\int_0^x f(v)\,.\,dv$ feet, where $f(v) = \int_0^v 0{\cdot}0005\,(10-u)\,du$; [or more shortly,

$$\int_0^x \int_0^v 0{\cdot}0005(10-u)\,du\,dv].$$

Find the deflection (i) when $x=10$, (ii) when $x=5$.

19. A lathe of length 10 feet is supported at its ends and loaded at the middle. The deflection at x feet from one end is $\int_0^x f(v)\,.\,dv$ feet, where $f(v) = \int_v^5 0{\cdot}0005u(10-u)\,du$. Find the deflection at the mid-point.

20. The moment of inertia of a cylinder, radius r in., height h in., about its axis is $2\pi\rho h\int_0^r x^3\,.\,dx$, where ρ lb. is the mass per cu. inch. If the mass of the cylinder is W lb., express the moment of inertia in terms of r, W.

21. The moment of inertia of a sphere, radius r in., about a diameter is $\pi\rho\int_0^r (r^2-x^2)^2 dx$, where ρ lb. is the mass per cu. inch. If the mass of the sphere is W lb., express the moment of inertia in terms of r, W.

22. A plane area is enclosed between the curve $y^2 = 2x^3$ and the line $x=3$. The distance of the centre of gravity of this area measured along the axis of x from the origin is given by the formula $S = \int_0^3 yx\,dx \div \int_0^3 y\,.\,dx$. Evaluate this expression.

23. A solid is formed by the rotation of the portion of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ lying between the lines $x=0$ and $x=5$ about the axis of x. The distance of the centre of gravity of this solid measured from the origin along its central axis is given by the formula $S = \int_0^5 \pi y^2 x\,dx \div \int_0^5 \pi y^2\,.\,dx$. Evaluate this expression.

24. A gas is compressed in a cylinder by a piston. If the face of the piston is initially a inches from the inside end of the cylinder, and if the piston travels forward until this distance is b inches, the work done is $-\int_a^b \frac{660}{x^{1.4}}\,dx$ ft.-lb. If $a=10$, find the work done in compressing the gas to half its volume.

25. A new cure for chilblains was tried on 8 people. It succeeded in 5 cases and failed in 3 cases. The chance that it succeeds in the next two cases is

$$\int_0^1 x^7(1-x)^3\,dx \div \int_0^1 x^5(1-x)^3\,dx.$$

Find what the chance is.

26. Two points are taken at random on a line 12 inches long. The chance that the distance between them does not exceed 3 inches is $\int_3^{12} (x-3)\,dx \div \int_0^{12} x\,dx$. Evaluate this chance.

27. The portion of the parabola $y=\frac{1}{4}x^2$ (the y-axis being drawn vertically downwards) above $y=5$ is taken and submersed in a liquid in a vertical plane with its vertex uppermost and 1 unit distance below the surface. Then the depth of the centre of pressure below the surface is

$$\int_0^5 (1+x)^2\sqrt{x}\,dx \div \int_0^5 (1+x)\sqrt{x}\,dx.$$

Evaluate this expression.

28. The density of a spherical body of radius a feet varies directly as its distance from the centre. If ρ is the density (*i.e.* the mass in lb. per cu. ft.) at the surface, (i) prove that the mass of the spherical shell whose inner and outer radii are x and $x+\delta x$ feet, is approximately $\frac{4\pi\rho x^3}{a}\,.\,\delta x$ lb.; (ii) find the total mass of the body.

29. The density of a chain AB varies as its distance from A (*i.e.* at a point x feet from A, the material weighs kx lb. per foot length); its length is 20 feet; the lighter half weighs 30 lb. Find the weight of the heavier half.

30. The velocity of a body after t sec. from its start is v feet per sec., where v is a given function of t, viz. $v=f(t)$; (i) what meanings can you give to $\int v\,dt$? (ii) Given the time-velocity graph of a moving particle, how could you find the distance travelled in a given time?

31. Fig. 65 represents a mound with the rectangle $ABCD$ as base and with its sides sloping up to the ridge EF, which is situated symmetrically. $AB = 30'$, $AD = 18'$, $EF = 10'$; and the height of EF above the base is 12 feet.

(i) Prove that the area of the section of the mound parallel to the base and h feet below EF is

$$\tfrac{5}{2}h\,(h+6) \text{ sq. ft.}$$

(ii) Find the volume of the mound.

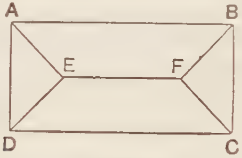


FIG. 65.

32. (i) Sketch the half of the ellipse $\frac{x^2}{4} + y^2 = 81$ which lies between $x = 0$ and $x = 18$.

(ii) This portion is rotated about Ox to form the explosive head of a torpedo; unit on each axis, 1 inch. Find the volume of the explosive in the head.

DUFTON'S RULE FOR APPROXIMATION OF AREAS.

PQ is an arc of a curve; PA, QB are the perpendiculars to a line AB.

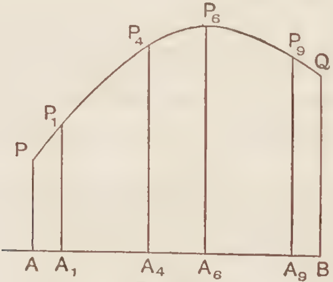


FIG. 66.

AB is divided into ten equal parts AA_1, A_1A_2, A_2A_3, … A_9B. Through the points A_1, A_4, A_6, A_9 perpendiculars A_1P_1, A_4P_4, A_6P_6, A_9P_9 are drawn to meet the curve.

Then the area contained by PA, AB, BQ, and the arc PQ is

$$\bumpeq \tfrac{1}{4} \,.\, AB \,.\, [A_1P_1 + A_4P_4 + A_6P_6 + A_9P_9].$$

SIMPSON'S RULE FOR APPROXIMATION OF AREAS.

PA, QB are the perpendiculars from P, Q to a line AB.

AB is divided into any even number, say $2n$, equal parts, and the ordinates $y_1, y_2, y_3, \ldots y_{2n+1}$ are drawn from each point of division and from the beginning and end of the line to the curve.

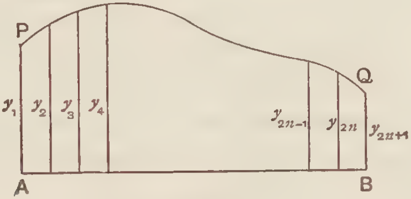


FIG. 67.

Let $$h = \frac{AB}{2n}.$$

Then the area contained by PA, AB, BQ and the arc PQ is

$$\simeq \tfrac{1}{3}h\,[y_1 + y_{2n+1} + 2(y_3 + y_5 + y_7 + \ldots + y_{2n-1}) + 4(y_2 + y_4 + y_6 + \ldots + y_{2n})].$$

Dufton's rule is obviously quicker to apply than Simpson's rule, as the numerical computation is less. The relative accuracy of the two rules depends on the nature of the curve, but in general Dufton's rule gives as high a degree of accuracy as is required.

EXERCISE XVIII. c.

1. (i) Write down the values of x^2 for $x = 0{\cdot}1$, $0{\cdot}4$, $0{\cdot}6$, $0{\cdot}9$, and then use Dufton's rule to find the area bounded by the x-axis, the graph of $y = x^2$ and the ordinate through $x = 1$.

(ii) Evaluate this area by calculation, and find the error per cent. in the result obtained from Dufton's rule.

(iii) Divide the area into 4 strips and apply Simpson's rule to find the area.

2. (i) Draw the graph corresponding to the following table of values :

$x=0$	1	2	3	4	5	6	7	8	9	10
$y=0$	7	8	11	15	18	21	18	14	6	0

(ii) Find its area by "counting squares."
(iii) Find its area by "Dufton's rule."
(iv) Find its area by "Simpson's rule."

3. Draw a semicircle, diameter 10 cm., and find its area

(i) by Dufton's rule,
(ii) by Simpson's rule (ten strips),
(iii) by formula.

Give the approximate error per cent. of (i) and (ii).

4. Find the area bounded by the graph of $y=\frac{60}{x}$, the x-axis, and the ordinates $x=5$ and $x=10$

(i) by Dufton's rule,
(ii) by Simpson's rule.

Assuming that the area is 41·6, find the approximate error per cent. in each case.

5. The velocity of train after leaving a station is shown in the following table :

t, time in minutes	0	2	4	6	8	10	12	14	16	18	20
v, speed in yards per minute -	0	50	110	160	230	290	360	410	470	530	570

Find in miles the distance travelled in the first 20 minutes

(i) by Dufton's rule,
(ii) by Simpson's rule.

6. In finding the volume of earth in a cutting between two sections A and B, the following measurements were obtained :

	Distance from A in yards.	Area of section of cutting in sq. feet.
At A - -	0	2000
	8	3000
	12	3420
	19	3900
	24	3900
	30	3700

(i) Represent these results graphically.

(ii) Find the volume of earth in the cutting in cu. ft. (*a*) by Dufton's rule, (*b*) by Simpson's rule. (C.S.C.)

7. The speed of a car, v miles per hour, was noted at intervals of 12 seconds over a period of 2 minutes.

Time in seconds =	0	12	24	36	48	60	72	84	96	108	120
$v =$	5	3	2·5	3	5·7	10·3	14	10·5	5·6	2	0

Find the distance travelled in feet by two methods. (C.S.C.)

8. Use Dufton's rule to find an approximate result for the volume of a hemisphere, radius 10 cm., taking sections parallel to the base of the hemisphere. Compare this with the result given by Simpson's rule and by the formula $\frac{2}{3}\pi r^3$ for a hemisphere.

9. (i) Sketch the graph of $y = \log x$ from $x = 1$ to $x = 10$.

(ii) Interpret geometrically $\int_1^2 \log x \, . \, dx$.

(iii) Use Dufton's rule and Simpson's rule to find an approximate value of this definite integral.

10. By using Dufton's and Simpson's rules, find an approximate value of $\int_0^{10} \sqrt{1 + x^2}\, dx$.

CHAPTER XIX.

SIMPLE SERIES.

Example I. The following sequence of numbers obeys a simple law; find (i) the 15th term, (ii) the nth term in the sequence

$$11, 15, 19, 23, 27, 31, \ldots.$$

We see that each term exceeds the term before it by 4.

The 2nd term $= 15 = 11 + 4$.

The 3rd term $= 19 = 11 + 4 + 4 = 11 + 2 \times 4$.

The 4th term $= 23 = 11 + 4 + 4 + 4 = 11 + 3 \times 4$,

and so on;

$\therefore$ the 15th term $= 11 + 14 \times 4 = 11 + 56 = 67$,

and the nth term $= 11 + (n-1) \times 4 = 11 + 4n - 4 = 4n + 7$.

Example II. Find the nth term in each of the following sequences:

(i) $7, 21, 63, 189, \ldots;$

(ii) $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \ldots.$

(i) We see that each term is three times the term before it.

The 2nd term $= 21 = 7 \times 3$.

The 3rd term $= 63 = 7 \times 3 \times 3 = 7 \times 3^2$.

The 4th term $= 189 = 7 \times 3 \times 3 \times 3 = 7 \times 3^3$,

and so on;

$\therefore$ the nth term $= 7 \times 3^{n-1}$.

(ii) We notice that the numerators are $3, 3^2, 3^3, 3^4, \ldots;$

$\therefore$ the numerator of the nth term is 3^n.

The denominators are $4, 4 + 3, 4 + 2 \times 3, 4 + 3 \times 3, \ldots;$

$\therefore$ the denominator of the nth term is

$$4 + (n-1) \times 3 = 4 + 3n - 3 = 3n + 1;$$

$$\therefore \text{ the } n\text{th term} = \frac{3^n}{3n+1}.$$

EXERCISE XIX. a.

1. The following sequences of numbers obey simple laws; write down in each case the 6th and the nth number in the sequence :

(i) 2, 4, 6, 8, ... ; (ii) 1, 4, 9, 16, 25, ... ;
(iii) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots$; (iv) 10, 200, 3000, 40000, ... ;
(v) 3, 6, 12, 24, ... ; (vi) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \frac{9}{32}, \dots$;
(vii) 5, 12, 19, 26, 33, ... ; (viii) 10, 7, 4, 1, -2, ... ;
(ix) 0·1, 0·02, 0·003, 0·0004, ... ; (x) -1, 1, -1, 1, -1, ... ;
(xi) 8, 27, 64, 125, ... ; (xii) 14, 108, 1012, 10016, ...

2. Write down the first three terms of the sequence in which the nth term is

(i) $2n-1$; (ii) $\frac{3^n}{18}$; (iii) $\frac{n+1}{n+2}$;
(iv) n^2+n ; (v) $3 . 10^{n-1}+5n$; (vi) n^2-1.

3. The sum of the first n terms of a sequence is $\frac{n(n+1)}{2}$; (i) what is the first term? (ii) what is the sum of the first two terms? (iii) what is the second term? (iv) what is the 10th term? (v) what is the nth term?

4. The sum of the first n terms of a sequence is $\frac{n(n+1)(2n+1)}{6}$; find (i) the first term, (ii) the 4th term, (iii) the nth term.

5. What is the average of the following 14 numbers?

3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47, 51, 55.

Is it necessary to add them all up?
When you know their average, how can you find their sum? What is it?

6. Find (i) the average, (ii) the sum of the following sequences :

(i) 8, 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 47, 50.
(ii) 60, 55, 50, 45, 40, 35, 30, 25, 20, 15, 10.
(iii) 20, 13, 6, -1, -8, -15, -22, -29, -36, -43, -50.
(iv) $3\frac{1}{2}$, $4\frac{3}{4}$, 6, $7\frac{1}{4}$, $8\frac{1}{2}$, $9\frac{3}{4}$, 11, $12\frac{1}{4}$, $13\frac{1}{2}$, $14\frac{3}{4}$, 16.

7. How many terms are there in the following sequences?

(i) 7, 9, 11, 13, 15, ..., 53, 55, 57, 59 ;
(ii) 22, 29, 36, 43, 50, 57, ..., 155, 162, 169;
(iii) 100, 97, 94, 91, 88, ..., 46, 43, 40 ;
(iv) $3\frac{1}{6}$, $4\frac{1}{8}$, $5\frac{1}{10}$, $6\frac{1}{12}$, ..., $20\frac{1}{40}$, $21\frac{1}{42}$.

8. Find the least number of three digits which belongs to the sequence:

(i) 6, 13, 20, 27, ... ;
(ii) 1, 2, 4, 8, 16, ... ;
(iii) 3, 14, 25, 36, ...

Find also the nth term in each case.

9. In a sequence the terms increase by a constant amount (positive or negative) [as, for example, in 6 (i), (iii)]; find the second term if

(i) the first term is 8 and the 10th term is 71;
(ii) the first term is 3 and the 15th term is 4;
(iii) the first term is 14 and the 8th term is -7;
(iv) the fourth term is 25 and the 12th term is 97.

10. Find the sum of the following:

(i) $25+28+31+34+\dots+94+97+100+97+94+91+\dots+52+49$;
(ii) $4-5+6-7+8-9+\dots+40-41+42$.

ARITHMETICAL PROGRESSIONS.

Definition. A sequence of numbers in which each term exceeds the preceding term by the same amount, positive or negative, is called an *Arithmetical Progression* or, for short, an A.P.

E.g. (i) 5, 12, 19, 26, 33, ...,
or (ii) 10, 2, -6, -14, ...,
or (iii) $6\frac{1}{4}$, $7\frac{3}{4}$, $9\frac{1}{4}$, $10\frac{3}{4}$, ...

In (i) each term exceeds the term before it by 7.

This amount 7 is called the *common difference.*

In (ii) the common difference is $2-10=-8$,
or $-6-2=-8$.

In (iii) the common difference is $7\frac{3}{4}-6\frac{1}{4}=1\frac{1}{2}$.

If three numbers are in arithmetical progression, the middle number is called the *arithmetic mean* of the other two.

E.g. 21, 29, 37 are in A.P.;
$\therefore$ 29 is the arithmetic mean of 21 and 37.

The arithmetic mean of two numbers is simply their average.

Example III. Find the sum of 24 terms of the sequence

$$-41,\ -33,\ -25,\ -17,\ \dots .$$

The common difference $= -33-(-41) = -33+41 = 8$;

$\therefore$ the 24th term $= -41+23\times 8 = -41+184 = 143$;

$\therefore$ the sum $\quad s = -41-33-25 \dots + 127+135+143.$

The average of this sequence is $\frac{-41+143}{2} = \frac{102}{2} = 51$;

$$\therefore\ s = 24\times 51$$
$$= 1224.$$

Another method.

$$s = -41-33-25-\dots+127+135+143.$$

Write the sequence backwards;

$$\therefore\ s = 143+135+127+\dots-25-33-41;$$

$\therefore$ adding $\qquad 2s = 102+102+102+\dots+102.$

But there are 24 terms;

$$\therefore\ 2s = 102\times 24;$$

$$\therefore\ s = \frac{102\times 24}{2} = 51\times 24 = 1224.$$

Example IV. The 10th term of an A.P. is 17 and the 25th term is 41; find the second term.

Let the first term be a and the common difference d;

$$\therefore\ \text{the 10th term is } a+9d;$$

$$\therefore\ a+9d = 17.$$

Similarly, $\qquad a+24d = 41$;

$$\therefore\ -15d = -24;$$

$$\therefore\ d = \tfrac{24}{15} = \tfrac{8}{5} = 1\tfrac{3}{5};$$

$$\therefore\ a = 17-9d = 17-\tfrac{72}{5} = 17-14\tfrac{2}{5} = 2\tfrac{3}{5};$$

$$\therefore\ \text{the second term} = a+d = 2\tfrac{3}{5}+1\tfrac{3}{5}$$
$$= 4\tfrac{1}{5}.$$

EXERCISE XIX. b.

1. The first term of an A.P. is 10 and the third is 11; find the 20th.

2. The first term of an A.P. is 11 and the 10th term is -4: find the 3rd term.

3. The first term of an A.P. is 21, the last term is 193, and the common difference is 2; find the number of terms.

4. Find the arithmetic mean of 23 and 90.

5. If 12, x, y, z, 41 are in A.P., find x, y, z. This question might be worded as follows: "insert three arithmetic means between 12 and 41."

Sum the series in Ex. 6-15:

6. $7+8+9+\dots+100$.

7. $22+19+16+\dots$ 20 terms.

8. $5{\cdot}3+7{\cdot}2+9{\cdot}1+\dots$ 25 terms.

9. $8\frac{1}{3}+6\frac{5}{6}+5\frac{1}{3}+\dots$ 18 terms.

10. $3+7+11+15+\dots$ n terms.

11. $a+(a+d)+(a+2d)+(a+3d)+\dots$ 12 terms.

12. $a+(a+d)+(a+2d)+\dots$ n terms.

13. $11+13\frac{1}{2}+\dots+36$.

14. $23+\dots+67$ (12 terms).

15. $a+\dots+l$ (n terms).

16. In the sequence

[1]; [2, 3]; [4, 5, 6]; [7, 8, 9, 10]; [11, 12, 13, 14, 15], ...,

what is the first number in (i) the 8th bracket, (ii) the nth bracket?

17. Find the sum of all numbers less than 100 which are not divisible by 7.

18. The first two terms of an A.P. are a, x; what is (i) the third term, (ii) the nth term?

19. In the sequence

[1]; [3, 5]; [7, 9, 11]; [13, 15, 17, 19]; ...,

what is the first number in (i) the 10th bracket, (ii) the nth bracket?

20. The nth term of a sequence is $5n-11$; what is the common difference and the sum of the first n terms?

21. (i) Each term of an A.P. is multiplied by 7; is the result an A.P.?

(ii) Each term of an A.P. is multiplied by itself; is the result an A.P.?

22. What is the sum of all numbers divisible by 12 between 1000 and 5000?

23. In a sequence of n terms, the last term is l and the common difference is d; what is the first term?

24. If a, b are integers, find the sum of all integers from a to b inclusive, assuming $a<b$.

SUMMARY OF RESULTS.

Any arithmetical progression can be written in the form

$$a, \quad a+d, \quad a+2d, \quad a+3d, \ \ldots;$$

(i) the nth term $=a+(n-1)d$;

or, if $l=$ the last term and $n=$ the number of terms,

$$l=a+(n-1)d;$$

(ii) the sum of n terms $=s=n\times$ average term;

or $$s=\frac{n}{2}(a+l)$$

and $$s=\frac{n}{2}[2a+(n-1)d];$$

(iii) the arithmetic mean of a, b is $\dfrac{a+b}{2}$.

GEOMETRICAL PROGRESSIONS.

Definition. A sequence of numbers in which each term bears a fixed ratio to the term preceding it is called a *Geometrical Progression* or, for short, a G.P.

E.g. (i) $4, \ 12, \ 36, \ 108, \ 324, \ \ldots,$

or (ii) $7, \ -14, \ 28, \ -56, \ 112, \ \ldots,$

or (iii) $12, \ 4, \ 1\frac{1}{3}, \ \frac{4}{9}, \ \frac{4}{27}, \ \ldots.$

In (i), the ratio is $\frac{12}{4}=3=\frac{36}{12}=\frac{108}{36}=\ldots.$

This amount 3 is called the *common ratio.*

In (ii) the common ratio $=\frac{-14}{7}=-2.$

In (iii) the common ratio $=\frac{4}{12}=\frac{1}{3}.$

If three numbers are in geometrical progression, the middle number is called the *geometric mean* of the other two.

E.g. 4, 12, 36 are in G.P.;

$\therefore$ 12 is the geometric mean of 4 and 36.

If a, x, b are in G.P., $\dfrac{x}{a}=\dfrac{b}{x}$, by definition;

$$\therefore \ x^2=ab;$$

$$\therefore \ x=\pm\sqrt{ab}.$$

Therefore the geometric mean of a, b is $\pm\sqrt{ab}$.

Example V. Find the sum of 10 terms of the sequence

$$18,\ 12,\ 8,\ 5\tfrac{1}{3},\ \ldots .$$

$$\text{The common ratio} = \frac{12}{18} = \frac{2}{3};\quad \left[\frac{8}{12} = \frac{2}{3} \text{ and } \frac{5\frac{1}{3}}{8} = \frac{2}{3}\right];$$

$$\therefore \text{ the 10th term} = 18 \times (\tfrac{2}{3})^9.$$

$$s = 18 + 12 + 8 + \ldots + 18 \times (\tfrac{2}{3})^7 + 18 \times (\tfrac{2}{3})^8 + 18 \times (\tfrac{2}{3})^9;$$

$$\therefore \tfrac{2}{3}s = 12 + 8 + 5\tfrac{1}{3} + \ldots + 18 \times (\tfrac{2}{3})^8 + 18 \times (\tfrac{2}{3})^9 + 18 \times (\tfrac{2}{3})^{10}.$$

The terms from 12 up to $18 \times (\frac{2}{3})^9$ are the same in both lines.

$$\therefore s - \tfrac{2}{3}s = 18 - 18 \times (\tfrac{2}{3})^{10} = 18[1 - (\tfrac{2}{3})^{10}];$$

$$\therefore 3s - 2s = 54[1 - (\tfrac{2}{3})^{10}];$$

$$\therefore s = 54[1 - (\tfrac{2}{3})^{10}].$$

An approximate value can be found by logarithms.

$$\tfrac{2}{3} = 0\cdot 6667 = 10^{1\cdot 8240};$$

$$\therefore (\tfrac{2}{3})^{10} = 10^{2\cdot 240} = 0\cdot 0174;$$

$$\therefore s = 54 \times 0\cdot 9826$$

$$= 53\cdot 1.$$

EXERCISE XIX. c.

1. Write down, using the index notation, the nth terms in geometrical progressions which start :

(i) $7,\ 35$; ii) $1,\ -2$; (iii) $8,\ 4$;

(iv) $-10,\ 6$; (v) $6,\ 7$; (vi) $2\frac{1}{3},\ 3$;

(vii) $-x,\ x^3$; (viii) $a,\ b$; (ix) $a,\ 1$.

2. Simplify (i) $8(-\frac{1}{2})^7$; (ii) $(-1)^{2n} - (-1)^{2n-1}$.

3. What is the first term in the sequence $1,\ \frac{3}{4},\ \frac{9}{16},\ \frac{27}{64},\ \ldots$, which is less than $0\cdot 001$?

4. If $s = 3 + 6 + 12 + 24 + 48 + 96 + 192 + 384 + 768$, write down $2s$ in full ; then subtract, and so find s.

5. If $s = 54 + 18 + 6 + 2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \frac{2}{243}$, write down $\frac{1}{3}s$ in full ; then subtract, and so find s.

6. If $s = a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6$, write down rs in full ; what is $rs - s$? Hence find s.

7. Sum the following sequences ; do *not* multiply out the result :

(i) $4,\ 4 \times 3,\ 4 \times 3^2,\ 4 \times 3^3,\ \ldots$ to 10 terms ;

(ii) $5,\ 5 \times \frac{2}{5},\ 5 \times (\frac{2}{5})^2,\ 5 \times (\frac{2}{5})^3,\ \ldots$ to 12 terms ;

(iii) $\frac{3}{4},\ \frac{1}{2},\ \frac{1}{3},\ \ldots$ to 8 terms ;

(iv) $7,\ -14,\ 28,\ -56,\ \ldots$ to 11 terms ;

(v) $27,\ -12,\ 5\frac{1}{3},\ \ldots$ to 20 terms.

8. (i) Divide $1-r^6$ by $1-r$.

(ii) What is the quotient when $1-r^{10}$ is divided by $1-r$?

(iii) Express $1+r+r^2+r^3+\ldots+r^{18}+r^{19}+r^{20}$ as a simple fraction.

(iv) Express $a+ar+ar^2+ar^3+\ldots+ar^{20}$ as a simple fraction.

9. If $s=a+ar+ar^2+\ldots$ to n terms,

(i) what is the nth term ?

(ii) what is the last-but-one term ?

(iii) write down the value of rs, putting in the first three and the last three terms ;

(iv) what is the value of $rs-s$? Hence find s.

10. (i) Prove that $1+\frac{1}{10}+\frac{1}{100}+\ldots$ to 8 terms $=\frac{10}{9}\left(1-\frac{1}{10^8}\right)$.

(ii) What is the value of this answer to 3 places of decimals ?

(iii) What is the approximate error per cent. in taking the sum of these 8 terms as $\frac{10}{9}$?

(iv) What is the sum of 50 terms of the same sequence, and what is the approximate error per cent. in taking the sum as $\frac{10}{9}$?

(v) What is the meaning of $1\cdot\dot{1}$, and what is its value ?

11. AB is a straight line, one inch long. Bisect AB at C_1 ; bisect C_1B at C_2 ; bisect C_2B at C_3 ; bisect C_3B at C_4 ; and so on.

(i) What are the lengths of AC_1, C_1C_2, C_2C_3, C_3C_4, C_4C_5, ... ?

(ii) What are the lengths of AC_2, AC_3, AC_4, AC_5, AC_{10}, AC_{100} ?

(iii) What is the limiting value of the length of AC_n when $n \to \infty$, and what series is this associated with ?

(iv) What is the difference between the sum of the first 10 terms and the sum of the first 50 terms of the sequence 8, 4, 2, 1, $\frac{1}{2}$, ... ?

(v) What is a fair approximation for the sum of the first 10 terms of this sequence ?

(vi) What meaning would you give to the expression "the sum to infinity" of this sequence ? Why is it better to call it the "limiting sum" ? [This name is due to Prof. T. P. Nunn.]

12. Which of the following geometrical progressions can be "summed to infinity" or in other words has a "limiting sum" ?

(i) $1, \frac{10}{9}, (\frac{10}{9})^2, (\frac{10}{9})^3, \ldots$; (ii) 100, 90, 81, ... ;
(iii) $4, -6, +9, \ldots$; (iv) $1, -\frac{1}{2}, \frac{1}{4}, \ldots$.

13. Sketch freehand a graph (it will consist of a set of disconnected points) representing the sum of the first 1, 2, 3, 4, ... terms of the sequence as a function of the number of terms

(i) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$; (ii) $1, 2, 4, 8, \ldots$.

14. Express $0{\cdot}\dot{4}\dot{3}$ as a geometrical progression, and then find its "limiting sum."

15. Find the "limiting sum" of the following:

(i) $500 + 400 + 320 + 256 + \ldots$;

(ii) $10, \ -2, \ 0{\cdot}4, \ -0{\cdot}08, \ldots$;

(iii) $a + ar + ar^2 + ar^3 + \ldots$ if $1 > r > -1$.

16. Find, using logarithms, an approximate value of

(i) $50 + 50\,(1{\cdot}05) + 50\,(1{\cdot}05)^2 + \ldots$ to 10 terms;

(ii) $8 + 12 + 18 + \ldots$ to 18 terms.

SUMMARY OF RESULTS.

Any geometrical progression can be written in the form

$$a, \quad ar, \quad ar^2, \quad ar^3, \ \ldots;$$

(i) the nth term $= ar^{n-1}$;

(ii) the sum of n terms $= \dfrac{a(r^n - 1)}{r - 1} = \dfrac{a(1 - r^n)}{1 - r}$;

(iii) if $1 > r > -1$, the "limiting sum" or "sum to infinity"

$$= \frac{a}{1 - r};$$

(iv) the geometric mean of a, b is $\pm\sqrt{ab}$.

MISCELLANEOUS EXAMPLES ON SERIES.

EXERCISE XIX. d.

1. A train travelling 12 m.p.h. gathers speed uniformly till five minutes later its speed is 40 m.p.h.; what was its speed at the end of the first minute, and how far did it go in the five minutes?

2. Thirty concentric circles are engraved at equal distances apart on a metal plate; the radii of the inner and the outer circles are $4{\cdot}5''$ and $7{\cdot}3''$; find the distance between two consecutive circles correct to $\frac{1}{1000}$ inch.

3. A ball thrown vertically upwards from the floor in a vacuum returns to the floor in 2 seconds; the times of successive rebounds are $1\frac{1}{2}$ sec., $1\frac{1}{8}$ sec., $\frac{27}{32}$ sec., ... ; how long elapses before the sixth bounce? Find also the time from the start till the ball comes to rest.

4. A man starting business loses £240 the first year, £160 the second year, £80 the third year; if the same improvement continues, what is his total gain or loss after 12 years?

5. Potatoes are planted in parallel rows in a field, each row containing 3 more than the row before; the first row holds 32 and the last 149. How many potatoes are there in the field?

6. The height of a tree is 20 feet and one year later is 24 feet; its growth each year is $\frac{3}{4}$ of its growth the previous year. What is the limiting height of the tree, however long it lives?

7. Find the sum of $0{\cdot}7 + 0{\cdot}71 + 0{\cdot}72 + 0{\cdot}73 + \ldots$ to 100 terms.

8. Find the sum of $2 + 22 + 222 + \ldots$ to 8 terms.

9. A square is divided into 25 equal squares, and the numbers 1, 2, 3, ... up to 25 are arranged in these squares so that the sum of the numbers in each row and each column is the same; what is the sum of the numbers in each row? [This arrangement is called a magic square.]

10. In a potato race, ten potatoes are placed in a line with the starting point at intervals of two yards, and the nearest is 20 yards from the starting point to which each potato must be brought separately. How many yards must a competitor run if he can reach one yard both picking up and putting down the potato?

11. A pendulum is set swinging; its first vibration is through 20°, and each succeeding vibration is $\frac{9}{10}$ of the one before it. What is the total angle through which it swings before coming to rest?

12. What is the sum of the first n odd integers?

13. Postal orders are issued for all multiples of sixpence from 6d. to 21s. inclusive, except for 20s. 6d.; the commissions are: 6d. to 2s. 6d., one penny; 3s. to 15s., three halfpence; 15s. 6d. to 21s., twopence. Find the cost of one complete set of postal orders.

14. In a group of n people every one writes once to every one else; how many letters are written altogether? [Count them as follows: how many letters does A write and receive? how many letters does B, not counting correspondence with A? how many letters does C, ignoring A, B? and so on.]

15. *A* starts with a salary of £500 a year and receives a yearly increase of £20 a year; *B* starts at the same time at £300 a year and receives a yearly increase of £50 a year. After how many years will the total sum received by *B* exceed that received by *A*, neglecting interest ?

16. *A* starts at a salary of £500 a year and receives a yearly increase of £100 a year; *B* starts at £500 a year and receives a half-yearly increase of £50 per half-year. How much has each received altogether after 10 years, neglecting interest ?

17. A gramophone record has a spiral line cut on it winding from the outer edge, which is a circle of 6 inches radius, to the inner edge, which is a circle of radius 2 inches. The record revolves 80 times a minute and takes 4 minutes to play. Assuming that the spiral path is equivalent to a number of equally spaced concentric circles, find the total length of the path in yards.

18. (i) To how much will £100 amount after 10 years at 5 per cent. compound interest, reckoned yearly ?

(ii) On January 1st, 1900, a man invested £100 at 5 per cent. compound interest, reckoned yearly. He invested in the same way £100 on January 1st of each year up to 1910 inclusive. What was the value of his investments on January 1st, 1911 ?

19. (i) How much must be paid now in order to receive £100 at the end of 10 years, allowing 5 per cent. compound interest, reckoned yearly ?

(ii) What lump sum was paid on January 1st, 1900, in order to secure a payment of £100 a year for ten years, starting on January 1st, 1915, allowing 5 per cent. compound interest ?

20. What single payment was required on January 1st, 1900, in order to secure a payment of £100 a year on every succeeding 1st of January for ever, allowing 4 per cent. compound interest ?

21. If $s = 1 + 2x + 3x^2 + \ldots + nx^{n-1}$, what is $s - xs$ equal to ? Hence find s.

22. The series $1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + \ldots$ is obviously greater than

$$1 + \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{4}) + (\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}) + \ldots \quad \text{or} \quad 1 + \tfrac{1}{2} + (\tfrac{2}{4}) + (\tfrac{4}{8}) + \ldots.$$

How many terms of the first series will give a sum greater than 100 ?

23. Show that the sum of any number of terms of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$ always lies between 1 and $\frac{1}{2}$.

24. Prove that the sum of any number of terms of the series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots$ is less than a sufficient number of terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$, and therefore is < 2.

[Group the terms of the first series as in Example 22 above.]

25. Find the sum of 20 terms of the series

$$\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\frac{1}{5\times 6}+\ldots.$$

[Note that $\frac{1}{2\times 3}=\frac{1}{2}-\frac{1}{3}$, etc.]

26. Find the sum of 30 terms of the series

$$\frac{1}{3\times 5}+\frac{1}{5\times 7}+\frac{1}{7\times 9}+\ldots.$$

[See hint for No. 25.]

27. The sum of n terms of the series $1^2+2^2+3^2+4^2+5^2+\ldots$ is of the form an^3+bn^2+cn, where a, b, c are constants; find a, b, c.

28. The sum of n terms of a series is n^3; find the first four terms.

29. Verify that $1^3+2^3+3^3+\ldots+n^3=(1+2+3+\ldots+n)^2$ for the case $n=5$.

30. Find the sum of 10 terms of the series

$$1+1\times 2^2+1\times 2\times 3^2+1\times 2\times 3\times 4^2+\ldots.$$

[Note that

$$\begin{aligned}1\times 2\times 3\times 4^2&=1\times 2\times 3\times 4\,(5-1)\\&=1\times 2\times 3\times 4\times 5-1\times 2\times 3\times 4.]\end{aligned}$$

31. If all rational numbers are written down as follows:

$$\frac{1}{1},\ \frac{2}{1},\ \frac{1}{2},\ \frac{3}{1},\ \frac{2}{2},\ \frac{1}{3},\ \frac{4}{1},\ \frac{3}{2},\ \frac{2}{3},\ \frac{1}{4},\ \frac{5}{1},\ \frac{4}{2},\ \frac{3}{3},\ \frac{2}{4},\ \frac{1}{5},\ \ldots,$$

(i) how many terms in the sequence are there before $\frac{11}{13}$?

(ii) what will be the 100th term?

32. Find the sum of 198 terms of the series

$$\log 1\tfrac{1}{2}+\log 1\tfrac{1}{3}+\log 1\tfrac{1}{4}+\log 1\tfrac{1}{5}+\ldots.$$

GROWTH FUNCTIONS.

Example VI. If £1 is invested at 3 per cent. compound interest, find its amount at the end of x years.

The interest on £100 is £3;

∴ the interest on £1 is £0·03;

∴ £1 amounts to £1·03 at the end of 1 year.

We therefore start the second year with £1·03.
This amounts to £(1·03)(1·03) at the end of the second year;

$\therefore$ the amount at the end of the second year is £$(1{\cdot}03)^2$.

Similarly, the amount at the end of the third year is £$(1{\cdot}03)^3$, and so on;

$\therefore$ the amount at the end of x years is £$(1{\cdot}03)^x$.

Such a function as $(1{\cdot}03)^x$ is called a *Growth function*, because it represents the magnitude of a quantity which increases in such a way that its growth over a definite period (in the above case one year) is proportional to its magnitude at the beginning of that period.

EXERCISE XIX. e.

1. (i) Draw the graph of the growth function $(1{\cdot}5)^x$ as x varies from 0 to 4.

(ii) A plant grows so that its height is $(1{\cdot}5)^x$ inches after $10x$ days, as long as $x < 4$; what is its height (1) to start with, (2) at the end of 4 days, 8 days, 12 days, 16 days?

(iii) In what ratio does its height increase (1) from the end of the 4th to the end of the 8th day, (2) from the end of the 8th day to the end of the 12th day, (3) from the end of the 12th day to the end of the 16th day?

2. Assuming that the population of a country is a growth function of the time, find the population in 1900, given that the population in 1880 was 23·6 million and in 1890 was 27·4 million.

3. The excess of the temperature of a liquid cooling in a room of fixed temperature above that of the room is a growth function of the time. If the temperature of the room is 30° C., and if at 1 p.m. the temperature of the liquid is 70° C. and at 1.10 p.m. it is 60° C., what is its temperature at 1.30 p.m.?

4. A merchant finds that his capital is a growth function of the time. After 10 years' work his capital is £50,000, and after 20 years' work it is £80,000; what will it probably be after 25 years' work?

5. If $r^x = 10$ and $r^{x+4} = 20$, what is r^{x+8} and what is r?

6. In an influenza epidemic it was found that the number of cases was a growth function of the time. In ten days the number had increased from 320 to 480; what was the daily increase per cent.?

7. (i) If $f(x)$ is a growth function of x, prove that

$$\frac{f(x)}{f(x+a)}=\frac{f(x+a)}{f(x+2a)}=\frac{f(x+2a)}{f(x+3a)}, \text{ etc.}$$

(ii) Given a graph of a function, how would you test whether the function is or is not a growth function?

CHAPTER XX.

PERMUTATIONS AND COMBINATIONS.

PERMUTATIONS.

Example I. Given five different letters a, b, c, d, e, how many different arrangements are there if each arrangement consists of two different letters ?

One way of answering this is to write down all the possible arrangements in a tabular form as follows :

Initial letter -	a	b	c	d	e
Arrangements -	ab ac ad ae	ba bc bd be	ca cb cd ce	da db dc de	ea eb ec ed

This shows that there are 5 groups each containing 4 arrangements ;

$\therefore$ the total number of arrangements is $5 \times 4 = 20$.

Note.—This number is called the number of *permutations* of 5 things taken 2 at a time, and is written ${}^5P_2$.

The argument can be put more concisely as follows, without writing down all the possible arrangements.

Imagine two compartments thus :

Initial Letter.	Second Letter.

Into the first compartment any one of the 5 letters can be put ; *i.e.* it can be filled in 5 ways.

And then, whatever letter has been put first, there are always 4 letters left, any one of which can be put in the second compartment.

Therefore the total number of possible arrangements is

$$5 \times 4 = 20.$$

Example II. How many different arrangements can be made with the 26 letters of the alphabet, if each arrangement is to consist of three different letters ?

Imagine three compartments :

First Letter.	Second Letter.	Third Letter.

The first can be filled in 26 ways.

The second can then be filled in 25 ways, however the first has been filled.

∴ there are 26×25 ways of filling the first two compartments.

The third compartment can then be filled in 24 ways, however the first two have been filled.

∴ there are $26 \times 25 \times 24$ ways of filling the three compartments, *i.e.* the number of permutations of 26 things taken 3 at a time, or ${}^{26}P_3$ is $26 \times 25 \times 24$.

Similarly, it could be shown that the number of permutations of 5 things taken 5 at a time is $5 \times 4 \times 3 \times 2 \times 1$.

This product is usually written $\underline{|5}$ or $5!$, and called "factorial five," so that

$${}^{5}P_5 = \underline{|5},$$

and

$${}^{n}P_n = n(n-1)(n-2) \ldots \times 3 \times 2 \times 1 = \underline{|n},$$

where n is any positive integer.

EXERCISE XX. a.

1. Write down in tabular form all the different arrangements that can be made with the six letters a, b, c, d, e, f, if each arrangement is to consist of two different letters. How many arrangements are there ? What is ${}^{6}P_2$?

2. How many different arrangements can be made with six letters, if each arrangement is to consist of three different letters ? What is the number of permutations of six things taken three at a time ?

3. What is the value of (i) ${}^4P_2$, (ii) ${}^{10}P_3$?

4. In how many different ways can six books be arranged on a shelf ?

5. How many different numbers between 100 and 1000 can be formed out of the digits 2, 3, 4, 5, 6, 7, 8, if no digit may occur twice in the same number ?

6. What is the value of (i) ${}^7P_4$, (ii) $\frac{\underline{|7}}{\underline{|3}}$?

7. (i) There are ten people, each of whom sends a Christmas card to each of the others ; how many cards are required altogether ?

(ii) Ten houses play in a league competition, each house playing every other house ; how many games are played altogether ?

8. In how many different ways can the algebraical expression *abcd* be written ?

9. There are 8 stations on a local line ; how many different kinds of single third-class tickets must be printed in order that it may be possible to book from any one station to any other ?

10. One pound notes are numbered as follows : $\genfrac{}{}{0pt}{}{\text{N}}{27}$ 803407. How many differently numbered pound notes can be issued if any one of the 26 letters of the alphabet may be used, if the number under the letter may be any number from 1 to 99 inclusive and the number following may be any number from 1 to 999999 inclusive ?

11. In how many ways can the order of batting of a cricket XI. be arranged if a particular batsman is to go in last and it is immaterial which of the first two batsmen is put down first ?

12. How many terms are there in the expansion of the products

(i) $(a+b)(x+y+z)$;

(ii) $(a+b)(c+d+e)(f+g+h+k)$;

(iii) $(1+a)(1+b)(1+c)(1+d)(1+e)$?

13. There are 12 houses on an open plain, no three of which are in a straight line. A man starts from one particular house and ends at another particular house ; what is the number of possible routes followed if he calls at 2 other houses on the way ?

14. A railway carriage has 8 seats ; in how many ways can 8 people arrange themselves (i) if there is no restriction, (ii) if three people have to sit on the side facing the engine, (iii) if two people have to sit in corner seats ?

15. How many numbers greater than 5000 can be formed out of the digits (i) 3, 4, 5, 6, 7, (ii) 0, 3, 4, 5, 6, if no digit is repeated ?

16. In how many ways can five people sit at a round table (i) if there is no restriction, (ii) if two particular people insist on sitting together, (iii) if two particular people refuse to sit together ?

17. In how many ways can 8 people be arranged in a row of 8 chairs if two particular people occupy the end seats ?

18. In how many ways can 6 men and 3 ladies be arranged at a round table so that no two ladies sit together ?

19. Six letters are written and the six corresponding envelopes are addressed ; in how many different ways can the letters be put one into each envelope so that (i) exactly two go wrong, (ii) at least two go wrong ?

20. In how many ways can five Latin books, four Greek books and four French books be arranged on a shelf so that the books in each language are together ?

21. How many even numbers between 100 and 1000 can be formed of the digits 2, 3, 4, 5, 6 ?

22. The guests at a dinner party consist of six ladies and six gentlemen : the host and hostess sit at opposite ends of an oblong table, the guests sit 6 on each side. In how many ways can the table be arranged if ladies and gentlemen sit in alternate places ?

23. (i) Show that $\dfrac{\underline{|11}}{\underline{|6}} = 11 \times 10 \times 9 \times 8 \times 7$.

(ii) Show that $14 \times 13 \times 12 = \dfrac{\underline{|14}}{\underline{|11}}$.

24. Express in factorial form as in Ex. 23 (ii) :

(i) $20 \times 19 \times 18 \times 17 \times 16$; (ii) ${}^{20}P_6$;

(iii) ${}^{30}P_{12}$; (iv) ${}^{n}P_{12}$; (v) ${}^{n}P_{r}$.

25. Simplify (i) $\underline{|n} \div \underline{|n-1}$;

(ii) $\underline{|n} - \underline{|n-1}$.

26. Simplify $\dfrac{\underline{|2n}}{1.3.5.7.9\ldots(2n-3)(2n-1)}$.

27. Use logarithms to find the value of ${}^{200}P_{10}$ to two significant figures.

28. If $\underline{|20}$ is expressed in prime factors, to what power will (i) 2, (ii) 3 occur ?

SUMMARY OF RESULTS.

The number of permutations of n things taken r at a time is written ${}^nP_r$ and is equal to $\dfrac{\underline{|n}}{\underline{|n-r}}$ or

$$n(n-1)(n-2)\ldots(n-r+1).$$

COMBINATIONS.

Example III. Find the number of ways in which from the seven letters a, b, c, d, e, f, g, a selection of three letters can be made, no regard being paid to the order.

Any selection of 3 letters can be rearranged in $\underline{|3}=6$ ways; *e.g.* the selection b, c, f yields the 6 distinct arrangements bcf, bfc, cbf, cfb, fbc, fcb.

The total number of arrangements of 7 things, taken 3 at a time, is $7\times 6\times 5$.

But each selection yields $\underline{|3}$ arrangements;

$$\therefore \text{ the number of selections} = \frac{7\times 6\times 5}{\underline{|3}} = 35.$$

Note.—This is called the number of *combinations* of 7 things taken 3 at a time, and is written ${}^7C_3$.

A difference in order makes a new *arrangement*, but does not alter the *selection*.

The number of different ways of selecting r things from n different things is called the number of *combinations* of n things taken r at a time, and is denoted by ${}^nC_r$.

The argument in Example III. shows that

$${}^nC_r = {}^nP_r \div \underline{|r} = \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}.$$

EXERCISE XX. b.

1. In how many ways can 2 men be chosen from 5 men? What is the value of ${}^5C_2$?

2. In how many ways can a football eleven be chosen from 14 men? What is ${}^{14}C_{11}$?

3. In how many ways can 3 cards be chosen from a pack of 52 cards ?

4. How many different hands is it possible for a player to have at bridge if his hand consists of 13 cards chosen from a pack of 52 ? Evaluate this number by logarithms to two significant figures.

5. There are 12 points in a plane, no 3 of which are collinear ; (i) how many lines can be drawn joining pairs of points ? (ii) how many triangles can be drawn each having 3 of these points as vertices ?

6. What is the greatest number of points of intersection of

(i) 12 lines with each other ;
(ii) 12 circles with each other ?

7. How many different sums of money can be made up if one of each of the following coins is available : a halfpenny, a penny, a sixpence, a shilling, a florin ?

8. In how many ways can a committee of 4 men and 2 ladies be appointed from a society of 12 men and 8 ladies ?

9. (i) £12 is distributed among six people so that two receive £3 each, two receive £2 each, and two receive £1 each ; in how many ways can this be done ?

(ii) In how many ways can six people be divided into three pairs ?

Why are the answers to (i) and (ii) different ?

10. (i) In how many ways can two sides of six players each be chosen from 12 men ?

(ii) In how many ways can three sides of four players be chosen from 12 men ?

11. In how many ways can the partners at a dance be arranged if there are 6 ladies and 6 gentlemen ?

12. (i) Four people play a game in which 13 cards are dealt to each of them. How many different distributions of the cards are possible ?

(ii) In how many ways can a pack of 52 cards be arranged in 4 hands of 13 cards each ?

13. A cricketing party on tour consists of 19 men, of whom 12 are primarily batsmen, 5 primarily bowlers, 2 primarily wicket-keepers : in how many ways can an eleven consisting of 7 batsmen, 3 bowlers, 1 wicket-keeper be selected ?

14. In how many ways can a committee of 5 be chosen from 10 candidates so as to include both the youngest and oldest ?

15. In how many ways can 5 dots and 3 dashes be arranged in a row ?

16. In how many ways can 6 dots and 4 dashes be arranged in a row if no two dashes are together ?

17. Prove that (i) ${}^7C_5 = {}^7C_2$;
(ii) ${}^nC_r = {}^nC_{n-r}$

18. Prove that (i) ${}^{10}C_4 + {}^{10}C_5 = {}^{11}C_5$;
(ii) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$.

19. Simplify (i) ${}^nC_r \div {}^nC_{r-1}$;
(ii) ${}^nC_r \div {}^{n-1}C_{r-1}$.

Prove that ${}^nC_r + 2 \cdot {}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$.

20. (i) If n is a positive integer > 2, prove that $n(n-1)(n-2)$ is divisible by 6.

(ii) If n is a positive integer $> r$, prove that

$$n(n-1)(n-2) \dots (n-r+1)$$

is divisible by $\underline{|r}$.

21. If $n = a + b + c$, and if n, a, b, c are positive integers, prove that $\underline{|n}$ is divisible by $\underline{|a} \times \underline{|b} \times \underline{|c}$.

22. A selection of r letters is made from the n letters, a_1, a_2, a_3, ... a_n ; (i) how many of these include a_1 ? (ii) how many of these do not include a_1 ? (iii) hence deduce that

$${}^{n-1}C_r + {}^{n-1}C_{r-1} = {}^nC_r.$$

MISCELLANEOUS EXAMPLES

(Including Simple Probability).

Example IV. There are 10 tickets in a bag which are numbered 1, 2, 3, ... up to 10. Two are drawn out at random. What is the chance that at least one of these is greater than 5 ?

The number of ways in which 2 tickets can be selected from 10 is ${}^{10}C_2 = \frac{10 \times 9}{2} = 45$.

The number of ways in which 2 tickets can be selected from those numbered 1, 2, 3, 4, 5 is ${}^5C_2 = 10$;

$\therefore$ the chance that *both* tickets are not greater than 5 is $\frac{10}{45} = \frac{2}{9}$;

$\therefore$ the chance that *at least* one ticket is greater than 5 is

$$1 - \tfrac{2}{9} = \tfrac{7}{9}.$$

Note.—The *odds* that at least one ticket is greater than 5 are 7 : 2.

When it is said that the probability of an event is, say, $\frac{3}{4}$, this does not mean that in every four trials the event will occur exactly 3 times; nor does it follow that in four million trials, it will occur exactly three million times. But it does mean that if it occurs r times in n trials, it will usually be found that the fraction $\frac{r}{n}$ approaches closer to $\frac{3}{4}$, the larger n becomes.

If the probability of an event occurring is $\frac{r}{n}$, then the probability against it occurring is $1-\frac{r}{n}$ or $\frac{n-r}{n}$,

and the odds on it occurring are $r:(n-r)$ on it,

or $(n-r):r$ against it.

EXERCISE XX. c.

1. Two cards are drawn from a pack of 52 cards.

(i) In how many ways can this be done?

(ii) What is the number of ways if none of the four aces are chosen?

(iii) What is the number of ways if at least one of the two cards is an ace?

(iv) What is the chance that at least one of the two cards is an ace?

(v) What are the odds against one of the two cards being an ace?

2. Two Americans and four Englishmen toss up to decide which four shall play in a set of tennis.

(i) In how many ways can the four players be chosen?

(ii) In how many of these ways will both Americans be chosen?

(iii) What is the chance that both Americans play in the first set?

3. Twelve boys are chosen at random from a division of twenty boys, which includes three boys from a particular house. What is the chance that these three boys are all among the chosen twelve?

4. You are one of 15 boys from whom an eleven is picked at random.

(i) In how many different ways can the eleven be picked?

(ii) In how many of these ways will you yourself be picked?

(iii) What is your chance of being picked?

5. Work out the general case of which Question 4 above is a particular example, and show that the method of Question 4 gives your chance of being one of m boys picked at random from a total of n boys as ${}^{n-1}C_{m-1} \div {}^{n}C_{m}$, and that this simplifies to the result $\frac{m}{n}$ which ordinary common sense would lead you to expect.

6. (i) Two coins are put down on a table. What is the chance that they both show ‘heads’ ?

(ii) Three coins are put down on a table. What is the chance that they all three show ‘heads’ ?

(iii) n coins are put down on a table. What is the chance that all the coins show ‘heads’ ?

7. Three coins are put down on a table.

(i) What is the chance that they are either all three heads or all three tails ?

(ii) What is the chance that they show either two heads and a tail or two tails and a head ?

(iii) What is the chance that they show two heads and a tail ?

8. Two people each draw a card from an ordinary pack of 52 cards ; what is the chance that their cards form a pair ?

9. The six faces of two dice are numbered from 1 to 6. The dice are thrown on the table.

(i) What is the chance that 1 is the upper face on both dice ?

(ii) What is the chance that the sum of the numbers on the two upper faces is 9 ?

10. There are ten entries for a mile race, and numbers are drawn at random for the starting positions. What is the chance that the slowest runner is one of the inside three ? What is the chance that the slowest runner is one of the inside three and also the fastest runner one of the outside three ?

11. Six boys, including yourself and a friend, are arranged at random in a line. What is the chance that you are next to your friend ?

12. Two sides of eleven are picked at random from twenty-two people. What is the chance that A and B are on opposite sides ?

13. How many different arrangements, using all five letters, can be made of the letters

(i) a_1, a_2, a_3, b_1, b_2 ;

(ii) a_1, a_2, a_3, b, b ;

(iii) a, a, a, b, b ?

14. (i) In how many different ways can fifteen differently coloured beads be arranged on a straight thread ?

(ii) What is the number of ways if five of the beads are all red and exactly alike ?

(iii) What is the number of ways if also the other ten are all blue and exactly alike ?

(iv) What is the number of ways if the thread is circular instead of being straight and the beads all of different colour ?

15. Three prizes are awarded to a division of twelve boys, and any boy may win all the prizes. In how many ways can each of the three prizes be awarded separately ? What is the total number of ways in which the three prizes can be awarded ?

16. How many numbers of three digits can be formed with the digits 1, 3, 5, 7, (i) if no repetitions are allowed, (ii) if the digits may be repeated ?

17. How many numbers of not more than four digits can be formed with the digits 1, 3, 5, 7, if repetitions are allowed ?

18. A certain periodical announces a competition, in which a prize is awarded to the reader sending in correct results to sixteen different football matches. If the result may be either a win, a loss or a draw in each match, what is the total number of different forecasts that may be sent in ? Use logarithms to evaluate this number to two significant figures, and state the chance of a reader hitting upon the correct solution if he fills up the results at random.

19. A cricket XI. is being photographed. Five of the eleven sit on chairs, four stand and two sit on the ground. How many different arrangements can be made, disregarding difference of order in standing or sitting, if the captain is to sit on a chair ?

20. In how many ways can 6 different books be distributed between A, B and C, so that A has one book, B has two and C has three ?

21. In how many ways can 6 people be divided into three groups, if there need not be more than 1 person in a group ?

22. In how many ways can $2n$ people be divided into n couples ?

23. A golfer wants to buy three golf balls at the professional's shop, and says that he wants either "Silver Kings," "Dunlops," "Colonels" or "Why Nots." The professional has only 3 Silver Kings, 2 Dunlops, 1 Colonel and 1 Why Not left in stock. In how many ways can the golfer choose 3 balls from these ?

24. Compute the chance that a player at bridge who has 13 cards dealt to him from a pack of 52 in the ordinary way holds all four aces in his hand.

25. A man has 3 pennies and 4 half-crowns in his pocket, and he pulls out 3 coins at random. How many different sums can he pull out, and which of these is the most likely ? What would you be prepared to give him for the three coins before learning what they were ?

26. A pessimist went during the war to France as one of a draft of 120 men. He calculated that in 3 months' time $\frac{1}{10}$th of the draft would be dead, $\frac{1}{5}$th would be in hospital and $\frac{1}{12}$th would be captured. At what amount should he have fixed the probability of his being still on active service in 3 months' time ?

CHAPTER XXI.

THE BINOMIAL THEOREM.

If we multiply out $(x+a)(x+b)$ we obtain various terms, each of which has two factors, one from the first bracket and the other from the second bracket.

Thus $$(x+a)(x+b) \equiv x^2+ax+bx+ab.$$

In the same way, if we multiply out $(x+a)(x+b)(x+c)$ we get various terms, each of which has three factors, one from each bracket. These terms can be conveniently grouped according to the number of x factors chosen in each term.

If the x is chosen from each bracket, we get the term x^3.

If the x is chosen from two brackets and the remaining letter from the third bracket, we get ${}^3C_1$ terms, namely the terms $ax^2+bx^2+cx^2$.

If the x is chosen from one bracket and the remaining letter from the other two brackets, we get ${}^3C_2$ terms (the number of ways in which the other two letters can be chosen), and these terms are $abx+bcx+cax$.

Finally, if no x term is chosen, we get one term abc.

Thus $(x+a)(x+b)(x+c)$ can be written down by inspection

as $$x^3+x^2(a+b+c)+x(ab+bc+ca)+abc.$$

Note that if we now make b and c equal to a, we find that

$$\begin{aligned}(x+a)(x+a)(x+a) &\equiv x^3+x(a+a+a)+x(a^2+a^2+a^2)+a^3\\ &\equiv x^3+3ax+3a^2x+a^3.\end{aligned}$$

EXERCISE XXI. a.

1. *Write down* the expansions of

(i) $(x+a)(x+b)(x+c)(x+d)$;
(ii) $(x+a)(x+a)(x+a)(x+a)$;
(iii) $(x+1)(x+1)(x+1)(x+1)$;
(iv) $(x-1)^4$.

2. (i) How many terms are there in the expansion of

$$(x+a)(x+b)(x+c)(x+d)(x+e)(x+f)\ ?$$

(ii) How many of these terms have a factor x and not x^2 ?

(iii) How many of these terms have x^2 as a factor and not x^3 ?

3. What is the coefficient of x^3 in the expansions of

(i) $(x+a)(x+b)(x+c)(x+d)(x+e)$;
(ii) $(x+a)^5$;
(iii) $(x-1)^5$?

4. How many terms in the following expansions contain x^3 but not x^4 as a factor :

(i) $(x+a)(x+b)(x+c)(x+d)$;
(ii) $(x+a_1)(x+a_2)(x+a_3)(x+a_4)(x+a_5)(x+a_6)(x+a_7)$

What is the coefficient of x^3 in $(x+1)^7$?

5. Expand in descending powers of x :

(i) $(x+1)(x+2)(x+3)$;
(ii) $(x+1)(x+2)(x+3)(x+4)$.

6. Expand in descending powers of x :

(i) $(x+1)(2x+1)(3x+1)$;
(ii) $(x+1)(2x+1)(3x+1)(4x+1)$.

The examples of the last exercise show how to write down the expansion of any expression of the form

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4)\dots(x+a_n).$$

It is, in fact,

$$x^n+x^{n-1}(a_1+a_2+a_3+\dots)+x^{n-2}(a_1a_2+a_1a_3+a_2a_3+\dots)$$
$$+x^{n-3}(a_1a_2a_3+a_1a_2a_4+\dots)+\dots.$$

The coefficient of x^{n-2} is the sum of the products, taken *two* at a time, of the n letters $a_1, a_2, \dots, a_n$.

The coefficient of x^{n-3} is the sum of these products taken *three* at a time.

The coefficient of x^{n-4} is the sum of these products taken *four* at a time, and so on.

Therefore, there are n terms in the coefficient of x^{n-1},
and ${}^nC_2$ .... x^{n-2},
and ${}^nC_3$ .... x^{n-3},
and so on.

Suppose now $a_1 = a_2 = a_3 = a_4 = \dots = a_n = a$.

Then each term in the coefficient of x^{n-1} equals a,
and .... x^{n-2} .... a^2,
and .... x^{n-3} .... a^3,
and so on.

$\therefore$ the coefficient of x^{n-1} is $n \, . \, a$,
.... x^{n-2} is ${}^nC_2 \, . \, a^2$,
.... x^{n-3} is ${}^nC_3 \, . \, a^3$,
and so on.

$\therefore$ if n is any positive integer,

$$(x+a)^n \equiv x^n + n \, . \, x^{n-1}a + {}^nC_2 \, . \, x^{n-2}a^2 + {}^nC_3 \, . \, x^{n-3}a^3 + \dots + {}^nC_r x^{n-r}a^r + \dots + a^n,$$

or

$$(x+a)^n \equiv x^n + n \, . \, x^{n-1}a + \frac{n(n-1)}{1 \, . \, 2} x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1 \, . \, 2 \, . \, 3} x^{n-3}a^3 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \, . \, 2 \, . \, 3 \dots r} x^{n-r}a^r + \dots + a^n$$

This result is called the *Binomial Theorem.*

EXERCISE XXI. b.

Write down the expansions of Nos. 1-8.

1. $(1+x)^3$. **2.** $(1+x)^4$. **3.** $(1-x)^4$. **4.** $(x-2)^3$.

5. $(x-y)^6$. **6.** $\left(x+\dfrac{1}{x}\right)^5$. **7.** $(2x-3y)^3$. **8.** $(a^3-2b)^4$.

9. What is the coefficient of x^3 in (i) $(x-1)^7$; (ii) $(1+2x)^n$?

10. What is the 4th term in the expansion of $(2-x)^{10}$?

11. What is the term independent of x in the expansion of $\left(x+\dfrac{2}{x}\right)^4$?

12. How many terms are there in the expansion of $(2x-3y)^{10}$?

13. What is the middle term in the expansion of $(x-y)^6$?

14. What is (i) the 4th term from the beginning; (ii) the 4th term from the end in $(x+y)^{10}$?

15. What is the rth term in the expansion of (i) $(1+2x)^n$; (ii) $(1-2x)^n$?

16. What is (i) the rth term from the beginning; (ii) the rth term from the end in $(x+y)^n$? Are the coefficients equal ?

17. Simplify $(x-1)^3+3(x-1)^2+3(x-1)+1$.

18. Simplify $1-4(x+1)+6(x+1)^2-4(x+1)^3+(x+1)^4$.

19. In the identity

$$(1+x)^n \equiv 1+{}^nC_1 \,.\, x+{}^nC_2 \,.\, x^2+\ldots+{}^nC_r x^r+\ldots+x^n,$$

what results are obtained by putting (i) $x=1$; (ii) $x=-1$?

20. What is the coefficient of x^4 in $(1-x)(1+x)^7$?

21. What is the coefficient of x^r in $(1-x)(1+x)^n$?

22. Expand $(1+x+x^2)^n$ as far as terms involving x^3.

23. Simplify $(\sqrt{2}+1)^4+(\sqrt{2}-1)^4$.

24. Use the binomial theorem to evaluate 101^4 and 99^5.

25. Use the binomial theorem to evaluate to 5 significant figures $1{\cdot}02^5$ and $0{\cdot}97^6$.

APPROXIMATION.

The binomial expansion

$$(1+x)^n \equiv 1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\ldots$$
$$+\frac{n(n-1)(n-2)\ldots(n-r+1)}{1\,.\,2\,.\,3\ldots r}x^r+\ldots$$

has been proved, if n is a positive integer.

If n is fractional or negative, this result still holds good, provided that x lies between $+1$ and -1. The general proof is, however, beyond the scope of this book.

If n is a positive integer, the expansion contains $n+1$ terms.

But if n is fractional or negative, the expansion in this form does not terminate: in this case, if x is small compared with 1, the first few terms of the expansion may give a useful approximation.

EXERCISE XXI. c.

1. Prove by long division that

$$\text{(i)}\ \frac{1}{1-x} \equiv 1 + x + \frac{x^2}{1-x};$$

$$\text{(ii)}\ \frac{1}{1-x} \equiv 1 + x + x^2 + \frac{x^3}{1-x};$$

$$\text{(iii)}\ \frac{1}{1-x} \equiv 1 + x + x^2 + x^3 + \frac{x^4}{1-x}.$$

2. Find R if $\dfrac{1}{1-x} \equiv 1 + x + x^2 + x^3 + x^4 + R$.

3. Assuming the binomial theorem, if x is a positive fraction less than 1, write down the first five terms of the expansion of

$$\frac{1}{1-x} \equiv (1-x)^{-1}.$$

4. If $x = 0{\cdot}1$, what is approximately the error in taking $\dfrac{1}{1-x}$ equal to (i) $1 + x$; (ii) $1 + x + x^2$?

5. Find R by long division if

$$\text{(i)}\ \frac{1}{1+x} \equiv 1 - x + R;$$

$$\text{(ii)}\ \frac{1}{1+x} = 1 - x + x^2 + R;$$

$$\text{(iii)}\ \frac{1}{1+x} = 1 - x + x^2 - x^3 + R.$$

6. Assuming the binomial theorem, if x is a positive fraction less than 1, write down the first five terms of the expansion of

$$\frac{1}{1+x} \equiv (1+x)^{-1}.$$

If $x = 0{\cdot}01$, what is approximately the error in taking $\dfrac{1}{1+x}$ equal to (i) $1 - x$; (ii) $1 - x + x^2$?

7. Find R if (i) $\dfrac{1}{(1-x)^2} \equiv 1 + 2x + R$;

$$\text{(ii)}\ \frac{1}{(1+x)^2} = 1 - 2x + R.$$

8. Assuming the binomial theorem, if x is a positive fraction less than 1, write down the first four terms of the expansions of

$$\frac{1}{(1-x)^2} \equiv (1-x)^{-2} \quad \text{and} \quad \frac{1}{(1+x)^2} \equiv (1+x)^{-2}.$$

What is approximately the error in taking $(1-x)^{-2} = 1 + 2x$ and $(1+x)^{-2} = 1 - 2x$ if $x = 0{\cdot}01$?

9. If x is a positive fraction less than 1, prove that

(i) $(1-\frac{1}{2}x)^2 > 1-x$; (ii) $(1-\frac{1}{2}x-\frac{1}{2}x^2)^2 < 1-x$;

(iii) $1-\frac{1}{2}x-\frac{1}{2}x^2 < \sqrt{1-x} < 1-\frac{1}{2}x$;

(iv) the error in taking $\sqrt{1-x}=1-\frac{1}{2}x$ is less than $\frac{1}{2}x^2$.

10. (i) Find by the square root rule the value of $\sqrt{1-x}$ as far as terms involving x^3.

(ii) What are the first four terms of the binomial expansion for $\sqrt{1-x} \equiv (1-x)^{\frac{1}{2}}$, assuming the Binomial Theorem?

(iii) If $x=0{\cdot}01$, what is the approximate error in taking $\sqrt{1-x}=1-\frac{1}{2}x$?

11. (i) Find by the square root rule the value of $\sqrt{1+x}$ as far as terms involving x^3.

(ii) If x is a positive fraction less than 1, what are the first four terms of the binomial expansion for

$$\sqrt{1+x} \equiv (1+x)^{\frac{1}{2}}$$

assuming the Binomial Theorem?

(iii) If $x=0{\cdot}1$, what is the approximate error in taking $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x$?

12. (i) Prove that $\dfrac{1}{x+h} \equiv \dfrac{1}{x}\left(1+\dfrac{h}{x}\right)^{-1}$.

(ii) If h is small compared to x, find an approximation for $\dfrac{1}{x+h}$.

(iii) Express approximately $\dfrac{1}{1{\cdot}02}$ as a decimal.

13. (i) Prove that $\dfrac{1}{\sqrt{x^2+h}} \equiv \dfrac{1}{x}\left(1+\dfrac{h}{x^2}\right)^{-\frac{1}{2}}$.

(ii) If h is small compared to x, find an approximation for $\dfrac{1}{\sqrt{x^2+h}}$.

(iii) Express approximately $\dfrac{1}{\sqrt{4{\cdot}01}}$ as a decimal.

(iv) Express approximately $\dfrac{1}{\sqrt{9{\cdot}05}}$ as a decimal.

SUMMARY OF RESULTS.

I. If x is small compared with 1,

$$(1+x)^n \simeq 1+nx,$$
$$(1-x)^n \simeq 1-nx.$$

In particular,
$$\frac{1}{1+x} \equiv (1+x)^{-1} \simeq 1-x,$$
$$\frac{1}{1-x} \equiv (1-x)^{-1} \simeq 1+x,$$
$$\sqrt{1+x} \equiv (1+x)^{\frac{1}{2}} \simeq 1+\tfrac{1}{2}x,$$
$$\sqrt{1-x} \equiv (1-x)^{\frac{1}{2}} \simeq 1-\tfrac{1}{2}x.$$

II. If h is small compared with x,
$$(x+h)^n \equiv x^n\left(1+\frac{h}{x}\right)^n \simeq x^n\left(1+\frac{nh}{x}\right).$$

Example I. If h is small, find an approximation for $\frac{(1-h)^2}{\sqrt{1+h}}$, and deduce an approximate value of $\frac{(0{\cdot}98)^2}{\sqrt{1{\cdot}02}}$.
$$\begin{aligned}\frac{(1-h)^2}{\sqrt{1+h}} &= (1-h)^2 \,.\, (1+h)^{-\frac{1}{2}}\\ &\simeq (1-2h)(1-\tfrac{1}{2}h) \simeq 1-2h-\tfrac{1}{2}h\\ &\simeq 1-\frac{5h}{2}.\end{aligned}$$

If $h=0{\cdot}02$, we have
$$\begin{aligned}\frac{(0{\cdot}98)^2}{\sqrt{1{\cdot}02}} &\simeq 1-\frac{5\times 0{\cdot}02}{2} \quad\text{or}\quad 1-0{\cdot}05\\ &\simeq 0{\cdot}95.\end{aligned}$$

Example II. The cost per hour of an electric lamp varies as $\frac{V^2}{R}$, where V is the voltage and R is the resistance. Find the approximate percentage change in the cost if the voltage is increased by 2 per cent. and the resistance is decreased by 3 per cent.

If the original cost per hour is $\frac{kV^2}{R}$ pence, the new cost is
$$\begin{aligned}\frac{k(V\times 1{\cdot}02)^2}{R\times 0{\cdot}97}\text{ pence} &= \frac{kV^2}{R}(1+0{\cdot}02)^2(1-0{\cdot}03)^{-1}\\ &\simeq \frac{kV^2}{R}(1+0{\cdot}04)(1+0{\cdot}03)\\ &\simeq \frac{kV^2}{R}(1+0{\cdot}04+0{\cdot}03) \quad\text{or}\quad \frac{1{\cdot}07kV^2}{R}\text{ pence :}\end{aligned}$$

$\therefore$ the increase of cost is $\frac{0{\cdot}07kV^2}{R}$ pence :

$\therefore$ the increase is 7 per cent.

EXERCISE XXI. d.

1. (i) Simplify $(1+a)(1+b)-(1+a+b)$.

(ii) What is the error in taking $1{\cdot}01 \times 1{\cdot}02$ equal to

$$1 + {\cdot}01 + {\cdot}02 \quad \text{or} \quad 1{\cdot}03\ ?$$

(iii) Write down the approximate value of $1{\cdot}03 \times 1{\cdot}05$.

2. (i) If h is small, what is an approximate value of $(1+h)^3$?

(ii) Write down an approximate value of $(1{\cdot}02)^3$.

3. Each linear dimension of a rectangular block of steel expands $0{\cdot}000{,}011$ of itself for every 1° C. rise of temperature. In what ratio does the volume increase if the temperature rises (i) 1° C., (ii) 3° C. ?

4. If an error of 1 per cent. is made in measuring the length and breadth of a field, what may be the approximate error per cent. in the calculated area ?

5. A man estimates the dimensions of a room, and makes an error of 1 per cent. in the length, 2 per cent. in the breadth and 3 per cent. in the height ; what may be the approximate error per cent. in the calculated volume ?

6. (i) If the diameter of a sphere increases by 1 per cent., what is the approximate increase per cent. (a) in its volume, (b) in its surface ?

(ii) If the superficial area of a spherical soap-bubble increases by 1 per cent., what is the approximate increase per cent. in its volume ?

7. If h is small,

(i) show that $\dfrac{1-h}{1+h} \simeq 1-2h$;

(ii) find an approximation for $\dfrac{1+h}{1-h}$, and hence for $\dfrac{1{\cdot}01}{0{\cdot}99}$.

8. Find approximations for

(i) $(1{\cdot}002)^5$; (ii) $(0{\cdot}98)^3$; (iii) $\dfrac{1}{0{\cdot}997}$;

(iv) $\sqrt{1{\cdot}008}$; (v) $\sqrt[3]{1{\cdot}015}$; (vi) $\dfrac{1}{5{\cdot}003}$.

9. If h is small, find approximate values of

(i) $\dfrac{(1+3h)(1+4h)}{1+5h}$; (ii) $\dfrac{(1+3h)(1+4h)}{1+12h}$.

What is the approximate value of $\dfrac{1{\cdot}03 \times 1{\cdot}04}{1{\cdot}12}$?

10. The time of oscillation of a pendulum of length l feet is approximately $1{\cdot}11\sqrt{l}$ seconds. Find approximately the increase in the time when the length is altered from (i) 4 feet to 4·1 feet, (ii) l feet to $(l+h)$ feet, where h is small compared with l.

11. A given mass of gas at constant temperature obeys the law $pv=\text{constant}$, where p is the pressure in lb. per sq. in. and v is its volume in cu. in.

(i) If the pressure is increased by 2 per cent., what is the approximate change in the volume?

(ii) If the pressure alters from p to $p+p_1$, what is approximately the change of volume if p_1 is small compared with p?

12. (i) What are the roots of $x^5-x^3=0$?

(ii) One root of $x^5-x^3=0{\cdot}01$ is nearly equal to 1; by putting $x=1+h$ and treating h as small, find a closer approximation to this root.

13. If h is small compared with x, find approximate values for

$$\text{(i)}\ \frac{(x+h)^3-x^3}{h}; \qquad \text{(ii)}\ \frac{(x+h)^{10}-x^{10}}{h};$$

$$\text{(iii)}\ \frac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}; \qquad \text{(iv)}\ \frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

14. The maximum deflection of a beam of length l feet and depth d inches, supported at its ends, is $\dfrac{kl^4}{d^2}$ in., where k is a constant; in what ratio is the deflection altered if l increases from 20 to 20·1 and d increases from 4 to 4·1?

15. A given mass of gas, absolute temperature $T°$, volume v cu. in. at pressure p lb. per sq. in., obeys the law $\dfrac{pv}{T}=\text{constant}$. Find the percentage change in p if v increases by 2 per cent. and T decreases by 3 per cent.

16. The sides containing the right angle of a right-angled triangle are of lengths x in. and h in.; if h is small compared with x, find approximately by how much the hypotenuse exceeds the longer side.

17. If θ is small, find an approximation for $\dfrac{\theta-\dfrac{\theta^3}{6}}{1-\dfrac{\theta^2}{2}}$.

18. The time of pneumatic transmission through a pipe l yards long of diameter d inches under a pressure of p lb. per sq. inch varies as $\dfrac{l^{\frac{3}{2}}}{d^{\frac{1}{2}}p^{\frac{1}{2}}}$. What is the approximate percentage change in the time if l increases by 1 per cent., d increases by 2 per cent. and p decreases by 1 per cent.?

19. In the Michelson-Morley experiment for the determination of the velocity of the earth relative to the ether, the effect to be observed was measured by $2(l_1+l_2)\left\{\frac{c}{c^2-v^2}-\frac{1}{\sqrt{c^2-v^2}}\right\}$, where v is the velocity required and c the velocity of light. Prove that if $\frac{v}{c}$ is small, this expression $\simeq (l_1+l_2)\cdot\frac{v^2}{c^3}$.

CHAPTER XXII.

EMPIRICAL FORMULAE.

An experiment was made to determine the extension produced in a spiral spring by weights of various amounts attached to its extremity and the results were tabulated as follows :

Length of spring in cm., l	28·3	29·5	31·5	35·6	39·7
Weight attached in gr., W	10	25	50	100	150

These observed results can be shown to obey *approximately* the law $l = 0{\cdot}081W + 27{\cdot}5$.

A law obtained in this way is called an *empirical formula.*

If the result of plotting y against x gives a straight line, y and x are connected by an equation of the form $y = ax + b$ where a, b are constants. The following example shows how these constants are found, if the straight-line graph is given.

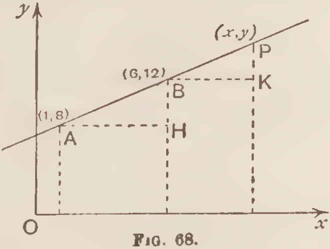


FIG. 68.

Example I. Two points on a straight-line graph which represents the relation between two variables x and y, are found to be $x = 1$, $y = 8$, and $x = 6$, $y = 12$.

Find the equation connecting x and y in the form $y = ax + b$.

A, B are the points (1, 8) and (6, 12).
Denote the coordinates of *any other* point P on the line by (x, y).

Through A, B, P draw lines parallel to Ox, Oy forming the triangles AHB, BKP, which are similar.

$$\therefore \frac{KB}{HA}=\frac{PK}{BH};$$

$$\therefore \frac{x-6}{6-1}=\frac{y-12}{12-8}; \qquad \therefore \frac{x-6}{5}=\frac{y-12}{4};$$

$$\therefore 4x-24=5y-60; \quad \therefore 5y=4x+36; \quad \therefore y=0{\cdot}8x+7{\cdot}2,$$

which is the functional relation required.

By exactly the same method, it may be shown that if (x_1, y_1) and (x_2, y_2) are the coordinates of any two points on a straight line graph, the relation which the graph represents is

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}.$$

Example II. The observations made in an experiment were tabulated as follows:

$x=1$	1·5	2	2·5	3	3·5	4
$y=2{\cdot}50$	2·20	1·92	1·65	1·32	0·94	0·60

Find the most probable relation which expresses y in terms of x.

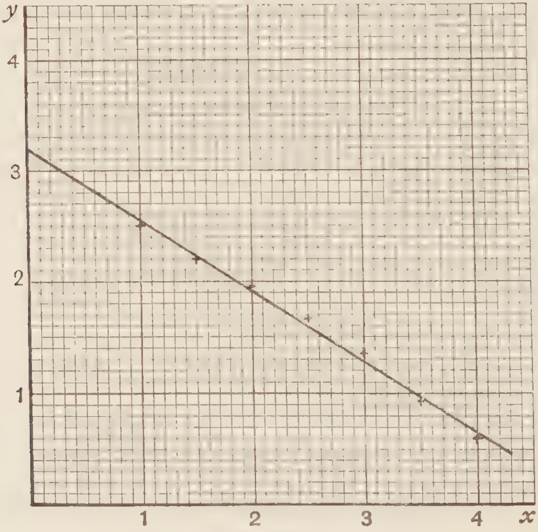


FIG. 69.

First plot the observations. We see that the points obtained, although not on a straight line, do not differ much from the straight. Now use a fine black thread to determine the "best-fit line" and read off the coordinates of two points at opposite ends: *e.g.* $x=0\cdot3$, $y=3$ and $x=3\cdot9$, $y=0\cdot7$.

The relation is therefore

$$\frac{x-3\cdot9}{3\cdot9-0\cdot3}=\frac{y-0\cdot7}{0\cdot7-3}$$

or

$$\frac{x-3\cdot9}{3\cdot6}=\frac{y-0\cdot7}{-2\cdot3};$$

$$\therefore\ y-0\cdot7=-\frac{2\cdot3}{3\cdot6}(x-3\cdot9);$$

$$\therefore\ y=-\frac{2\cdot3}{3\cdot6}x+\frac{2\cdot3\times3\cdot9}{3\cdot6}+0\cdot7;$$

$$\therefore\ y=-0\cdot64x+3\cdot2,$$

where each constant is calculated correct to two significant figures.

Example III. A load of 100 lb. is fastened to one end of a rope which passes round a rough circular peg and measurements are made of the least pull which must be applied at the other end of the rope in order to raise the load slowly. The magnitude of the pull required depends on the amount of the rope in contact with the peg. Suppose the least pull is P lb. when the portion of the rope in contact with the peg is θ right angles. The following observations were made:

θ = right angles	1	2	4	6	8
P lbs. - -	140	181	359	672	1190

Find the most probable relation expressing P in terms of θ.

If we plot these results as they stand, the graph will be a curve. If, however, we plot $\log P$ against θ we see that the points do not diverge far from a straight line. The table is

$\theta=$	1	2	4	6	8
$\log P=$	2·1461	2·2577	2·5551	2·8274	3·0755

These results are plotted in Fig. 70: and a fine thread is then used to discover the "best-fit line." We see that two points on this line are

$$\left.\begin{array}{r}\theta=7\cdot6\\ \log P=3\cdot038\end{array}\right\}\ \text{and}\ \left.\begin{array}{r}\theta=1\cdot3\\ \log P=2\cdot179\end{array}\right\};$$

$\therefore$ the relation is $\dfrac{\log P - 3\cdot038}{3\cdot038 - 2\cdot179} = \dfrac{\theta - 7\cdot6}{7\cdot6 - 1\cdot3}$;

$\therefore$ $\dfrac{\log P - 3\cdot038}{0\cdot859} = \dfrac{\theta - 7\cdot6}{6\cdot3}$ or $\log P = \dfrac{0\cdot859}{6\cdot3}\,\theta - \dfrac{0\cdot859 \times 7\cdot6}{6\cdot3} + 3\cdot038$;

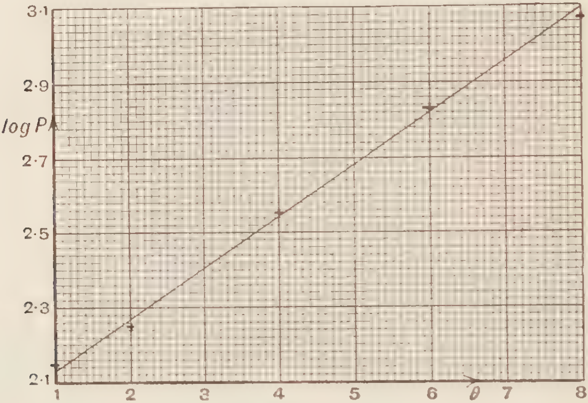


FIG. 70.

$\therefore$ $\log P = 0\cdot136\theta + 2\cdot002$; $\quad\therefore$ $P = 10^{0\cdot136\theta + 2\cdot002}$;

$\therefore$ $P = 10^{2\cdot002} \times 10^{0\cdot136\theta}$; $\quad\therefore$ $P = 100 \times 10^{0\cdot136\theta}$.

This approximate formula is the required relation.

EXERCISE XXII. a.

1. The following observations were made:

$x = 1$	2	4	5	7
$y = 0\cdot95$	2·8	5·8	7·7	10·7

Find the best fit equation for y in terms of x.

2. The length of a spiral spring is measured when different weights are suspended from its end; the results, when tabulated, are as follows:

Weight in gr., W	10	15	30	50	75
Length in cm., l -	22·1	24·1	30·2	38·4	48·7

Find the best fit equation for l in terms of W.

3. The following results were obtained with a screw jack:

Load in lb., W -	10	20	30	50	75
Effort in lb., P -	3·28	4·51	5·72	8·11	11·12

Find the best fit equation for P in terms of W.

4. If $y=\frac{100}{\sqrt{100+x}}$, make a table showing corresponding values of x and y between $x=10$ and $x=50$, and find a relation of the form $y=a+bx$, which best fits this function in this range. What is the error when $x=44$?

5. The following observations were made:

x	1	2	3	4	5
y	8	13	18	28·5	34

It is believed that they obey a linear law. In which observation is the error probably greatest ? Ignoring this, what is the best fit equation for y in terms of x ? And how would the table then run ?

6. The pressure P lb. per sq. foot exerted by a wind of v miles an hour is measured in the following cases:

v	5	10	15	20	25	30
P	0·12	0·51	1·15	1·88	2·95	4·58

Plot P against v^2, and hence find the probable relation expressing P in terms of v.

7. In Giffard's Injector, the following values are given for the delivery of Q gallons of water per hour under a pressure of 60 lb. per sq. inch for a throat diameter D inches.

D	0·1	0·15	0·2	0·25	0·3
Q	300	730	1200	2000	2700

Plot D against $\sqrt{Q}$, and hence find the probable relation expressing Q in terms of D

8. The following table gives the resistance R lb. to a train of weight 100 tons running at V miles an hour:

V	10	20	30	40	50	60
R	680	980	1380	2050	2880	3800

Plot R against V^2, and then express R in terms of V.

9. A body of density d, weighing 1 gr. in air (when brass weights are used), weighs 1 gr. $+k$ mgr. in a vacuum. The following observations were recorded:

$d=0{\cdot}5$	1	3	5	6
$k=2{\cdot}26$	1·06	0·26	0·10	0·06

Plot k against $\frac{1}{d}$, and express k in terms of d.

10. A marble rolls down a groove, at the side of which are markers which are so adjusted as to point the position of the marble at intervals of one second. The following observations were made:

Time in seconds, t	1	2	3	4	5
Distance rolled in inches, s	4·1	16·3	36·8	65·4	102·1

Find the probable relation expressing s in terms of t.

11. The following results were recorded in an optical experiment:

x	10	11	12	15	20	30
y	39·9	33·1	28·9	22·4	18·5	15·7

Plot $\frac{1}{x}$ against $\frac{1}{y}$, and find the best fit equation expressing y in terms of x.

12. The following list of British amateur running records is taken from *Whitaker's Almanack*:

Distance in yd., d	150	200	220	440	880	1000
Time in sec., t	14·6	19·4	21·8	48·4	112·2	132·4

Plot $\log t$ against $\log d$, and hence find a relation expressing t in terms of d.

Professor Perry has pointed out that all forms of race records (men and animals) conform to a law of the same kind.

13. The following observations were made for the pressure and volume of saturated steam:

Volume, cu. ft. per lb. water, v	297	173	82·4	55·1	21·3
Pressure, lb. per sq. in., p	1·13	2·02	4·42	6·77	18·7

Plot $\log p$ against $\log v$, and hence express p in terms of v.

14. The following census returns for England and Wales, correct to 4 significant figures, are taken from *Whitaker's Almanack*:

Year	1851	1861	1871	1881	1891	1901	1911
Population in millions	17·93	20·07	22·71	25·97	29·00	32·53	36·07

If the population in the year $(1851 + 10t)$ is P millions, show that an approximate formula of the type $P = a \times 10^{bt}$ exists, and find it. What, according to your formula, was the population in 1841, and what will it be in 1921?

15. In an experiment to determine the relation between the emissive power at a given temperature of a black body for the wave length for which the power is a maximum and that temperature, the following results were recorded by *Lummer* and *Pringsheim*: (See Preston's *Theory of Heat.*)

Absolute temp., T	621·2	723	908·5	1094·5	1259	1460·4
Emissive power, E	2·026	4·28	13·66	34·0	68·8	145·0
Wave-length, λ	4·73	4·08	3·28	2·71	2·35	2·04

Plot $\log E$ against $\log T$, and express E in terms of T.

16. With the data of Ex. 15, plot $\log \lambda$ against $\log T$, and express λ in terms of T.

17. In a radiation experiment, the following records were made:

Absolute temp., T	373·1	492·5	723	810	1378	1497
Deviation of galvanometer, δ	156	638	3320	5150	44700	61600

Plot δ against T^4, and express δ in terms of T. [Stefan's Law.]

18. The following table gives the age to which a woman expects to live at various ages :

Present age, x	10	15	20	30	35
Age probably reached, y	62·0	62·6	63·4	65·4	66·5
Present age, x	40	50	55	60	
Age probably reached, y	67·7	70·6	72·1	73·9	

Show that y can be expressed in terms of x by a relation of the form $y = 61 + ax + bx^2$ approximately ; find a and b. To what age would a woman 25 years old expect to live ?

$$\left[\text{Plot } \frac{y-61}{x} \text{ against } x.\right]$$

19. The following table for the specific heat of water at various temperatures is due to *Prof. Callendar* :

Temperature centigrade, $60+t$	60°	70°	80°	90°
Specific heat, S	1·0	1·0016	1·0033	1·0053
Temperature centigrade, $60+t$	100°	140°	180°	
Specific heat, S	1·0074	1·0176	1·0308	

Show that S can be expressed in terms of t by a relation of the form $S = 1 + at + bt^2$ approximately. Find a, b. What is the probable value of S at 160° C. ?

20. For a slope of 1 in 100, the velocity of flow in a full pipe, diameter d inches, is v feet per sec., where v, d are connected as follows :

d	4	10	15	30	48	60
v	0·56	1·2	1·6	2·7	3·7	4·3

It is thought that a relation of the form $v = \dfrac{ad}{1 + b\sqrt{d}}$ exists, where a, b are constants. What method of plotting would you adopt to try this ? Find a, b.

SUMMARY OF RESULTS.

(i) If the graph obtained by plotting y against x is, or approximates to, a straight line, and if (x_1, y_1), (x_2, y_2) are two points on it, preferably at opposite ends, then y can be expressed as a function of x by the equation

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}.$$

(ii) If the graph obtained by plotting y against x is a curve, it may be possible to obtain a straight line by plotting

(*a*) $\log y$ against $\log x$,
(*b*) $\log y$ against x,
(*c*) y against $\log x$,
(*d*) y against $\frac{1}{x}$, etc.

(iii) Where the data are experimental, coefficients in the functional relation should be expressed as decimals, and must not be given to more significant figures than the data or the method justify.

CHAPTER XXIII.

NOMOGRAPHY.

LINE CHARTS.

Example I. Construct a line chart connecting inches and centimetres, and read off from it, (i) the number of centimetres corresponding to 2·5 inches, (ii) the number of inches corresponding to 7 cm.

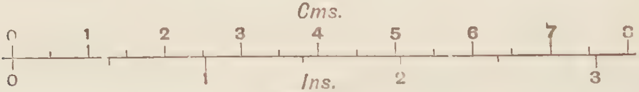


FIG. 71.

Draw a straight line; take an initial point O near the left end of the line; graduate the upper side of the line in cm. and the lower side in inches, each scale beginning at O.

The result is the required line chart.

Opposite the 2·5 graduation on the lower (inch) scale can be read off the number 6·3 on the upper (cm.) scale, so that $2 \cdot 5'' \equiv 6 \cdot 3$ cm. approx.

Opposite the 7 cm. graduation on the upper (cm.) scale can be read off the number 2·75 on the lower (inch) scale, so that 7 cm. $= 2 \cdot 75''$ approx.

Example II. Construct a line chart connecting numbers with their logarithms.

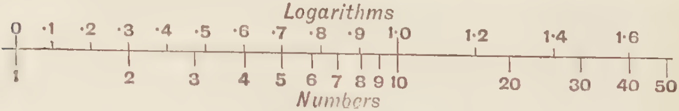


FIG. 72.

This chart embodies the principle of the slide rule. It is simply constructed by graduating the upper side of the line uniformly on a scale of (say) 5 cm. to a unit and marking the graduations 1, 2, 3, 10, 20, ... etc. on the lower side at distances from the *O* graduation on the upper side equal to 5 log 1, 5 log 2, 5 log 3, 5 log 10, 5 log 20 cm., etc., the values of the logarithms being taken from the tables.

EXERCISE XXIII. a.

1. Construct a simple line chart connecting numbers and their squares from 1 to 9.

Read off from your chart (i) the square of 7·2;

(ii) $\sqrt{72}$;

(iii) $(2{\cdot}6)^2$.

2. In the figure of Example I. above, read off the number of inches in 4 cm., and the number of cm. in 1·5 inches.

3. Construct a simple line chart connecting numbers and their cube roots from 1000 to 2000.

Read off from your chart (i) $\sqrt[3]{1420}$;

(ii) the cube of **112**.

4. Construct a simple line chart connecting the circumferences of circles and their diameters for diameters from 1″ to 10″.

Read off from your chart (i) the circumference of a circle whose diameter is 7·5″; (ii) the diameter of a circle whose circumference is 25″.

5. Construct, using sine tables, a simple line chart connecting the length of a chord, and the angle it subtends at the centre of a circle of radius 10 cm., for angles from 0 to 60°.

6. Construct a simple line chart connecting x and y, where $y = 3{\cdot}2x + 2{\cdot}6$ for $0 < x < 5$. Read off the value of x when $y = 10$.

7. Construct a simple line chart for reducing marks which run from 250 to 500 to the scale 0 to 100. To what does 340 reduce?

We will now illustrate another method of obtaining, graphically, a large number of approximate numerical results from a simple formula. Under the ordinary system of uniform graduation of the axes, a graph could be drawn to illustrate any formula involving one dependent variable. The graph would usually be a smooth curve. By adopting a system of non-uniform graduation, it is often possible to obtain a straight-line graph, and this method is particularly useful when approximate numerical results are required for a large number of closely-related formulae. Under the ordinary system a "family" of curves would be required; by adopting the method explained below, a "family" of straight lines can be drawn with very little more trouble than is required to draw one.

In cases of non-uniform graduation, the axes must be graduated at sufficiently small intervals to allow proportional differences to be read without introducing undue error.

Example III. If a current of C ampères fuses a wire of diameter d mm., $C^2 = k \cdot d^3$, where k is a constant depending on the material. For copper wire, $k = 6400$; for lead, $k = 1250$, etc.

Represent the functional relation by a straight-line chart for the range $d = 0$ to $d = 1$.

Read off from the chart (i) the current which will fuse a copper wire of diameter 0·8 mm., (ii) the current which will fuse a lead wire of diameter 0·95 mm.

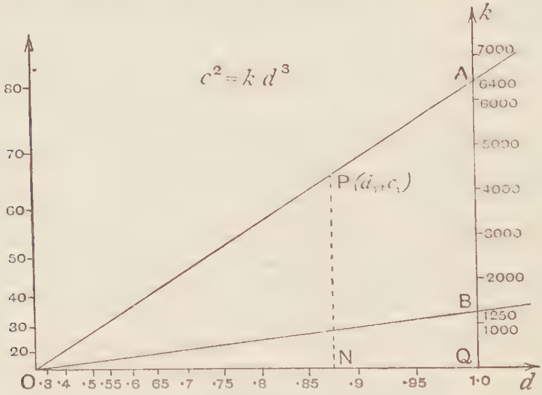


FIG. 73.

The d-axis (horizontal) must be graduated so that distances from O vary as the cube of the values of d: the C-axis (vertical) must be graduated so that distances from O vary as the square of the values of C. [Notice that both axes are thus simple line charts as in Examples I. and II., with the uniform set of graduations omitted.]

Choose any convenient unit for the d-axis, say 10 cm. Measure 10 cm. along the d-axis from O and mark the point Q, so obtained, 1. [*N.B.*—Fig. 73 is printed on a reduced scale.]

To obtain the graduation 0·9 on the d-axis, measure off

$$10 \times (0{\cdot}9)^3 = 7{\cdot}29 \text{ cm.}$$

from O, and so on.

For the C-axis, take any convenient unit, say $\frac{1}{1000}$ cm. To obtain 80 on the C-axis, measure off $\frac{1}{1000} \times (80)^2 = 6{\cdot}4$ cm. from O, and so on.

For copper, $k = 6400$;

$$\therefore \text{ when } d = 1, \; C^2 = 6400 \text{ or } C = 80.$$

Mark the point A, $d = 1$, $C = 80$, and join it to O.

This is the required line chart for copper wire. For, if P be any point on this line with coordinates (d_1, C_1), as taken from the graduations marked, the actual distances of the point P from these axes are not d_1 and C_1, but $10 \times d_1^3$ and $\frac{1}{1000} \times C_1^2$ cm.

Draw PN perpendicular to OQ, then, by the similar triangles PNO, AQO,

$$\frac{ON}{OQ} = \frac{PN}{AQ},$$

$$\frac{10 \, . \, d_1^3}{10} = \frac{\frac{1}{1000} C_1^2}{6{\cdot}4};$$

$$\therefore \; C_1^2 = 6400 d_1^3,$$

so that the reading (d_1, C_1) from the scales marked does give a solution of the formula $C^2 = 6400 d^3$.

We can now give the first numerical result required, for the point on the line OA with coordinate 0·8 on the d-scale has coordinate 57 approximately on the C-scale.

Therefore a current of 57 ampères will fuse a copper wire of diameter 0·8 mm.

To obtain the line chart for lead, we have $k = 1250$;

$$\therefore \text{ when } d = 1, \quad C = \sqrt{1250} = 35{\cdot}4.$$

Mark the point $d = 1$, $C = 35{\cdot}4$, and join it to O.

This is the required line chart for lead, and the reader can easily see how this would be proved.

[To obtain the point whose coordinates are $d = 1$, $C = \sqrt{1250}$, it is only necessary, with the scale chosen for C, to measure upwards from Q a distance

$$QB = \tfrac{1}{1000}(\sqrt{1250})^2 = \tfrac{1}{1000} \times 1250 = 1{\cdot}25 \text{ cm.};$$

so there is no need to find the square root.]

Any member of the family $C^2 = k \, . \, d^3$ can be drawn as follows:

Draw a line, parallel to the C-axis, through Q and call it the k-axis.

Graduate this k-axis by measuring off distances 1 cm., 2 cm., 3 cm., ... and marking the points obtained 1000, 2000, 3000, ...

To obtain the line chart for any value of k, say 4000, join the point graduated 4000 on the k-axis to O and this is the line required.

Example IV. Represent the relation $C^2=6400d^3$ by a straight-line chart, using logarithmic graduation. Read off the value of C when $d=0{\cdot}8$.

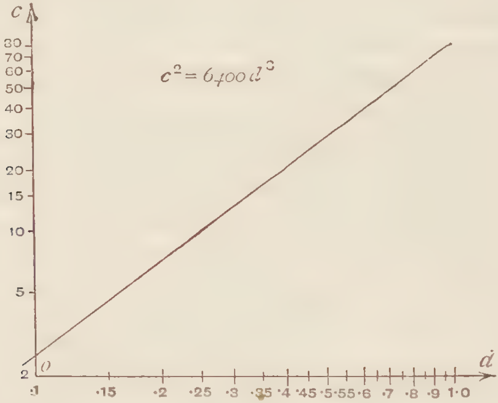


FIG. 74.

Since
$$\log C^2=\log(6400d^3)$$
$$=\log 6400+\log d^3,$$
$$\therefore\ 2\log C=3{\cdot}806+3\log d.$$

This is of the first degree in $\log C$ and $\log d$.

Therefore if the C-axis and the d-axis are graduated logarithmically, the graph of this relation will be a straight line.

We will represent d for the range 0·1 to 1, so that the greatest value of C is 80.

On the d-axis mark $d=0{\cdot}1$ and $d=1$ at a convenient distance apart, say 10 cm. [*N.B.*—Fig. 74 is printed on a reduced scale.] Call the point O at which $d=0{\cdot}1$.

Then 10 cm. on this scale represents $\log 1-\log(0{\cdot}1)=0-(\bar{1})=1$.

To find the position of any other graduation, say $d=0{\cdot}6$, we have $\log 0{\cdot}6-\log 0{\cdot}1=\bar{1}{\cdot}7782-(\bar{1})=0{\cdot}7782$.

$\therefore$ distance of "0·6" graduation from 0 is $10\times 0{\cdot}7782=7{\cdot}78$ cm.

Similarly, on the C-axis, put $C=2$ (say) at 0 and $C=20$ at a convenient distance, 5 cm., from it.

Then on this scale 5 cm. represents $\log 20-\log 2=\log 10=1$.

For any other value of C, say $C=80$,

$$\log 80-\log 2=\log 40=1{\cdot}6021\,;$$

∴ distance of the "80" graduation from 0 is

$$5 \times 1{\cdot}6021 = 8{\cdot}01 \text{ cm., and so on.}$$

Both scales have now been graduated in the same way as the simple line chart in Example II., with the uniform graduations omitted.

Now the relation $C^2 = 6400d^3$ is satisfied by the pairs of values

$$\left.\begin{array}{l} d=1, \\ C=80 \end{array}\right\} \text{ and } \left.\begin{array}{l} d=0{\cdot}25, \\ C=10. \end{array}\right\}$$

Mark these points ; the line joining them is the required chart. Any numerical result required can then be read from the chart. For example, when $d=0{\cdot}8$, $c=57$.

It is easy to see that, with this system of graduation, the family

$$C^2 = k \,.\, d^3, \quad \text{or} \quad 2 \log C = \log k + 3 \log d,$$

is represented by a system of parallel lines. And it is easy to draw a k-axis, as in Example III., so that any member of the system can be drawn at once.

Note.—The labour of graduation can be lightened by using specially prepared logarithmically scaled paper (which can be bought) or by taking the distances from a slide rule.

It may also save time to remember that the distance between two graduations depends only on their ratio ; *e.g.* the distance from 2 to 6 must be the same as from 7 to 21.

Thus if 0·1 and 0·2 are marked on the scale, the positions of 0·4, 0·8, 1·6, etc., can be marked without any further calculation.

EXERCISE XXIII. b.

1. From the figure of Example III. above, read off

(i) the current which will fuse a lead wire of diameter 0·95 mm., and of 0·85 mm. ;

(ii) the current which will fuse a copper wire of diameter 0·95 mm., and of 0·85 mm.

[Only a rough result can be expected, as not many secondary graduations are marked.]

2. From the figure of Example IV. above, read off

(i) the current which will fuse a copper wire of diameter 0·25 mm., and of 0·45 mm ;

(ii) the thickness of copper wire required if a current of 30 ampères does not fuse the wire.

3. The distance between the graduations 1 and 2 on a slide rule is 2·5 inches; find the distances between the graduations, (i) 2 and 4; (ii) 2 and 8; (iii) 4 and 16; (iv) 2 and 5; (v) 2 and 10.

4. The distance between the graduations 1 and 10 on a slide rule is 10 inches; what is the distance between the graduations (i) 2 and 3; (ii) 1 and 5?

5. A scale Ox is graduated so that the distances from O vary as the squares of the values of x. The distance of the graduation "5" from O (the zero graduation) is 10 cms.; what is the distance between the graduations "1" and "2"?

6. A scale Ox is graduated so that, for a uniform graduation of Oy, the graph of $y=\sqrt{x}$ is a straight line. The graduation $x=10$ is 5 inches from O; find the distances of the graduations $x=1$ and $x=5$ from O.

7. Take two axes Ox, Oy graduated uniformly, unit for each 1 cm., and draw the graphs of $y=2x$ and $y=3x$. Now graduate the x-axis so that the distances from O of the graduations 1, 2, 3, 4, ... are 1, 4, 9, 16, ... cm. What functional relations do these graphs now represent?

8. The time, t seconds, taken by a marble to roll s feet down a certain plane satisfies the equation $s=3{\cdot}4t^2$. Represent this by a straight-line chart for $0<t<4$. Show how you would represent the family $s=kt^2$. Read off the value of s if $t=2{\cdot}9$.

9. The diameter d inches of Cornish boilers is connected with the indicated horse-power H by the relation $d=11{\cdot}4\sqrt{H}$.
Represent this by a straight-line chart for $9<H<49$.
What is d when $H=30$?
Show how you would represent the family $d=k\sqrt{H}$.

10. Represent by straight-line charts on the same diagram:

(i) $y=3\sqrt{x}+1$; (ii) $y=4\sqrt{x}+1$; (iii) $y=3\sqrt{x}+5$;
(iv) $y=4\sqrt{x}+5$; for $0<x<25$.

11. Represent by a straight-line chart the formula

$$y=\frac{1{\cdot}2}{x}-1{\cdot}4 \text{ for } 1<x<5.$$

What is y when $x=3{\cdot}5$?

In Ex. 12-16, use logarithmic graduation.

12. Represent by a straight-line chart the formula

$$p\,.\,v^{1\cdot 06}=475 \text{ for } 10<p<50.$$

What is v when $p=38$?

13. Represent by a straight-line chart the formula

$$t=0{\cdot}28d^{1\cdot 6} \text{ for } 200<d<500.$$

What is t when $d=320$?

14. Thomson's formula for the discharge of water through a notch is $D^2 = 3{\cdot}91H^5$; draw a straight-line chart for

$$0{\cdot}1 \leqslant H \leqslant 0{\cdot}6.$$

What is D when $H = 0{\cdot}44$?

15. Represent by a straight-line chart the relation

$$y = (2{\cdot}7)^x \text{ for } 0 < x < 2.$$

What is y when $x = 1{\cdot}2$?

16. Represent by a straight-line chart the relation

$$xy^2 = 25 \text{ for } 1 < y < 5.$$

What is y when $x = 12$?

NOMOGRAMS—THREE OR MORE VARIABLES.

The methods already illustrated are unsuitable for formulae containing more than two variables.

Different types of methods have been evolved for such formulae, and are illustrated in the examples which follow; the diagrams constructed for this object are called *Nomograms.*

Example V. To construct a nomogram for the function $ax + by$, where x, y are variable and a, b are given constants.

Let $$z = ax + by.$$

The problem is to construct a nomogram from which we can read off the values of z corresponding to any values of x and y.

We shall measure values of x, y and z along three *parallel* lines, which we shall call the x-axis, y-axis and z-axis.

Suppose 1″ on x-axis $= \xi$ units,
1″ on y-axis $= \eta$ units.

Draw a line EGF making

$$EG = \lambda \, . \, b \, . \, \eta \text{ in.},$$
$$GF = \lambda \, . \, a \, . \, \xi \text{ in.},$$

where λ is any convenient number.

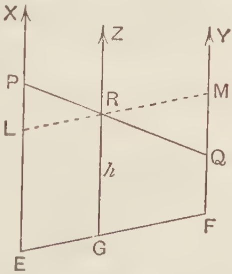


FIG. 75.

Through E, G, F draw any three parallel lines, EX, GZ, FY; these are to be the three axes.

Along EX, FY, measure off lengths EP, FQ to represent any definite values (x_1, y_1) of x and y respectively. Then

$$EP = \frac{x_1}{\xi} \text{ inches,}$$

$$FQ = \frac{y_1}{\eta} \text{ inches.}$$

Join PQ, cutting GZ in R; let $GR = h$ inches.

Draw LRM parallel to EGF cutting EX, FY at L, M.

By similar triangles and parallels,

$$\frac{PL}{QM} = \frac{LR}{RM} = \frac{EG}{GF};$$

$$\therefore \frac{\frac{x_1}{\xi} - h}{h - \frac{y_1}{\eta}} = \frac{\lambda . b\eta}{\lambda . a\xi} = \frac{b\eta}{a\xi};$$

$$\therefore a\xi\left(\frac{x_1}{\xi} - h\right) = b\eta\left(h - \frac{y_1}{\eta}\right);$$

$$\therefore ax_1 - a\xi h = b\eta h - by_1;$$

$$\therefore ax_1 + by_1 = h(a\xi + b\eta);$$

$$\therefore z_1 = h(a\xi + b\eta).$$

Choose the unit on the z-axis so that 1″ on z-axis $= a\xi + b\eta$ units.

Then z equals the number represented by GR on this scale. And this is true, however x_1 and y_1 vary, for the position of GZ and the scale on GZ do not depend on x_1 and y_1.

This result may be expressed in the following rule:

To chart $z = ax + by$.

(1) Choose convenient scales for the x-axis and y-axis.
Suppose 1″ on x-axis $= \xi$ units,
1″ on y-axis $= \eta$ units.

(2) Take three points E, G, F on a line so that

$$EG : GF = b\eta : a\xi.$$

(3) Draw three parallel lines EX, GZ, FY.

(4) GZ is the z-axis, and its scale is 1″ $= a\xi + b\eta$ units.

(5) If a straight edge crosses the parallel lines cutting off lengths representing x and y from EX, FY, then it cuts off a length representing $ax + by$ or z from GZ.

The z-axis is called the *support* line or *Reference* line.

Example VI. For screw propellers, if the horse-power is increased by x per cent. and the number of revolutions per minute by y per cent., the diameter of the propeller should be increased by z per cent., where $z=0{\cdot}2x-0{\cdot}6y$ (adapted from Doig's formula): chart z for variations of x from 0 to 10 per cent. and of y from 0 to 15 per cent.

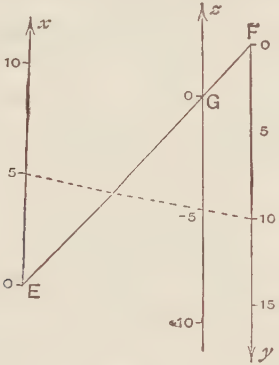


FIG. 76.

Take for unit on x-axis $1''=4$ units, and on y-axis $1''=5$ units. [*N.B.*—Figure 76 is printed on a reduced scale.]

$$\frac{EG}{GF}=\frac{5\times 0{\cdot}6}{4\times 0{\cdot}2}=\frac{15}{4}=\frac{3}{0{\cdot}8};\ \text{draw } EG=3'',\ GF=0{\cdot}8''.$$

Unit on z-axis is $1''=0{\cdot}2\times 4+0{\cdot}6\times 5$ units $=3{\cdot}8$ units.

Owing to the minus sign in $0{\cdot}2x-0{\cdot}6y$, the y-axis must be graduated in the opposite sense to that of the x-axis.

With these data, the chart can now be constructed.

The dotted line on the chart gives the reading $z=-5$ corresponding to $x=5$, $y=10$, and any number of numerical results can be similarly read from the chart.

NOMOGRAMS CAN BE CONSTRUCTED FOR ANY NUMBER OF VARIABLES.

Example VII. Chart $z=\frac{1}{2}u+3v+9x+5y+4$ for the ranges $10<u<40$, $10<v<15$, $0<x<5$, $0<y<10$.

Let $p=\frac{1}{2}u+3v$ and $q=9x+5y$;

$$\therefore\ z=p+q+4.$$

We first construct the "p" and "q" supports, and from them construct the "z" support.

Take as units :

1" on u-axis	v-axis	x-axis	y-axis
10 units	2 units	2 units	4 units

[*N.B.*—Figure 77 is printed on a reduced scale.]

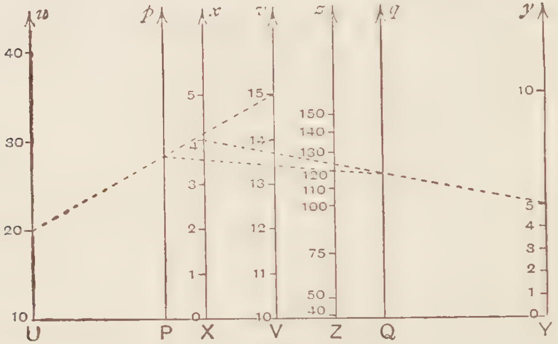


FIG. 77.

(1) For "p" support, $\frac{UP}{PV}=\frac{3\times 2}{\frac{1}{2}\times 10}=\frac{6}{5}=\frac{1\cdot5}{1\cdot25}$; take $UP=1\cdot5''$, $PV=1\cdot25''$.
Scale on p-axis is $1''=3\times 2+\frac{1}{2}\times 10=11$ units.

(2) For "q" support, $\frac{XQ}{QY}=\frac{5\times 4}{9\times 2}=\frac{20}{18}=\frac{2}{1\cdot8}$; take $XQ=2''$, $QY=1\cdot8''$.
Scale on q-axis is $1''=5\times 4+9\times 2=38$ units.

(3) For "z" support, $\frac{PZ}{ZQ}=\frac{1\times 38}{1\times 11}=\frac{38}{11}$; divide PQ at Z in ratio 38 : 11.
Scale on z-axis is $1''=1\times 38+1\times 11=49$ units.

(4) By substitution, when $u=10$, $v=10$, $x=0$, $y=0$, we have $z=5+30+4=39$.

$\therefore$ the graduation at the point Z in the figure is 39.

$\therefore$ we can now graduate the z-axis.

(5) The dotted lines in the figure show how z is obtained for $u=20$, $v=15$, $x=4$, $y=5$; giving $z=120$.

Note.—(*a*) In actually drawing the figure, it is best to try to arrange so that axes are not too near together.

Take the points in the following order:

(i) U, P, V at the proper distances apart.

(ii) Q. The line PQ has to be divided in ratio 38 : 11; so choose PQ a convenient length, say $\frac{1}{2}$ of $\frac{(38+11)}{10} = 2{\cdot}45$ inches. Then Z is half of $3{\cdot}8 = 1{\cdot}9''$ from P.

(iii) X, Y, Z.

(*b*) There is no need to graduate the support lines p, q.

(*c*) It is usually best to graduate the z-axis by taking special values, and so obtaining two points of known graduations on it.

EXERCISE XXIII. c.

1. It has been suggested that the Chancellor of the Exchequer should regulate the tax on motor-cars by the following formula. For a car of H horse-power and weight W cwt., the tax should be £T, where $T = 0{\cdot}4H + 0{\cdot}8W$. Construct a nomogram to show the amount of the tax for cars of 10 to 40 H.P. and of weights 10 cwt. to 30 cwt. Proceed as follows: (i) Draw two parallel lines AH, BW 6 cm. apart and 15 cm. long and a line perpendicular to them cutting them at A, B; (ii) on AB, take a point C, so that $AC = 4$ cm., and through C draw CT parallel to AH; (iii) the lines AH, BW, CT drawn in the same sense form the H, W, T axes; take as scale for the H-axis, 1 cm. = 2 H.P., and as scale for the W-axis, 1 cm. = 2 cwt.; then the scale for the T-axis is

$$1 \text{ cm.} = 2 \times 0{\cdot}4 + 2 \times 0{\cdot}8 = £2{\cdot}4;$$

(iv) when $H = 10$ and $W = 10$, we have $T = 12$; mark at A, B, C the graduations 10, 10, 12 respectively. Graduate the H-axis and W-axis according to the units given in (iii). Measure off 5 cm. along CT, and mark the point K so obtained $12 + 5 \times 2{\cdot}4 = £24$. Divide CK into 12 equal parts, so that each graduation now represents £1.

This is the required nomogram. Use it to read off the tax on a car of

(*a*) 20 H.P. weighing 18 cwt.,
(*b*) 25 H.P. weighing 30 cwt.,
(*c*) 30 H.P. weighing 30 cwt.

2. Draw a line ACB so that $AC = 4$ cm., $CB = 3$ cm.; draw three parallel lines AX, CZ, BY (in the same sense); regarding this as a nomogram, take A, B, C as zero graduations on the X, Y, Z axis; as units take 1 cm. = 5 units on X-axis, 1 cm. = 4 units on Y-axis, 1 cm. = 10 units on Z-axis. Graduate the axes, and read off the values of Z for (i) $X = 20$, $Y = 20$; (ii) $X = 15$, $Y = 30$; (iii) $X = 25$, $Y = 12$. Calculate the function Z in terms of X and Y.

Next suppose that the graduations at A, B, C are respectively 10, 20, 30, but that the units remain the same, read off the new answers to (i), (ii), (iii), and calculate the new function Z in terms of X and Y.

3. Work out all the results asked for in Question 2, taking BY in the opposite sense to that of AX and CZ.

4. Draw the nomogram for the relation $z = 4{\cdot}3x + 8{\cdot}6y$ for $0 < x < 5$, $0 < y < 5$, and read off the values of z for (i) $x = 2$, $y = 4$; (ii) $x = 3$, $y = 1$.

5. For snapped rivets, $l = d + \frac{5t}{4} + \frac{1}{32}$, where l = length of rivet before closing, d = diameter of hole, t = thickness of plate, all in inches. Construct a nomogram for $0 < d < 1\frac{1}{4}$, graduated to $\frac{1}{8}$ inch, and $0 < t < 4$ graduated to $\frac{1}{2}$ inch. Read off the values of l for (i) $d = \frac{3}{4}$, $t = 1\frac{1}{2}$; (ii) $d = 1\frac{1}{8}$, $t = 2\frac{1}{2}$.

6. Draw three lines OU, OV, OF, so that $\angle UOF = 60° = \angle FOV$. Call these the u-axis, the v-axis, the f-axis, and graduate each on a scale 1 cm. equals 1 unit. Regarding this as a nomogram, read off the values of f corresponding to (i) $u = 4$, $v = 4$; (ii) $u = 10$, $v = 6$.

If any straight line cuts OU, OV, OF at A, B, C, it can be proved that $\frac{1}{OA} + \frac{1}{OB} = \frac{1}{OC}$. Use this to write down the relation represented by the above nomogram.

By producing VO to T and taking OT as the t-axis, construct a nomogram for $\frac{1}{z} = \frac{1}{u} + \frac{1}{v} + \frac{1}{t}$.

7. Draw the nomogram for the relation $z = 5{\cdot}1x - 3{\cdot}4y + 20$ for $10 < x < 20$, $0 < y < 10$, and read off the values of z if (i) $x = 12$, $y = 6$; (ii) $x = 18$, $y = 9$.

8. Draw the nomogram for the relation $z = 3x - 2y + 5t$ for $0 < x$ or y or $t < 10$, and read off the values of z if (i) $x = 1$, $y = 3$, $t = 2$; (ii) $x = 3{\cdot}5$, $y = 7{\cdot}5$, $t = 4{\cdot}5$.

9. Draw the nomogram for the relation $z = 2x + y - 3t - 4w$ for $50 < x$ or $y < 100$, $10 < t$ or $w < 30$, and read off the values of z if (i) $x = 60$, $y = 80$, $t = 20$, $w = 25$; (ii) $x = 75$, $y = 65$, $t = 25$, $w = 28$.

10. Draw a line $CABZ$ such that $CA = 3$ cm., $AB = 5$ cm.; draw two lines AX, BY *perpendicular* to AB in *opposite* senses. Regarding this as a nomogram, take C, A, B as the zero graduations for the Z, X, Y axes respectively; as units take 1 cm. = 2 units on X-axis, 1 cm. = 3 units on Y-axis, 1 cm. = 10 units on Z-axis. Graduate the axes and read off the values of Z for (i) $X = 8$, $Y = 12$; (ii) $X = 6$, $Y = 20$; (iii) $X = -5$, $Y = 15$. Calculate the function Z in terms of X and Y.

Suppose next that the graduations at A, B for AX, BY are 10, 40 respectively, calculate the new function Z in terms of X and Y.

11. Draw two parallel lines XAW, ZBY in the same sense 6 cm. apart and a line ABT perpendicular to them cutting them at A, B. Regarding this as a nomogram and AX, BY BZ, AW as the x-axis, y-axis, z-axis, w-axis with zero graduations at A, B and ABT as a support line, graduate each axis on a scale 1 cm. equals 1 unit. Suppose the straight edge joining any two graduations x, y cuts the support line AT at K and the straight edge joining any graduation z to K cuts AW at w, then w depends on x, y, z. (i) What function is w of x, y, z? (ii) Read off from the nomogram the values of w corresponding to $x=4$, $y=3$, $z=5$. (iii) Use the nomogram to read off the values of $\frac{3 \cdot 5 \times 2 \cdot 7}{4 \cdot 2}$ and $\frac{6 \cdot 3 \times 5 \cdot 4}{7 \cdot 1}$.

USE OF NON-UNIFORM GRADUATION.

In the nomograms so far considered, the graduations of the axes have been uniform. By using non-uniform systems of graduation, the method can be extended to more complicated formulae. For example, the formula $z = 3x^2 + 2y^2$ could be charted by graduating the axes so that the distance of any graduation marked "x" is at a distance up the scale proportional to x^2, etc., and then proceeding as in Example VI. We add one further example to illustrate this idea, the data of which are due to Professor Perry.

Example VIII. For the Thomson turbine, if P is the total horse-power of the waterfall, H the height of the fall in feet, n the number of revolutions per minute, $n = 22 \cdot 75 H^{1 \cdot 25} \times P^{-0 \cdot 5}$. Construct a nomogram for n, for $50 < H < 200$ and $50 < P < 100$.

We have $\log n = 1 \cdot 357 + 1 \cdot 25 \log H - 0 \cdot 5 \log P$.

[The number 1·357 in the formula merely affects the position of the origin on the n scale: and this we shall fix by taking a special case.]

$\log H$ varies from $\log 50$ to $\log 200$ or $1 \cdot 699$ to $2 \cdot 301$,

$\log P$ varies from $\log 50$ to $\log 100$ or $1 \cdot 699$ to 2.

For $\log H$ axis, take $1'' = 0 \cdot 15$ unit.

For $\log P$ axis, take $1'' = 0 \cdot 1$ unit.

If any line cuts the $\log H$, $\log P$, $\log n$ axes at E, F, G,

$$\frac{EG}{GF} = \frac{0 \cdot 5 \times 0 \cdot 1}{1 \cdot 25 \times 0 \cdot 15} = \frac{0 \cdot 1}{0 \cdot 375} = \frac{1}{3 \cdot 75}.$$

Scale for log n axis, $1'' = 0 \cdot 5 \times 0 \cdot 1 + 1 \cdot 25 \times 0 \cdot 15 = 0 \cdot 2375$ units or 1 unit $= 4 \cdot 210''$.

The log P axis must be graduated in the opposite sense to that of the log H axis.

On the log H axis, the distance between (say) the 50 and 100 graduation is $(\log 100 - \log 50) \div 0{\cdot}15$ in. $= \dfrac{0{\cdot}301}{0{\cdot}15} = 2{\cdot}007''$: in this way we can graduate the log H axis, and similarly the log P axis. [*N.B.*—Figure 78 is printed on a reduced scale.]

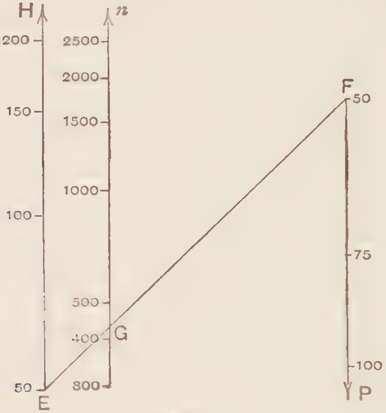


FIG. 78.

By calculation, when $H = 50$ and $P = 50$, we have $n = 427{\cdot}6$. Join these two points on the H and P scale; where the join cuts the "n" support, we mark the graduation 427·6, and graduate the scale upwards from this point.

EXERCISE XXIII. d.

1. If a car of H horse-power and weight W cwt. is taxed £T, a possible way of choosing T would be to use the formula $T = \sqrt[4]{HW^3}$. Construct a nomogram to show the amount of the tax for cars of 10 to 40 H.P. and of weights 10 cwt. to 30 cwt. Proceed as follows: (i) Write down $\log T$ in terms of $\log H$ and $\log W$. (ii) Draw a line ACB so that $AC = 3''$, $CB = 1''$, and draw three lines AH, CT, BW perpendicular to AB and on the same side of it; these lines form the H, T, W axes, which will be graduated logarithmically. (iii) Take as unit both for $\log H$ and $\log W$, 1 inch $= 0{\cdot}1$ unit; then the unit for $\log T$ is

$$1 \text{ inch} = \tfrac{1}{4} \times 0{\cdot}1 + \tfrac{3}{4} \times 0{\cdot}1 = 0{\cdot}1 \text{ unit.}$$

(iv) Mark at A the graduation 10 H.P. and at B the graduation 10 cwt. What is T when $H=10$, $W=10$? What is the graduation at C? (v) $\log 20 - \log 10 = 0{\cdot}301$; but 1 inch $= 0{\cdot}1$ unit; therefore the 20 graduation is $\frac{0{\cdot}301}{0{\cdot}1}$ or 3·01 inches from A. Graduate in this way the three axes. (vi) From the nomogram read off the tax on a car of

(a) 20 H.P. weighing 20 cwt.,
(b) 20 H.P. weighing 30 cwt.,
(c) 15 H.P. weighing 25 cwt.

2. The horse-power H transmitted by each inch width of a belt $\frac{1}{4}$ inch thick is given by $H = 0{\cdot}0036dn$, where d = diameter of pulley in feet, n = number of revolutions per minute. Construct a nomogram for $100 < n < 300$, $1 < d < 10$, and read off from it (i) the value of H if $n = 250$, $d = 6$; (ii) the value of n if $H = 5{\cdot}5$, $d = 8$.

3. Represent by a nomogram Edwards' torsional resistance formula $d^3 = \frac{m}{0{\cdot}196f}$ for $10 < m < 50$, $4 < f < 7$, and read off (i) the value of d if $m = 35$, $f = 5{\cdot}5$, (ii) the value of f if $m = 20$, $d = 2{\cdot}7$.

4. Represent by a nomogram the formula $z = 2xyt^2$ for $1 < x$ or y or $t < 3$, and read off the values of z if (i) $x = 1{\cdot}5$, $y = 2{\cdot}5$, $t = 2{\cdot}1$; (ii) $x = 2{\cdot}5$, $y = 1{\cdot}8$, $t = 1{\cdot}4$.

5. The discharge of gas in pipes is given by $Q = 20000\sqrt{\frac{D^5H}{L}}$, where Q cu. ft. of gas are discharged per hour through a pipe of diameter D inches and length L yards under a pressure equivalent to a head H inches of water. Construct a nomogram for

$$80 < L < 100, \quad 1 < D < 5, \quad 0{\cdot}5 < H < 1$$

Read off the values of Q if (i) $L = 85$, $D = 2$, $H = 0{\cdot}8$; (ii) $L = 92$, $D = 3{\cdot}5$, $H = 0{\cdot}65$.

6. [*D'Ocagne's nomogram for any quadratic equation.*]

To construct a nomogram for solving the quadratic $t^2 - at - b = 0$, proceed as follows:

(i) Make an accurate drawing of the graph of $y = \frac{x^2}{1-x}$ from $x = 0$ to 0·9 (see Fig. 79), taking the unit on the x-axis as 5″ and the unit on the y-axis as $\frac{1}{2}$″.

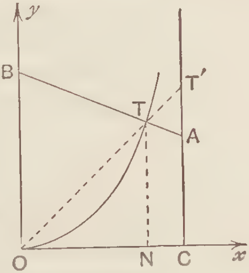


FIG. 79.

(ii) Take C on Ox at unit distance from O (*i.e.* 5″), and draw CA parallel to Oy; draw any line cutting Oy, CA and the curve at B, A, T, and let OT produced meet CA at T'. Draw TN perpendicular to OC.

(iii) Let $ON = x$ units, $NT = y$ units, $OB = b$ units, $CA = a$ units, and let $\frac{y}{x} = t$.

Prove that (1) $t = \frac{x}{1-x}$, $x = \frac{t}{1+t}$, $y = \frac{t^2}{1+t}$,

(2) $\frac{y-a}{1-x} = \frac{b-a}{1}$,

(3) $t^2 - at - b = 0$,

(4) $CT' = \frac{y}{x} = t$.

(iv) Regarding the figure as a nomogram, take OB, CA as the b-axis and a-axis, and graduate them in the same way as Oy, viz. $\frac{1}{2}$ inch equals 1 unit.

To graduate the curve, join O to any point on CA, *e.g.* to $a = 2$, and mark the point where this line meets the curve 2. This means that for this point on the curve $t = \frac{y}{x} = CT' = 2$.

In this way, the different values of t are marked all along the curve.

(v) Use the nomogram to solve $t^2 - 2t - 3 = 0$. Join the points $a = 2$, $b = 3$, and read off the value of t where this line cuts the curve.

(vi) Use the nomogram to find one root of

(1) $t^2 - 2t - 5 = 0$,
(2) $t^2 + 2t - 3 = 0$,
(3) $t^2 + 2t - 5 = 0$.

What inference about the roots of the equation

$$t^2 + 2t + 2 = 0$$

can be drawn from the nomogram?

(vii) The sum of the roots of $t^2 - at - b = 0$ is a; use this fact to write down the second root in the equations given in (v) and (vi).

(viii) How could you use the nomogram to solve the equation

$$x^2 - 23x - 485 = 0\ ?$$

In the preceding pages, it has been impossible to do more than give a brief introductory account: those who wish to pursue the subject further should consult either Brodetsky's *First Course in Nomography*, or d'Ocagne's *Traité de Nomographie*, or one of the other books mentioned in the Introduction.

CHAPTER XXIV.

FURTHER DEVELOPMENTS FOR THE SPECIALIST.

THEORY OF QUADRATICS.

[*Note.*—It is assumed in this section that the letters a, b, c, p, q, r, stand for real numbers, unless otherwise stated.]

If α, β are the roots of $ax^2+bx+c=0$, then [see Part I. p. 153]

$$ax^2+bx+c \equiv a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \equiv a(x-\alpha)(x-\beta)$$
$$\equiv a[x^2-x(\alpha+\beta)+\alpha\beta];$$

$$\therefore \text{ the sum of the roots} = \alpha+\beta = -\frac{b}{a},$$

$$\text{the product of the roots} = \alpha\beta = \frac{c}{a}.$$

[These two results should be learnt by heart.]

By formula, the solution of the quadratic is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Hence (i) if b^2-4ac is a perfect square, the roots are rational and unequal;

(ii) if $b^2-4ac>0$, but not a perfect square, the roots are real, irrational and unequal;

(iii) if $b^2-4ac=0$, the roots are real and equal;

(iv) if $b^2-4ac<0$, the roots are imaginary.

The expression b^2-4ac is called the *discriminant* of the quadratic function ax^2+bx+c. Its sign determines the nature of the factors of this function; and the factors are rational if the discriminant is a perfect square.

Example I. For what range of values of x is the function $5+7x-6x^2$ positive? Find also its greatest value.

(i) $$5+7x-6x^2 \equiv (1+2x)(5-3x) \equiv 2(\tfrac{1}{2}+x)(-3)(x-\tfrac{5}{3})$$
$$\equiv -6(x-\tfrac{5}{3})(x+\tfrac{1}{2}).$$

The function is positive if $(x-\frac{5}{3})(x+\frac{1}{2})$ is negative.
If $x > \frac{5}{3}$, each factor is positive;

$\therefore$ the product is positive.

If $\frac{5}{3} > x > -\frac{1}{2}$, the first factor is negative, the second is positive;

$\therefore$ the product is negative.

If $-\frac{1}{2} > x$, each factor is negative;

$\therefore$ the product is positive;

$\therefore$ the range of values is $\frac{5}{3} > x > -\frac{1}{2}$.

(ii) $$5+7x-6x^2 \equiv 5-6\left(-\frac{7x}{6}+x^2\right)$$
$$\equiv 5-6\left(\frac{7}{12}-x\right)^2+\frac{49}{24}$$
$$=7\tfrac{1}{24}-6\left(\frac{7}{12}-x\right)^2.$$

Now the least value of $\left(\frac{7}{12}-x\right)^2$ is zero.

$\therefore$ the greatest value of $5+7x-6x^2$ is $7\frac{1}{24}$, and the function has this value when $x=\frac{7}{12}$.

Example II. Prove that the roots of

$$(1+a^2)x^2-2(1+ab)x+1+b^2=0$$

are imaginary.

The discriminant $=4(1+ab)^2-4(1+a^2)(1+b^2)$
$$=4[1+2ab+a^2b^2-1-a^2-b^2-a^2b^2]$$
$$=-4(a^2-2ab+b^2)=-4(a-b)^2,$$

which is negative;

$\therefore$ the roots are imaginary.

Example III. If α, β are the roots of $ax^2+bx+c=0$, find (i) the value of $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}$, (ii) the equation whose roots are 2α, 2β.

(i) $\alpha+\beta=-\dfrac{b}{a}$; $\alpha\beta=\dfrac{c}{a}$.

$$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)[(\alpha+\beta)^2-3\alpha\beta]}{\alpha\beta}$$

$$=\frac{\dfrac{-b}{a}\left[\dfrac{b^2}{a^2}-\dfrac{3c}{a}\right]}{\dfrac{c}{a}}=\frac{-b(b^2-3ac)}{a^2c}.$$

(ii) Put $y=2x$, so that when $x=\alpha$, $y=2\alpha$;

$$\therefore\ x=\frac{y}{2};\quad \therefore\ a\left(\frac{y}{2}\right)^2+b\left(\frac{y}{2}\right)+c=0;$$

$$\therefore\ ay^2+2by+4c=0$$

is the required equation.

EXERCISE XXIV. a.

1. State whether the roots of the following equations are real, coincident or imaginary; and if real, what sign. The actual roots are *not* required.

(i) $x^2-3x-5=0$; (ii) $x^2+97x-173=0$;
(iii) $x^2-2x+5=0$; (iv) $x^2-14x+49=0$;
(v) $3x^2+105x+1=0$; (vi) $3x^2+x+105=0$.

2. Form the equations whose roots are

(i) $2,\ -1$; (ii) $0,\ \frac{3}{4}$; (iii) $a+b,\ a-b$; (iv) $a,\ b,\ c$;
(v) $\pm\sqrt{2}$; (vi) $\sqrt{2}+1,\ \sqrt{2}-1$; (vii) $\dfrac{a}{b},\ -\dfrac{c}{d}$.

3. Express as simply as possible the roots of $ax^2+2bx+c=0$.

4. Find the greatest value of a for which $x^2-5x+a=0$ has real roots.

5. What must be added to $4x^2-5x$ to make the result a perfect square? Find the least value of this function.

6. If x is real, find (i) the greatest value of $1+2x-x^2$, (ii) the least value of $5x^2-x$.

7. If x is real, prove that x^2-3x+3 is positive.

8. For what range of values of x is (i) $3-2x-8x^2$ positive? (ii) $2x^2-x-3$ negative?

9. How is x limited if x^2+3x-4 is positive?

10. For what range of values are the following functions positive?

(i) $(x+1)(x-1)(x-3)$; (ii) $(2x-3)(x+2)(3x-1)$;
(iii) $(1-x)(x+4)(2x+1)$; (iv) $(x-1)(x-3)(x+1)(x+3)$.

11. State the values of x for which $\dfrac{(x-2)(x+3)}{x+1}$ is (i) positive, (ii) negative.

12. State the values of x for which $\dfrac{(1-2x)(x+4)}{(x+3)(1-3x)}$ is (i) positive, (ii) negative.

13. (i) One root of $5x^2-17x+k=0$ is $1{\cdot}4$; what is the other?
(ii) One root of $5x^2-kx-39=0$ is $2{\cdot}6$; what is the other?

14. One root of $5x^2+9x+c=0$ is double the other; find them: also find c.

15. Find b if $3x^2+bx+4=0$ has equal roots.

16. One root of $x^2-px+12=0$ is three times the other; find p.

17. If α, β are roots of $x^2+qx+r=0$, find the condition that

(i) $\alpha=2\beta$; (ii) $\alpha=-\beta$; (iii) $\dfrac{\alpha}{\beta}=\dfrac{2}{3}$;
(iv) $\alpha=\dfrac{1}{\beta}$; (v) $\alpha=0$; (vi) $\alpha-\beta=1$.

18. Prove that the roots of $x^2-2ax-b^2=0$ are real.

19. (i) Prove that $(a+b-c)^2+4c(a+b)$ is a perfect square.
(ii) Prove that $(a+b)x^2+x(a+b-c)-c=0$ has rational roots.

20. Prove that $a(x^2-y^2)+bxy$ has real factors.

21. Prove that the roots of $(a^2+b^2)x^2-2(a+b)x+2=0$ are imaginary.

22. Prove that the roots of $(a+4b)x-2(a+b)x+a-2b=0$ are rational.

23. Prove that the roots of $(x-a)(x-b)=c^2$ are real.

24. Prove that the roots of $ax^2+bx+c=0$ are rational if $a+b+c=0$.

25. Prove that $(a^2+bc)x^2+(a-b)(a-c)x-a(b+c)=0$ has rational roots.

26. Prove that $\dfrac{1}{x+a}+\dfrac{1}{x+b}+\dfrac{1}{x+c}=0$ has real roots.

27. Find k if $(3x-2y)^2+k(x-y)(x-2y)$ is a perfect square.

28. Find the sum of the squares of the roots of $3x^2-7x-5=0$.

29. If α, β are the roots of $2x^2 - 5x = 8$, find the values of

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$; (ii) $\alpha^2 - \alpha\beta + \beta^2$; (iii) $(1+\alpha^2)(1+\beta^2)$.

30. Find the conditions that the roots of $x^2 + bx + c = 0$ are respectively $\frac{2}{3}$ of the roots of $x^2 + qx + r = 0$.

31. Find the condition that the roots of $x^2 + bx + c = 0$ differ by $5b$.

32. (i) Form the equation whose roots are ten times the roots of $x^2 - 4x = 9$.

(ii) Form the equation whose roots are 1 less than the roots of $2x^2 - x = 4$.

33. (i) Form the equation whose roots are the squares of the roots of $2x^2 - 4x = 3$.

(ii) Form an equation whose roots exceed by 2 the roots of $x^2 - 4x = 7$.

34. What is the connection between the roots of $ax^2 + bx + c = 0$ and those of $ax^2 + bpx + cp^2 = 0$?

35. Find a condition, independent of p, that the roots of $ax^2 + bx + c = 0$ are (i) p and $p+1$; (ii) $1 + \frac{1}{p}$ and $1 + \frac{1}{p+1}$.

36. What can you say about the nature of the roots of

$$x^2 + 2(a+b)x + 2a^2 + 2b^2 = 0 \text{ ?}$$

37. Prove that $3x^2 + 7xy + 2y^2 - 2x + y - 1$ has rational factors.

38. The sum of the roots of $x^2 - (a+7)x + 2(2a+1) = 0$ is half their product. Find a.

39. Find a if the equation $x^2 - 4(1-2a)x + 4 - 5a = 0$ has equal roots.

40. What is the condition that $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have one and only one common root ?

41. What is the condition that $x^2 + xy - 2y^2$ and $ax^2 + 2hxy + by^2$ have a common factor ?

42. Given that $3x^2 - 29x - 44 = 0$ and $3x^2 + 73x + 92 = 0$ have a common root, find it.

43. What is the condition that $x^2 + bx + c = 0$ and $x^2 - cx - b = 0$ have a common root ?

44. If $a + b + c = 0$, prove that $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root.

45. What is the condition that $ax^2 + ax + c$ is positive for all real values of x ?

46. If x, y, z are real, prove that $x^2+y^2+z^2-xy-yz-zx$ cannot be negative. What follows if the expression is zero?

47. If x is real, what limits are there to the value of $x+\frac{1}{x}$?

48. (i) Solve the equation $\frac{x}{x^2+1}=k$.

(ii) What is the condition that the roots of this equation are real?

(iii) If x is real, what is the least value of $\frac{x}{x^2+1}$?

49. If x is real, find the limits within which $\frac{x+4}{(x+1)(x-8)}$ cannot lie.

50. If x is real, find the limits within which $\frac{3x^2+2}{2x^2-2x+1}$ must lie.

51. If x is real, prove that $\frac{x+1}{x^2-4}$ is capable of any real value.

52. If α, β are the roots of $x^2+qx+r=0$, form the equation whose roots are (i) $\frac{1}{\alpha}$, $\frac{1}{\beta}$; (ii) $\alpha^2-\alpha\beta$, $\beta^2-\alpha\beta$.

53. Find the condition that the roots of $x^2+qx+r=0$ differ by the same amount as those of $x^2+bx+c=0$.

54. Can real values of x and y be found such that

(i) $x^2+y^2-4x+2y+5=0$;
(ii) $x^2+y^2-6y+10=0$?

55. (i) Write down the equation whose roots are α, β, γ, and express it in the form $x^3+px^2+qx+r=0$.

(ii) If α, β, γ are the roots of $ax^3+bx^2+cx+d=0$, express $\alpha+\beta+\gamma$ and $\alpha\beta+\beta\gamma+\gamma\alpha$ and $\alpha\beta\gamma$ in terms of a, b, c, d.

56. If α, β, γ, δ are the roots of $x^4+px^3+qx^2+rx+s=0$, express p, q, r, s in terms of α, β, γ, δ.

57. What can you say about the roots of the equations

(i) $x^3+qx+r=0$;
(ii) $x^3+px^2+qx+p=0$;
(iii) $x^4+qx^2+s=0$;
(iv) $x^3+px^2+qx=0$.

58. The roots of $x^3+px^2+qx+r=0$ are 3, 1, -2; what is the value of q?

59. Two of the roots of $x^3+qx+r=0$ are 4, 3; what is the other root?

ALGEBRAIC FORM.

The ideas included under this heading are of too general a nature for illustrative examples.

EXERCISE XXIV. b.

1. Use the identity $x^3+y^3 \equiv (x+y)(x^2-xy+y^2)$ to factorise
(i) x^3-y^3; (ii) $(a+b)^3+(b+c)^3$; (iii) $(a+b)^3-(b+c)^3$.

2. Use the identity $a^4+a^2b^2+b^4 \equiv (a^2+ab+b^2)(a^2-ab+b^2)$ to factorise
(i) a^4+4a^2+16; (ii) $(x+y)^4+(x^2-y^2)^2+(x-y)^4$.

3. Given $(x+y)^4 \equiv x^4+4x^3y+6x^2y^2+4xy^3+y^4$, write down the expansion of $(2x-3y)^4$.

4. Given that the square root of $1+x(x+1)(x+2)(x+3)$ is $1+3x+x^2$, write down the square root of

(i) $1+(x+1)(x+2)(x+3)(x+4)$;
(ii) $y^4+(x+y)(x+2y)(x+3y)(x+4y)$.

5. Prove that

$$(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3 \equiv (y+z)^3+3(y+z)^2x+3(y+z)x^2+x^3.$$

6. If $x+y+z=0$, prove that

(i) $x^2+y^2+z^2=2(x^2-yz)$;
(ii) $y^2-zx=z^2-xy$.

7. Given that the area of the triangle whose sides are of lengths x, y, z is

$$\tfrac{1}{4}\sqrt{\{(x+y+z)(x+y-z)(y+z-x)(z+x-y)\}},$$

find the area of the triangle whose sides are $x+y$, $y+z$, $z+x$.

8. Given that $x=1$ is a root of $x^6+2x^4+2x^2=5$, write down a second root.

9. Given that $x=\frac{2}{3}$ is a root of $6x^3-7x^2-7x+6=0$, write down a second root.

10. Given $x+2y$ is a factor of $4x^4-17x^2y^2+4y^4$, write down the other three factors.

11. Given $\dfrac{3}{(x-4)(x-1)} \equiv \dfrac{1}{x-4}-\dfrac{1}{x-1}$, express $\dfrac{3}{(x+4)(x+1)}$ as the difference of two fractions.

12. Given that $(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x)$, factorise $x^3y^3+y^3z^3+z^3x^3-(xy+yz+zx)^3$.

13. Factorise $(a+3b)^2-5(a+3b)(7a-11b)-14(7a-11b)^2$.

14. Given that $a+b+c$ is a factor of

$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2,$$

write down the other three factors.

15. What is the effect of writing px for x and py for y in the expressions

$$\text{(i)}\ \frac{ax^n-by^n}{xy(cx^{n-2}-dy^{n-2})};\quad \text{(ii)}\ \frac{\sqrt[3]{x^3+y^3}}{xy}?$$

16. Simplify $\dfrac{2\,.\,4\,.\,6\,.\,8\ldots(2n-2)}{1\,.\,2\,.\,3\,.\,4\ldots n}$.

17. Generalise

$$(x-y)(x+y)(x^2+y^2)(x^4+y^4)(x^8+y^8)\equiv x^{16}-y^{16}.$$

18. Given that $x+y=a$ and $x^2+y^2=b^2$, state with reasons which of the following statements are obviously untrue:

(i) $xy=\frac{1}{2}(a^2-b^2)$; (ii) $\frac{1}{x}+\frac{1}{y}=\frac{2a^2}{a^2-b^2}$;

(iii) $\frac{x}{y}+\frac{y}{x}=\frac{2b^2}{a^2-b^2}$; (iv) $x^3+y^3=\frac{ab}{2}(3b^2-a^2)$.

19. Given that $(ab+xy)^2+(ay-bx)^2\equiv(a^2+x^2)(b^2+y^2)$,

(i) write down the factors of $(ab-xy)^2+(ay+bx)^2$;

(ii) express $(a^2+x^2)(b^2+y^2)(c^2+z^2)$ as the sum of two squares.

20. How many terms are there in the expansion of

$$(a+b)(c+d)(e+f)(g+h)(k+l)?$$

Which of the following terms belong to this expansion (i) *acfhk*, (ii) *aceg*, (iii) *bdefk*, (iv) *bdfgk*, (v) *lgfcb*?

21. Find the coefficient of x^7 in

$$(1-2x+3x^3-x^5+x^9)(1+x^2+2x^3-4x^6).$$

22. Find the coefficient of x^{20} in

$$(1-x^3+x^6-x^9+x^{12}-\ldots)(1+x^5+x^{10}+x^{15}+\ldots).$$

23. Find the coefficient of x^n in

(i) $(1+x^{n-1})(1+2x+3x^2+4x^3+\ldots)$;

(ii) $(1+x)(1+x^2+x^4+x^6+\ldots)$;

(iii) $(1+x^2)(1+x^2+x^4+x^6+\ldots)$;

(iv) $(1-x)(1+x+x^2+x^3+\ldots)$.

24. Find the coefficient of x^{100} in

$$(1+2x^3+3x^6+4x^9+5x^{12}+\ldots)^2.$$

25. Find the coefficient of (i) x^6, (ii) x^n in

$$(1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots).$$

26. What is the coefficient of x^{2n} in

(i) $1+2x+3x^2+4x^3+\dots$;
(ii) $(1-x)(1+2x+3x^2+4x^3+\dots)$;
(iii) $(1-2x+x^2)(1+2x+3x^2+4x^3+\dots)$?

27. What is the coefficient of x^n (n odd or n even) in

(i) $1-x+x^2-x^3+x^4-\dots$;
(ii) $1+x-x^2-x^3+x^4+x^5-x^6-x^7+\dots$?

28. Given that, if n is a positive integer,

$$(1+x)^n \equiv 1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\dots$$
$$+\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3.4\dots r}x^r+\dots,$$

write down the expansion of (i) $(1+x)^5$; (ii) $(a+b)^4$.

29. Assuming the identity in Ex. 28, write down the coefficient of a^4b^3 in $(a+b)^7$.

30. Assuming that the identity in Ex. 28 is true for *all* values of n, provided that $1 > x > -1$, find approximate values for (i) $\sqrt{1+x}$, (ii) $\dfrac{1}{1+x}$, if x is so small that x^3 and higher powers of x can be neglected.

31. If h is so small compared with x that $\left(\dfrac{h}{x}\right)^3$ can be neglected, use the statement in Ex. 30 to find an approximate value for $\sqrt{x+h}$.

32. If $1 > x > -1$, use Ex. 30 to expand (i) $(1+x)^{-2}$; (ii) $(1-x)^{-n}$.

33. It can be proved that, if x, y, z are any positive quantities.

$$\frac{x+y}{2} \nless \sqrt{xy} \quad \text{and} \quad \frac{x+y+z}{3} \nless \sqrt[3]{xyz}.$$

Assuming these results, prove that

(i) $(x+y)(y+z)(z+x) \nless 8xyz$;
(ii) $(x+y+z)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right) \nless 9$.

34. What is the connection between the graphs of

$$\frac{x(x-3)}{(x-4)(x-5)} \quad \text{and} \quad \frac{x(x+3)}{(x+4)(x+5)}?$$

35. Prove that the graphs of

$$\frac{x(x-1)}{(x+1)(x-2)} \quad \text{and} \quad \frac{2}{(x+1)(x-2)}$$

are of the same size and shape.

36. Given that

$$(a^2-pb^2)(c^2-pd^2) \equiv (ac+pbd)^2 - p(ad+bc)^2,$$

express $(a^2-pb^2)(c^2-pd^2)(e^2-pf^2)$ in the form $Y^2 - p \,.\, Z^2$.

37. Given that $x = \dfrac{br-cq}{aq-bp}$ satisfies the equations $ax+by+c=0$, $px+qy+r=0$, write down the value of y given by the equations.

38. Given $ax^2+2hxy+by^2=0$, $lx+my=1$, it can be proved that $x^2(am^2-2hlm+bl^2)+2x(mh-bl)+b=0$; write down an equation in y independent of x.

39. Given that $x = \dfrac{k}{(a-b)(a-c)}$ is the value of x satisfying $x+y+z=0$, $ax+by+cz=0$, $a^2x+b^2y+c^2z=k$, write down the values of y, z.

40. Given that $x = \dfrac{(a+\lambda)(a+\mu)(a+\nu)}{(a-b)(a-c)}$ is the value of x arising from $\dfrac{x}{a+\lambda}+\dfrac{y}{b+\lambda}+\dfrac{z}{c+\lambda} = \dfrac{x}{a+\mu}+\dfrac{y}{b+\mu}+\dfrac{z}{c+\mu} = \dfrac{x}{a+\nu}+\dfrac{y}{b+\nu}+\dfrac{z}{c+\nu} = 1$, write down the values of y, z.

PROPORTION.

Example IV. If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, prove that each fraction is equal to $\dfrac{pa+qc+re}{pb+qd+rf}$ and to $\sqrt[3]{\left(\dfrac{pa^3+qc^2e}{pb^3+qd^2f}\right)}$.

Let $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=k$; $\quad\therefore\ a=bk, \quad c=dk, \quad e=fk$;

$$\therefore\ \frac{pa+qc+re}{pb+qd+rf}=\frac{pbk+qdk+rfk}{pb+qd+rf}=k=\frac{a}{b},$$

and
$$\sqrt[3]{\left(\frac{pa^3+qc^2e}{pb^3+qd^2f}\right)}=\sqrt[3]{\left(\frac{pb^3k^3+qd^2fk^3}{pb^3+qd^2f}\right)}=\sqrt[3]{k^3}=k=\frac{a}{b}.$$

Example V. If $\dfrac{x+y}{4}=\dfrac{y+z}{5}=\dfrac{z+x}{6}$, find $x:y:z$.

$$\text{Each fraction} = \frac{(x+y)+(y+z)-(z+x)}{4+5-6}=\frac{2y}{3}.$$

Similarly, each $=\dfrac{2x}{4+6-5}=\dfrac{2x}{5}$ and $=\dfrac{2z}{5+6-4}=\dfrac{2z}{7}$;

$$\therefore\ \frac{x}{5}=\frac{y}{3}=\frac{z}{7}\quad\text{or}\quad x:y:z=5:3:7.$$

Example VI. If a, b, c, d are in continued proportion prove that

(i) $b=dk^2$, $a=dk^3$, where $\dfrac{c}{d}=k$;

(ii) $\dfrac{a-b}{b-c}=\dfrac{b-c}{c-d}$.

(i) $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$ and $\dfrac{c}{d}=k$;

$$\therefore\ c=dk\ ;\ b=ck=dk\,.\,k=dk^2\ ;$$
$$a=bk=dk^2\,.\,k=dk^3.$$

(ii) $\dfrac{a-b}{b-c}=\dfrac{dk^3-dk^2}{dk^2-dk}=k$ and $\dfrac{b-c}{c-d}=\dfrac{dk^2-dk}{dk-d}=k$;

$$\therefore\ \frac{a-b}{b-c}=\frac{b-c}{c-d}.$$

Or, as follows:

$\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$; $\therefore\ \dfrac{a}{b}=\dfrac{b-c}{c-d}$, by Example IV., p. 288, and $\dfrac{a}{b}=\dfrac{a-b}{b-c}$;

$$\therefore\ \frac{a-b}{b-c}=\frac{b-c}{c-d}.$$

CROSS-MULTIPLICATION.

Example VII.

$$\text{Given}\ \begin{cases} a_1x+b_1y+c_1z=0,\\ a_2x+b_2y+c_2z=0,\ \text{find}\ x:y:z.\end{cases}$$

We have
$$a_1c_2x+b_1c_2y+c_1c_2z=0,$$
$$a_2c_1x+b_2c_1y+c_2c_1z=0\ ;$$
$$\therefore\ x\,(a_1c_2-a_2c_1)+y\,(b_1c_2-b_2c_1)=0\ ;$$
$$\therefore\ x\,(c_1a_2-c_2a_1)=y\,(b_1c_2-b_2c_1)\ ;$$
$$\therefore\ \frac{x}{b_1c_2-b_2c_1}=\frac{y}{c_1a_2-c_2a_1}=\text{similarly}\ \frac{z}{a_1b_2-a_2b_1}.$$

Determinant Notation.

The symbol $\begin{vmatrix} a & b\\ c & d\end{vmatrix}$ is used to represent $ad-bc$, and is called a ***determinant***.

The relation found above for $x:y:z$ can therefore be written

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

EXERCISE XXIV. c.

1. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $\frac{a}{d}=\frac{b^3}{c^3}$.

2. If $\frac{x^2}{p}=\frac{x}{q}=\frac{1}{r}$, prove that $q^2=pr$.

3. If $\frac{x}{y}=\frac{y}{z}$, prove (i) $\frac{x-y}{y-z}=\frac{y}{z}$; (ii) $\frac{x^2+y^2}{y^2+z^2}=\frac{x}{z}$.

4. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{a}$, prove that $\frac{c^2}{d^2}=\frac{e}{b}$.

5. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, state which of the following equations are identities:

(i) $\frac{a+c}{b+d}=\frac{c-e}{d-f}$; (ii) $\frac{a-c}{c-e}=\frac{b+d}{d+f}$; (iii) $\frac{ac}{bd}=\frac{e^2}{f^2}$;

(iv) $\frac{2a+c}{2b+d}=\frac{e^2}{f^2}$; (v) $\frac{3a+c}{3b+e}=\frac{2a-d}{2b-f}$; (vi) $\frac{a^2+7c}{b^2+7d}=\frac{e^2}{f^2}$;

(vii) $\frac{a^3-a^2c}{b^3-b^2d}=\frac{c^3+e^3}{d^3+f^3}$; (viii) $\sqrt{\frac{a^2-c^2}{b^2-d^2}}=\frac{c}{d}$;

(ix) $\frac{b-5f}{a-5e}=\frac{d-b}{c-a}$; (x) $\frac{ac-e^2}{a^2+c^2}=\frac{bd-f^2}{b^2+d^2}$.

6. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that $b+c$ is a mean proportional between $a+b$ and $c+d$.

7. If v, w, x, y are in continued proportion, prove that

$$(v+w+x+y)^2=(v+w)^2+2(w+x)^2+(x+y)^2.$$

8. Solve $\frac{x+y}{14}=\frac{y+z}{17}=\frac{z-t}{8}=\frac{x+t}{11}$, $x+y+z+t=7$.

9. If $\frac{a}{b}>\frac{c}{d}$, and if a, b, c, d are positive, prove that

$$\frac{a}{b}>\frac{a+c}{b+d}>\frac{c}{d}.$$

10. Solve $\frac{x^2-2x+5}{3x^2+4x-1}=\frac{x^2+2x-5}{3x^2-4x+1}$.

11. Given $2x + y + 5z = 0$, $3x + 2y + z = 0$, write down $x : y : z$.

12. (i) Given $3x - y - 2z = 0$, $5x + y - 3z = 0$, write down $x : y : z$.
(ii) Solve the equations $3x - y - 2 = 0$, $5x + y - 3 = 0$.

13. Given $x + by - cz = 0$, $ax - y + z = 0$, write down $x : y : z$.

14. Write down the solution of the equations $ax + by + c = 0$, $px + qy + r = 0$.

15. Write the equations

$$a_1x^2 + 2h_1xy + b_1y^2 = 0, \quad a_2x^2 + 2h_2xy + b_2y^2 = 0$$

in the form

$$\frac{x^2}{\;} = \frac{2xy}{\;} = \frac{y^2}{\;},$$

and hence find a relation independent of x and y.

16. Solve the equations

$$x + 2y + 5z = 0, \quad 5x - 4y + 4z = 0, \quad x^2 - y^2 - z^2 = 3.$$

17. If $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$, $a_3x + b_3y + c_3z = 0$, prove that

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

This is usually written $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

18. If $5x - 2y + z = 3x + y - 4z = 0$, find the value of $\dfrac{xy - z^2}{x^2 + z^2}$.

19. If $\dfrac{a}{x - y} = \dfrac{b}{y - z} = \dfrac{c}{z - x}$, prove that $a + b + c = 0$.

20. What number must be added to each of the numbers 27, 60, 3, 18 so as to make them in proportion ?

21. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, fill up the gaps in $\dfrac{a^4}{b^4} = \dfrac{a^4 + 3c^3e}{\;} = \dfrac{a^4 + 3ac^2e - 5ace^2}{\;}$.

[Notice the different dimensions.]

22. If $\dfrac{a}{b} = \dfrac{c}{d}$, fill up the gap in $\dfrac{a + c}{b + d} = \sqrt[4]{\left(\dfrac{(a^5 - c^5)}{\;}\right)}$.

23. Complete $\dfrac{x^2 + y^2}{a} = \dfrac{xy}{b} = \dfrac{x^2 - y^2}{\;}$.

24 If $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{a}{b}$, express $\dfrac{x + y}{x - y}$ in terms of a, b.

25. If $p^2 + q^2 = 1$, complete $\dfrac{p}{a} = \dfrac{q}{b} = \dfrac{1}{\;}$.

26. If $\dfrac{x}{a} = \dfrac{y}{b}$, prove that $\dfrac{x^2 + y^2}{ax + by} = \dfrac{xy(y - x)}{ay^2 - bx^2}$.

Σ NOTATION.

$f(x, y)$ is called a *homogeneous* function of x and y of *degree* n if $f(kx, ky) \equiv k^n f(x, y)$.

E.g. $x^3 - 7x^2y + 5y^3$ is a homogeneous function of x and y of degree 3, because

$$(kx)^3 - 7(kx)^2(ky) + 5(ky)^3 \equiv k^3(x^3 - 7x^2y + 5y^3).$$

$f(x, y)$ is called a *symmetrical* function of x and y if

$$f(x, y) \equiv f(y, x).$$

E.g. $3x^2 - 7xy + 3y^2 + \frac{2}{5}x + \frac{2}{5}y - 3\frac{1}{4}$ is a symmetrical function of x and y, because the expression remains unaltered when x and y are interchanged.

The Σ notation is a short-hand method of representing the sum of a number of things of the same type.

E.g. if we are using 3 letters a, b, c,

$$\begin{aligned} \Sigma\, 5a &\equiv 5a + 5b + 5c, \\ \Sigma\,(ab) &\equiv ab + bc + ca, \\ \Sigma\,(a^2b) &\equiv a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2. \end{aligned}$$

In an analogous fashion,

$\sum_{1}^{n} r^2$ is used to represent $1^2 + 2^2 + 3^2 + \ldots + n^2$,

and $\sum_{k}^{n} r^2$ is used to represent $k^2 + (k+1)^2 + (k+2)^2 + \ldots + n^2$,

where n, k are positive integers and $n > k$.

Sometimes it is necessary to show what letters are involved in the expression.

Thus $$\sum_{a,\, b,\, c,\, d} (ab) \equiv ab + ac + ad + bc + bd + cd.$$

Example VIII. Prove that

(i) $\sum_{a,\, b,\, c} (a^3) - 3abc \equiv \Sigma\,(a)\,.\,\Sigma(a^2 - bc)$;

(ii) $\Sigma\,(y^3z^3) + 3xyz\,.\,\Sigma\,(x)\,.\,\Sigma\,(yz) - [\Sigma\,(yz)]^3 \equiv 3x^2y^2z^2$, where Σ extends to x, y, z.

(i) $\Sigma(a) \equiv a + b + c$; $\quad \Sigma(a^2 - bc) \equiv a^2 - bc + b^2 - ca + c^2 - ab$
$$\equiv a^2 + b^2 + c^2 - bc - ca - ab.$$

By direct multiplication,

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab) \equiv a^3+b^3+c^3-3abc\,;$$

$$\therefore\ \Sigma(a)\,.\,\Sigma(a^2-bc) \equiv \Sigma(a^3)-3abc.$$

(ii) We have $\Sigma(a^2-bc) \equiv \Sigma(a^2+2bc)-3\Sigma(bc)$

$$\equiv [\Sigma(a)]^2-3\Sigma(bc)\,;$$

$\therefore$ from (i), $\Sigma(a^3)-3abc \equiv \Sigma(a)\,.\,\{[\Sigma(a)]^2-3\Sigma(bc)\}$

$$\equiv [\Sigma(a)]^3-3\Sigma(a)\,.\,\Sigma(bc).$$

Put $a=yz,\ b=zx,\ c=xy$.

Then $\Sigma(y^3z^3)-3x^2y^2z^2 \equiv [\Sigma(yz)]^3-3\Sigma(yz)\,.\,\Sigma(x^2yz)$

$$\equiv [\Sigma(yz)]^3-3xyz\,\Sigma(yz)\,.\,\Sigma(x),$$

which is equivalent to the required result.

EXERCISE XXIV. d.

1. Simplify (i) $\sum\limits_{x,y,z}(x-y)$; (ii) $\sum\limits_{x,y,z}(x^2-y^2)$;

(iii) $\sum\limits_{a,b,c}a(b-c)$; (iv) $\sum\limits_{1}^{n}[(r+1)^2-r^2]$.

Prove that $\sum\limits_{a,b,c}[bc(b-c)]+\Sigma[a(b^2-c^2)]=0$.

2. How many terms are there in the expansion of

(i) $\sum\limits_{a,b,c,d}ab$; (ii) $\sum\limits_{a,b,c}(a)\,.\,\Sigma(ab)$; (iii) $\sum\limits_{a,b,c,d}(a^2b)$; (iv) $\sum\limits_{a,b,c,d}\left(\frac{a}{b}\right)$;

$\left[\Sigma\left(\frac{a}{b}\right)\right.$ and $\Sigma\left(\frac{b}{a}\right)$ represent the same thing.$\left.\right]$

3. Express as simply as possible in the Σ notation:

(i) $[\Sigma(a)]^2-\Sigma(a^2)$; (ii) $\sum\limits_{a,b,c,d}(a+b)^2$;

(iii) $\sum\limits_{x,y,z}(x)\,.\,\Sigma\left(\frac{1}{x}\right)-\Sigma\left(\frac{x}{y}\right)$; (iv) $\sum\limits_{a,b,c,d}(a)\,.\,\Sigma(ab)-\Sigma(a^2b)$;

and simplify (v) $\sum\limits_{1}^{n}\log\left(1+\frac{1}{r}\right)$; (vi) $\sum\limits_{k}^{n}\log\left(1+\frac{1}{r}\right)$.

4. The identity $(r+1)^2-r^2 \equiv 2r+1$ leads at once to

$$\sum_{1}^{n}[(r+1)^2-r^2] \equiv \sum_{1}^{n}(2r+1) \equiv 2\sum_{1}^{n}(r)+\sum_{1}^{n}(1).$$

(i) What is the value of $\sum\limits_{1}^{n}(1)$?

(ii) What does $\sum\limits_{1}^{n}(r)$ represent ?

(iii) Use the identity to prove that $\sum\limits_{1}^{n}(r)=\dfrac{n(n+1)}{2}$.

5. (i) Prove that $\sum_1^n[(r+1)^3-r^3]\equiv\sum_1^n(3r^2+3r+1)$.

(ii) Use this identity and the result of Ex. 4 (iii) to prove that

$$\sum_1^n r^2=\frac{n(n+1)(2n+1)}{6}.$$

6. (i) Prove that $\sum_1^n[(r+1)^4-r^4]\equiv\sum_1^n(4r^3+6r^2+4r+1)$;

(ii) Use Ex. 4, 5 to prove that $\sum_1^n r^3=\left[\frac{n(n+1)}{2}\right]^2$.

7. Given that $\sum_1^n r=\frac{n(n+1)}{2}$, find the value of $\sum_1^{2n}(r-1)$.

8. Given that $\sum_1^n 2^{r-1}=2^n-1$, find the value of $\sum_1^{2n}(2^{r-1}-2)$.

9. (i) Find the value of $\sum_1^n\left(\frac{1}{r}-\frac{1}{r+1}\right)$;

(ii) What series does $\sum_1^n\frac{1}{r(r+1)}$ represent, and what is its sum?

10. Given that $\sum_1^n r^2\equiv\frac{n(n+1)(2n+1)}{6}$, find the value of $\sum_p^q r^2$, where p, q are positive integers, $p<q$.

11. Given that $\sum_1^n r(r+1)\equiv\frac{n(n+1)(n+2)}{3}$, find the value of $\sum_p^q r(r+1)$ where p, q are positive integers, $p<q$.

12. Given that $\sum_1^n\frac{1}{r(r+1)}=\frac{n}{n+1}$, find the value of $\sum_n^{2n}\frac{1}{r(r+1)}$.

13. (i) Use the $\sum$ notation to write down the most general homogeneous symmetrical function of x, y, z of the second degree.

(ii) Find the function if its value is 3 when $x=y=z=1$, and also 3 when $x=y=0$, $z=1$.

14. If $f(x, y, z)\equiv a\sum(x^3)+b\sum(x^2y)$, find a, b, given that

$$f(1, 1, 1)=12 \quad\text{and}\quad f(0, 0, 1)=2.$$

15. (i) Express $f(x, y, z)$ in the $\sum$ notation, if $f(x, y, z)$ is the general symmetrical homogeneous function of the third degree.

(ii) Find the constants if $f(0, 0, 2)=8$; $f(0, 1, 1)=6$; $f(1, 1, 1)=3$.

16. Given $\sum_{x,y,z} x=a$ and $\sum(x^2)=b^2$, express in terms of a, b

(i) $\sum(xy)$; (ii) $\sum(x^3)-3xyz$.

17. Given $\sum_{x,y,z}(x)=a$ and $\sum(xy)=b^2$, express $\sum(x-y)^2$ in terms of a, b.

18. If $\sum_{x,y,z} x=0$, prove that

$$\text{(i)}\ (x+y)(y+z)(z+x)+xyz=0\,;$$
$$\text{(ii)}\ \sum(x^2y)=-3xyz.$$

19. Prove that $\sum_{x,y,z}(y-z)^3=3(x-y)(y-z)(z-x)$.

20. If $f(x, y, z)\equiv\sum(x^2)-\sum(yz)$, prove that

$$f(x+h,\ y+h,\ z+h)\equiv f(x,\ y,\ z).$$

21. If $\sum_{x,y,z}(x)=a$, $\sum x^2=3b^2$, $\sum x^3=3c^3$, express $(x-a)(y-a)(z-a)$ in terms of a, b, c.

22. (i) Express $2\sum_{x,y,z}(x^2-yz)$ as the sum of three squares.

(ii) If x, y, z are real, and if $\sum(x^2)=\sum(yz)$, prove $x=y=z$.

23. If $\sum_{x,y,z}(x)\,.\,\sum\left(\frac{1}{x}\right)=1$, prove that $\sum x=x$ or y or z.

24. If $\sum_{x,y,z}(x)=0$, express $xy+2yz$ in the form $ax^2+by^2+cz^2$.

REMAINDER THEOREM—FACTORS.

If the function $f(x)\equiv a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ is divided by $x-k$, it is required to prove that the remainder is $f(k)$.

Suppose the quotient is $Q(x)$ and the remainder is R, where R is independent of x.

Then $$f(x)\equiv(x-k)\,.\,Q(x)+R.$$

In this identity, put $x=k$;

$$\therefore\ f(k)=0+R\,;$$
$$\therefore\ \text{the remainder}=f(k).$$

In particular, if k is such that $f(k)=0$, the remainder is 0, and therefore $x-k$ is a factor of $f(x)$.

Example IX. Find the value of a if $x+2$ is a factor of $x^3-ax^2+7x+10$.

If $x+2$ is a factor, the expression vanishes if $x=-2$;

$$\therefore\ (-2)^3-a(-2)^2+7(-2)+10=0\,;$$
$$\therefore\ -8-4a-14+10=0\,;$$
$$\therefore\ -4a=12\,;$$
$$\therefore\ a=-3.$$

Example X. Factorise $\sum\limits_{x,y,z} x(y-z)(y+z-x)^4$.

The given expression $= f(x, y, z)$

$$\equiv x(y-z)(y+z-x)^4 + y(z-x)(z+x-y)^4 + z(x-y)(x+y-z)^4.$$

If $x=0$, the function $= yz(z-y)^4 + z(-y)(y-z)^4$

$$= yz(y-z)^4 - yz(y-z)^4 = 0;$$

$$\therefore \; x \text{ is a factor of } f(x, y, z).$$

By symmetry, y and z are also factors.

Again, if $y=z$, the function $= z(z-x)x^4 + z(x-z)x^4$

$$= x^4z(z-x) + x^4z(x-z) = 0;$$

$$\therefore \; y-z \text{ is a factor of } f(x, y, z).$$

By symmetry, $z-x$ and $x-y$ are also factors;

$$\therefore \; f(x, y, z) \equiv A \, . \, xyz(y-z)(z-x)(x-y).$$

But $f(x, y, z)$ is a homogeneous function of degree 6, and $xyz(y-z)(z-x)(x-y)$ is a homogeneous function of degree 6.

$\therefore$ A must be purely numerical, independent of x, y, z;

$\therefore$ putting $x=1$, $y=2$, $z=3$, we have

$$(1)(-1)(4)^4 + 2(2)(2)^4 + 3(-1)(0)^4$$
$$= A(1)(2)(3)(-1)(2)(-1);$$

$$\therefore \; -256 + 64 = 12A;$$

$$\therefore \; 12A = -192 \quad \text{or} \quad A = -16;$$

$$\therefore \; \sum x(y-z)(y+z-x)^4 \equiv -16xyz(y-z)(z-x)(x-y).$$

Note.—Sometimes the value of A is best found by equating coefficients of a particular term.

EXERCISE XXIV. e.

1. Prove that $x+1$ is a factor of $x^3 - 2x - 1$.

2. Prove that $x+2y$ is a factor of $x^4 + 10xy^3 + 4y^4$.

3. Find a if $x-2$ is a factor of $x^3 + ax - 4$.

4. Find the factors of $2a^3 + a^2b + b^3$.

5. Find a, b if $x^2 + x - 6$ is a factor of $x^3 - ax^2 - bx - 6$.

6. If n is a positive integer, find the condition that

(i) $x-1$ is a factor of $x^n - 1$;

(ii) $x+1$ is a factor of $x^n + 1$;

(iii) x^2+1 is a factor of $x^n + 1$.

And prove that if $2^n + 1$ is a prime number, n must be a power of 2.

7. Find by inspection one factor of

$$(x+b+c)(x+c+a)(x+a+b) + abc.$$

8. Factorise $\sum\limits_{a,\,b,\,c} bc(b-c)$.

9. Factorise $[\sum\limits_{x,\,y,\,z}(x)]^3 - \sum(x^3)$.

10. Factorise $\sum\limits_{x,\,y,\,z}(yz)\,.\,\sum(x) - xyz$.

11. Factorise $\sum\limits_{x,\,y,\,z}(y-z)^3$.

12. Factorise $\sum\limits_{x,\,y,\,z}(x^3+y^3)(x-y)$.

13. Factorise $[\sum\limits_{a,\,b,\,c}(a)]^4 + \sum(a^4-[b+c]^4)$.

14. Prove that x^2-yz is a factor of $[\sum(x)]^3\,.\,xyz - [\sum(yz)]^3$: hence factorise it.

15. Prove that $xy-zw$ is a factor of $[\sum(xyz)]^2 - xyzw\,.\,[\sum(x)]^2$: hence factorise it.

16. Prove that

$$\sum_{a,\,b,\,c}(a[b+c-a]^2) + (b+c-a)(c+a-b)(a+b-c) \equiv 4abc.$$

17. Prove that $(x-1)^2$ is a factor of $x^n - nx + n - 1$.

18. If $x+y+z$ is a factor of

$$a(x^3+y^3+z^3) + b(y+z)(z+x)(x+y) + cxyz,$$

prove that $3a-b+c=0$.

MISCELLANEOUS EQUATIONS.

No general rule can be given. Suitable transformations and substitutions can be learnt only by experience. Whenever possible, symmetrical methods of working should be adopted.

If expressions are squared, or if any other step is taken which is not reversible, it is necessary to verify by substitution that the final solutions satisfy the original equation.

The symbol $\sqrt{x}$ is used for the *positive* square root of x.

EXERCISE XXIV. 1.

Solve the following equations :

1. $x+y=8xy,\ x^2+y^2=40x^2y^2$.

2. $xy=a^2,\ yz=b^2,\ zx=c^2$. 3. $ax=by+c=xy$.

4. $xy-x^2=2,\ 2y^2-3x^2=6$. 5. $xy+x-y=7=x^2-xy+y^2$.

6. $\frac{1}{x^2}+\frac{1}{xy}=10,\ \frac{1}{y^2}+\frac{1}{xy}=15.$ **7.** $\frac{x}{a^2}=\frac{y}{b^2}=\frac{z}{c^2}=xyz.$

8. $x(y+z)=4,\ y(z+x)=3,\ z(x+y)=-6.$

9. $xy=1+\frac{30}{xy},\ \frac{3}{x}+\frac{4}{y}=3.$ **10.** $x^2+cy=y^2+cx=7c^2.$

11. $x^3+y^3=9=3x+3y.$

12. $2xy=x+2y=\frac{x}{y}+\frac{2y}{x}.$

13. $x+\frac{1}{y}=y+\frac{1}{x},\ x+\frac{2}{y}=3.$

14. $(x-2)(y-2)=2\frac{1}{2}=\frac{1}{2x}+\frac{1}{2y}.$ [Put $x+y=u,\ xy=v.$]

15. $xy+y=6,\ xz+z=8,\ yz=3.$

16. $\frac{x^3}{y}=10-xy,\ \frac{y^3}{x}=40-xy.$ [Form an equation in xy.]

17. Solve $x+\frac{1}{y}=\frac{8}{x},\ y+\frac{1}{x}=\frac{x}{2}.$ [Multiply the equations together.]

18. Solve $\frac{x}{3}+\frac{y}{4}=6,\ \frac{3}{x}+\frac{4}{y}=\frac{2}{3}.$

19. $(x-1)(y+2)=10,\ (y+2)(z+5)=15,\ (z+5)(x-1)=6.$

20. $x-6=y+4=z-3=\sqrt[3]{(xyz)}.$

21. $xy-\frac{x}{y}=9,\ xy-\frac{y}{x}=11\frac{2}{3}.$

22. $x^3-y^3=19,\ x^2y-xy^2=6.$ [Divide one equation by the other.]

23. $x(1-y^2)=3y,\ y(1-x^2)=3x.$ [Subtract and factorise.]

24. $x^3=5x-y,\ y^3=5y-x.$

25. $x^2+y^2=10,\ (x-y)(x^3-y^3)=52.$

26. $xy+x+y=-5,\ yz+y+z=7,\ zx+z+x=-3.$
[Note $xy+x+y\equiv(x+1)(y+1)-1.$]

27. $2x-\sqrt{x}=6.$ [Put $x=y^2$.]

28. $\sqrt{3x+1}-\sqrt{x+8}+1=0.$ [Test your answers.]

29. (i) $\sqrt{x+1}+\sqrt{x-2}=3$; (ii) $\sqrt{x+1}-\sqrt{x-2}=3.$

30. $5+\sqrt{x-2}=2x.$ **31.** $\sqrt{12+x}-\sqrt{4+x}=\sqrt{1-x}.$

32. $\sqrt{3x+1}+\sqrt{2x+2}=\sqrt{2x+3}+\sqrt{3x}.$

33. $\sqrt{3x+1}-\sqrt{2x-1}=\sqrt{x}$.

34. $x^2+3x+3=\sqrt{2x^2+6x+5}$.
(Put $x^2+3x=y$ *or* $2x^2+6x+5=z^2$.)

35. $\dfrac{2}{1+\sqrt{x}}+\dfrac{3}{1-\sqrt{x}}=5$. **36.** $(x^2-2x)(x^2-2x+5)=24$.

37. $(x^2+4x)^2-4x^2-16x-5=0$.

38. $\left(x-1+\dfrac{2}{x}\right)\left(x-2+\dfrac{2}{x}\right)=2$.

39. $xy=x+y+2,\ yz=y+z-4,\ zx=z+x-2$.

40. $xy+x+y=-1,\ yz+2y+2z=24,\ zx+2z+2x=3$.

41. $x^2+y^2+z^2=133,\ x+y+z=19,\ y^2=xz$.

42. $(x+1)(x+3)(x+7)(x+9)=45$. [Put $x+5=y$.]

43. $x^4-5x^3-12x^2-5x+1=0$. $\left[\text{Divide by } x^2\text{, put } x+\dfrac{1}{x}=y.\right]$

44. $6x^4-35x^3+62x^2-35x+6=0$.

45. $x^2+y^2+z^2=14,\ x+y+z=6,\ xyz=6$.
[Eliminate x, y, z from $(t-x)(t-y)(t-z)=0$.]

46. $(4-6x)^{\frac{1}{3}}+(2-x)^{\frac{1}{3}}+(7x-6)^{\frac{1}{3}}=0$.
[Note that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$.]

47. $\sqrt{x+a-b}+\sqrt{x-a}=\sqrt{2x-b}$.

48. $x^2+xy+xz=p^2,\ xy+y^2+yz=q^2,\ xz+yz+z^2=r^2$.

49. $\dfrac{x+y}{2a}=\dfrac{y+z}{2b}=\dfrac{z+x}{2c}=yz$.

50. $x=\dfrac{y+2a-6}{ay-3y+a-2},\quad y=\dfrac{x+2a-6}{ax-3x+a-2}$.

Consider the cases $a=1$, $a=3$, $a=4$, $a=2\frac{1}{2}$.

51. $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}=\dfrac{xyz}{x+y+z}$.

52. $(y+z)^2-x^2=a^2,\ (z+x)^2-y^2=b^2,\ (x+y)^2-z^2=c^2$.

ELIMINATION.

Example XI. If
$$a+\frac{1}{x}=b+\frac{1}{y}=c+\frac{1}{z},$$
$$x+y+z=0,\quad x^2+y^2+z^2=0,$$
prove that a relation exists between a, b, c, and find it.

Let $a+\frac{1}{x}=\lambda$; $\therefore\ b+\frac{1}{y}=\lambda=c+\frac{1}{z}$;

$$\therefore\ \frac{1}{x}=\lambda-a,\quad \frac{1}{y}=\lambda-b,\quad \frac{1}{z}=\lambda-a.$$

Now $\Sigma(x)=0$ and $\Sigma(x^2)=0$;

$$\therefore\ \Sigma(xy)=\tfrac{1}{2}[\{\Sigma(x)\}^2-\Sigma(x^2)]=0;$$

$$\therefore\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{\Sigma(yz)}{xyz}=0;$$

$$\therefore\ \lambda-a+\lambda-b+\lambda-c=0;$$

$$\therefore\ \lambda=\tfrac{1}{3}(a+b+c);$$

$$\therefore\ x=\frac{1}{\lambda-a}=\frac{3}{3\lambda-3a}=\frac{3}{b+c-2a}.$$

But $\Sigma(x)=0$; $\quad\therefore\ \Sigma\left(\frac{1}{b+c-2a}\right)=0,$

which is the required relation.

This process is called *eliminating* x, y, z from the given equations.

EXERCISE XXIV. g.

1. If $x+ay=1$, $x+by=2$, $x+cy=3$, show that there must be a relation between a, b, c, and find it.

2. If $xy+x=a$, $xy+y=b$, $x+y=c$, show that a relation exists between a, b, c, and find it.

3. Eliminate m, given $x=am^2$, $y=2am$.

4. Eliminate t, given $x=t+\frac{1}{t}$, $y^2=t^2+\frac{1}{t^2}$.

5. Eliminate t, given $x=t+\frac{1}{t}$, $y=3t-\frac{1}{t}$.

6. Eliminate x, y, given $\frac{x+2y}{a}=\frac{2x+y}{b}=\frac{x+y}{c}$.

7. Eliminate x, given $\frac{x^2}{a+b}=\frac{x}{b+c}=\frac{1}{c+a}$.

8. Eliminate a, b, c, given $\frac{2x-y}{a}=\frac{2y+z}{b}=\frac{3x+y}{c}$, $a+b+c=0$.

9. Eliminate p, q, given $\frac{p+q}{x}=\frac{p-q}{y}=\frac{a}{b}$, $p^2+q^2=1$.

10. Eliminate t, given $x=\dfrac{a(1-t^2)}{1+t^2}$, $y=\dfrac{2at}{1+t^2}$.

11. Given $\dfrac{x}{y}+\dfrac{y}{x}=a$, $\dfrac{1}{x}+\dfrac{1}{y}=b$, $x+y=c$, show that a relation exists between a, b, c, and find it.

12. Given $\dfrac{x}{y}+\dfrac{y}{x}=c$ and $ax^2+bxy+ay^2=0$, must there be any relation between a, b, c? If so, find it.

13. Given $x+y=axy$, $y+z=byz$, $z+x=czx$, $xyz=1$, find the relation which must connect a, b, c.

14. Eliminate t, given $x=a+bt+ct^2$, $y=b+ct$.

15. Find the relation between a, b, c, d if

$$\sum_{x,\,y,\,z}(x)=a, \quad \sum(xy)=b^2, \quad \sum(x^3)=c^3, \quad xyz=d^3.$$

16. Eliminate x, given $\lambda=ax^2+bx+c=cx^2+bx+a$ and $c\neq a$.

17. Eliminate x, y, z, given $xy=a$, $yz=b$, $zx=c$, $x^2+y^2+z^2=d$.

18. If $a_1x^2+b_1x+c_1=0$ and $a_2x^2+b_2x+c_2=0$, prove that

$$\frac{x^2}{b_1c_2-b_2c_1}=\frac{x}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1};$$

hence eliminate x.

19. Eliminate x, y, z, given $x=p(y+z)$, $y=q(z+x)$, $z=r(x+y)$.

20. Eliminate b, c, given $b=\dfrac{ax}{x-a}$, $c=\dfrac{bx}{x-b}$, $d=\dfrac{cx}{x-c}$.

REVISION PAPERS.

Papers.	
E. 1–10.	(Part I.)
F. 1–16.	(Chap. XII., XIII.)
G. 1–10.	(„ XII.–XIV.)
H. 1–10.	(„ XII.–XVI.)
K. 1–16.	(„ XII.–XVII.)
L. 1–10.	(„ XII.–XVIII.)
M. 1–10.	(„ XII.–XIV. and XIX.)
N. 1–10.	(„ XII.–XIV. and XX., XXI.)
P. 1–8.	(„ XII.–XVIII. and XXII., XXIII.)
Q. 1–10.	(„ XXIV.)

REVISION PAPERS.

E.

(PART I.)

E. 1.

1. If an oak beam l feet long, b inches wide and d inches thick is fixed at one end, and a weight W cwt. is placed at the other end, it will break if W is larger than $\frac{5bd^2}{4l}$. Find whether such a beam 10 feet long, 1 foot wide and 9 inches thick will break when a load of 6 tons is placed at the free end.

2. Simplify the following expressions :

(i) $3(x+y)-(3x+y)$. (ii) $(3x)^2-3x^2$.

(iii) $\left(\frac{x}{2}\right)^3-\frac{x^3}{2}$. (iv) $\left(a+\frac{b}{2}\right)-\left(\frac{a+b}{2}\right)$.

3. Make w the subject of the formula $P=\frac{W+nw}{8}$.

4. If the square of $(3x+7)$ is equal to the square of $(3x+8)$, find the value of x.

5. Which of the numbers 0, 1, 2, 3, 4 (if any), are roots of the equation $(2-x)^5+(4-x)^5=(6-2x)^5$?

6. A farmer has 60 hurdles, each 2 yards long, with which to make a rectangular enclosure of which one side is already formed by the fence of the field. Complete the following table :

Number of hurdles in side parallel to fence - - -	10	20	30	40	50	60
Number of hurdles in each of the other sides - - -	25					0
Area of enclosure in square yards - - - - - -	1000					0

Draw a graph showing how the area depends on the number of hurdles placed in the side parallel to the fence, and find the number which gives the largest area possible.

E. 2.

1. Find the value of a, if $x=2$ is a solution of the equation

$$3x^2+2ax-20=0.$$

2. A flexible rod AB is built in at A, and supported at B; its length is l in. and the sag x in. from A is y in., where

$$y=\frac{1}{10l^3}x^2(l-x)(3l-2x).$$

(i) Find the sag halfway between A, B; (ii) compare the sags at $\frac{1}{3}$ of the distance from each end.

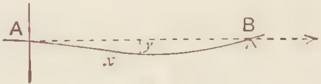


FIG. 80.

3. In a certain town eggs are advertised for sale at $3x$ shillings a dozen, and in another town they are being sold at x eggs for a shilling. What does one egg cost in each town? Find x if these two prices are the same.

4. Simplify the following expressions:

(i) $(x^3)^2 \div (x^2)^3$.   (ii) $(x^3)^2 \div (x^3 \times x^2)$.

(iii) $(a^2b^3)^2 \div (ab^2)^3$.   (iv) $6^{2x} \div 3^x$.

5. For what values of x is $(x+1)^3=x^3+1$?

6. The rule for finding the tax on the earned income of a bachelor is as follows: Deduct one-tenth from the income, then deduct £135 from the result; of the remainder £225 is taxed at 3s. in the £ and the rest at 6s. in the £. Find a formula for the tax on an income of £x, where $x > 1000$. What is the least value of x for which this formula holds good?

E. 3.

1. A working man gives a shillings a week to his wife for housekeeping expenses, spends b shillings a week on beer and tobacco, c shillings a month on tram fares, £d a year on boots and clothes, and £r a year on rent. How much will he have left from a wage of £4 a week at the end of the year, allowing exactly 52 weeks in a year? Give the answer in £ in as simple a form as possible.

2. Solve the equations: $\begin{cases} 2{\cdot}6x+0{\cdot}16y=4{\cdot}7, \\ 3{\cdot}8x-0{\cdot}48y=3{\cdot}3. \end{cases}$

3. Make V the subject of the formula $\frac{1}{R}=\frac{CL}{t}\cdot\frac{V}{V-V_1}$.

4. Draw a graph to illustrate the relation between P and W, if $P=\frac{1}{5}W+4{\cdot}6$, for values of W from 0 to 40. Read off the value of W when P is 10.

5. $\frac{x-1}{2}$ and $\frac{2x-1}{5}$ are consecutive integers, $\frac{x-1}{2}$ being the greater of the two. What integers are they ? Could they be consecutive integers if $\frac{2x-1}{5}$ were the greater ?

6. The formula $h=\frac{12dV^2}{5R}$ gives the height, h inches, which the outer rail of a curved railway should be raised, where R yards is the radius of the curve, d feet the distance between the rails and V miles per hr. the maximum velocity of trains on the curve. Rewrite the formula with V as subject, and find the safe speed on a curve of radius 2160 yards when the outer rail is raised 2 inches and the distance between the rails is 4 feet 6 inches.

E. 4.

1. At a school of 600 boys, 300 boys subscribe on the average a shillings each, 200 boys subscribe on the average b pence each, and 100 boys subscribe nothing to a certain charity. Find in pence the average subscription for the whole school ? Evaluate your answer when $a=2$ and $b=9$.

2. Prove that a triangle whose sides are (m^2+n^2) inches, (m^2-n^2) inches and $2mn$ inches is a right-angled triangle with (m^2+n^2) as its longest side whatever numbers m and n are. Give a numerical result when $m=5$ and $n=3$.

3. Factorize : (i) $16x^2-25$. (ii) $3x^2+7x-6$. (iii) $3x^2+x$.

4. An effort of P lb. is required to lift a weight of W lb. by a certain machine, where P and W are known to be connected by the equation $P=aW+b$, where a and b are constants, *i.e.* independent of the values of P and W. It is found that when $W=10$, $P=15$, and when $W=14$, $P=16{\cdot}2$. Find the values of a and b.

What is the value of W when $P=9$ and what does this mean ?

5. If $x=y^3$ and $y^2=z^5$, express x^2 in terms of z only.

6. A spring 4 inches long is lengthened $\frac{W}{4}$ inches when a weight W lb. is hung on it. Another spring $3\frac{1}{2}$ inches long is lengthened $\frac{W}{3}$ inches when a weight W lb. is hung on it. For what value of W will the two springs have the same length ?

E. 5.

1. A farmer keeps pigs and poultry only, and boasts that he has in his farmyard 200 heads and 464 legs, excluding those of himself and his household. How many pigs has he got in his farmyard ?

2. A rectangular piece of paper $ABCD$ is $5''$ by $3''$. The corner A is folded over the line BG so that it falls on the point F in the side DC, the dotted line BG being the position of the crease formed.

Calling AG x'' long, find the lengths of GF and GD in terms of x, and show that DF is 1 inch. Hence obtain an equation to find x and determine the position of the point G.

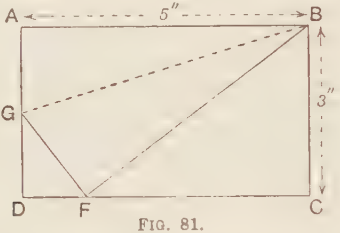


FIG. 81.

3. (i) Write down the L.C.M. of $2xy(x-y)^2$, $3y^2(x^2-y^2)$ and $4x^2(x+y)^2$.

(ii) Find the factors and H.C.F. of x^3-2x^2-3x and $x^4-5x^3-6x^2$.

4. Solve the equation $\dfrac{1}{x-1}+\dfrac{1}{x-2}=\dfrac{2}{x}$.

5. The weight of a foot of iron piping is $2{\cdot}45(D^2-d^2)$ lb., where D is the external and d is the internal diameter of the pipe in inches.

(i) Find the internal diameter of a pipe of external diameter $5{\cdot}2''$, which weighs $9{\cdot}8$ lb. per foot length ; (ii) Express the weight in terms of the mean radius r in. and the thickness t in.

6. A man walks a certain distance, x miles, up hill at 3 miles an hour, and walks back at 6 miles an hour. How long does he take going uphill ? How long does he take coming back ? How far has he been altogether, and how long has he taken ? Show that his average speed for the whole journey is 4 miles an hour.

[Lewis Carroll.]

E. 6.

1. A rectangle l'' long and b'' broad is such that its area would be increased by 44 square inches, if its length and breadth were both increased by 1 inch. Express this fact by an equation connecting l and b, and prove that the perimeter of the rectangle is 86 inches.

2. Find the value of $\frac{x^{n+1}}{y^{n+1}} - \frac{x^n}{y^n}$ when $x=3$, $y=2$ and $n=2$.

3. A weight of 100 lb. falls freely 20 inches and is then brought to rest by stretching a spring a further distance of x inches. The work done by gravity is $100(20+x)$ inch-lb., and this equals the work done in stretching the spring which is $50x^2$ inch-lb. How much is the spring stretched ?

4. Express the following statements in a single formula :

The tax on £200 is £$\frac{200-160}{8}$.

The tax on £380 is £$\frac{380-160}{8}$.

The tax on £460 is £$\frac{460-160}{8}$.

Use your formula to calculate the tax on the same system on £240.

5. Simplify the following expressions :

$$\text{(i)}\ \frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right], \qquad \text{(ii)}\ \frac{1}{h}\left[\frac{1}{(x+h)^2}-\frac{1}{x^2}\right],$$

and find approximately their values when $x=1$, $h=0{\cdot}001$.

6. Find the value of the constants a and b in the formula

$$\frac{1}{w}=\frac{a}{u}+\frac{b}{v}$$

if $w=5$, when $u=4$ and $v=10$; and $w=6$, when $u=3$ and $v=9$.

E. 7.

1. $2n$ is the middle one of five consecutive even numbers. What are the others in terms of n ?

If the sum of all five is 60, find the value of the middle one.

2. Find the value of $\frac{2a+3b}{2a-5b}$, when $a=5b$.

Construct any similar expression in a and b, whose value will be independent of the values of a and b, provided $a=5b$; and find its value if (i) $a=10$, $b=2$; (ii) $a=15$, $b=3$.

3. Solve the equations : $\begin{cases} 4P=5Q, \\ 14P-13Q=1{\cdot}8. \end{cases}$

4. a and b are two positive numbers $(a>b)$ such that

$$a^2+ab+b^2=79 \quad \text{and} \quad a^2+b^2=58.$$

Calculate in turn the values of (i) ab, (ii) $a^2+2ab+b^2$, (iii) $a+b$, (iv) $a^2-2ab+b^2$, (v) $a-b$. Hence find the values of a and b.

5. If $x=\sqrt{yz^3}$, and $y=2t^4$, $z=2s^2$, express x in terms of s and t.

6. In Fig. 82, $ABCD$ represents a rectangle with $AB=10''$, $BC=8''$. The lengths AF, AK, CG, CH are all equal to x''. What is the length of (i) FB, (ii) BG? Calculate the areas of the four triangles in the figure, and so express the area of the parallelogram $FGHK$ in terms of x. Give x the values 0, 1, 2 ... 8 in turn and calculate the corresponding values of the area $FGHK$. Show these results in a graph and from your graph find the value of x, which makes the area $FGHK$ a maximum.

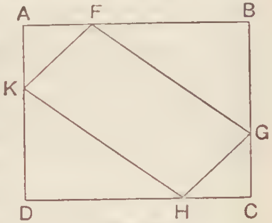


FIG. 82

E. 8.

1. Fig. 83 represents some smoothed statistics for a seven year period. Curve AA shows the percentage of unemployment, curve BB shows number charged for drunkenness, curve CC shows felonies committed, drawn on separate scales. What general conclusions, if any, could be deduced from these graphs?

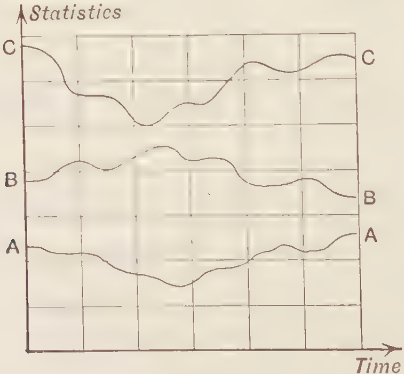


FIG. 83.

2. A boy is told to think of a number, subtract 7 from it and divide the remainder by 6. By mistake he subtracts 6 and divides by 7, but gets the same answer as he otherwise would have done. Find the number he thought of, and the answer he obtained.

3. Generalise the result of Question 2 by considering a boy who subtracts a and divides by b, instead of subtracting b and dividing by a. Prove that if he gets the same answer as he otherwise would have done, then he must have thought of the number $(a+b)$, and the answer he gets is 1.

4. (i) Simplify $\frac{2}{x^2-1}+\frac{3}{x-1}+\frac{5}{x+1}$.

(ii) Solve the equation $\frac{2}{x^2-1}+\frac{3}{x-1}+\frac{5}{x+1}=0$.

5. Solve the equations : (i) $6x^2-x-15=0$.

(ii) $2x+\frac{2}{x}=5$.

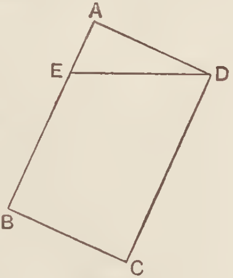


FIG. 84.

6. A rectangular tank of which $ABCD$ is a vertical section, is full of water and is tilted about the edge through C so that the water runs out. AB is 3 feet, BC is 2 feet and the length of the tank (not shown in the figure) is 5 feet.

DE represents the level of the water in the tank and AE is x ft.; find an expression in terms of x for the volume, V cubic feet, of water remaining in the tank, and also find the value of x in terms of V. What are the limits in the values of x and V, for which these formulae will hold ?

E. 9.

1. Make W the subject of the formula $t=\frac{2aW}{W+w}\left(1+\frac{V}{v}\right)$.

2. A ship is ordered to make a run of 240 sea-miles at 16 knots. After steaming for m hours, she is x miles short of the distance she ought to have traversed in m hours. Find a formula for the number of hours she will be late if she cannot increase her average speed.

3. ABC is an acute-angled triangle having $AB=7$ cm., $BC=8$ cm., $CA=9$ cm., and AD is drawn perpendicular to BC meeting it at D. Draw a rough figure and take BD to be x cm. long. Express CD in terms of x and show that $AD=\sqrt{(49-x^2)}$. Obtain an equation for x from the right-angled triangle ACD, and hence find the value of x and the area of the triangle.

4. Solve the equation $\frac{1}{2x-1}=\frac{3x-4}{3x}$.

5. It is known that $\frac{6a-7}{2}$ is greater than $\frac{5a+4}{3}$. If a is an integer, find the smallest possible value of a.

6. Sealed cylindrical tins of volume 8 cubic inches are to be made to hold two ounces of tobacco.

If the volume of a cylinder is $\pi r^2 h$ cubic inches, and the surface of the tin used is $2\pi r(r+h)$ square inches, where r'' is the radius and h'' is the height of the cylinder, find h in terms of r when the volume is 8 cubic inches and hence show that the area of the surface is $\left(2\pi r^2+\frac{16}{r}\right)$ square inches.

Tabulate the values of this expression, when r has the values $\frac{1}{2}$, 1, 2, 3, taking $\pi=3{\cdot}14$, and represent the result graphically. Hence find the value of r, so that the quantity of tin used may be a minimum.

E. 10.

1. A bath is filled by the cold-water tap in a minutes, what fraction of it is filled in 1 minute ?

It is filled by the hot-water tap in b minutes ; what fraction of it is filled in 1 minute ?

What fraction of the bath is filled, if both taps are turned on, in 1 minute, and how long will it be before the bath is full ?

The waste-pipe would empty the bath in c minutes.

Find an equation connecting a, b and c, if the level of the water remains unchanged when both taps are on, and the waste-pipe is open.

2. The volume of a sphere is $\frac{4}{3}\pi r^3$ cubic inches, if the radius is r''. Find the volume of metal in a spherical shell whose inner radius is $4''$ and outer radius $5''$, taking $\pi=\frac{22}{7}$.

3. If $t=2x+1$, write x in terms of t. Hence write $4x^2+4x-3$ in terms of t, and find the value of t if $4x^2+4x-3=0$.

4. Factorize (i) $5x^3-20x$, (ii) $7a^2-14a-21$, (iii) $b(b-4)-12$.

5. Solve the equations : $\begin{cases} y=x+2. \\ x^2+2y=7. \end{cases}$

6. In Fig. 85 ABC is a right-angled triangle in which $AB=6$ cm., $BC=4$ cm. A rectangle $PQRB$ is drawn as shown. If PQ is x cm., what is the length of RC ? It is known that

$$\frac{RC}{BC}=\frac{QR}{AB}.$$

Use this fact to show that $y=6-\frac{3x}{2}$, where y cm. is the length of QR. Hence express the area of the rectangle $PQRB$ in terms of x, and tabulate the values of the area of the rectangle when x has integral values from 0 to 4.

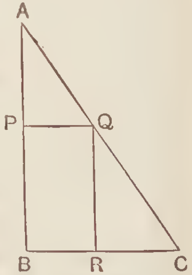


FIG. 85.

F.

(CHAPTERS XII.-XIII.)

F. 1.

1. A railway fare is increased by half; it is then increased again by $\frac{1}{6}$th of its new value. Prove that the final result is the same as if the original fare had been increased by three-quarters of its value.

2. Using the data of Question 1, find the original cost of a ticket which after the two increases costs 15s. 2d.

3. The following observed values of x and y will give a straight-line graph, subject to the errors of experiment:

x	1	2	3	4	5	6	7	8
y	7·2	12·1	16·7	21·8	26·6	31·5	36·3	41·2

Plot these values and draw the straight line which fits them best. Read off the most probable value of y when $x=4{\cdot}7$, and of x, when $y=30$.

4. If $x=\sqrt{y}$, $y^3=4z^2$, $z=4t^2$, (i) express y in terms of t, (ii) express x in terms of t.

5. If $2x-x^2=\frac{1}{2}$, find the value of (i) $(x-1)^2$, (ii) $(x-1)^6$.

6. Write the following numbers without using the index notation:

(i) $1{\cdot}8\times10^6$. (ii) $1{\cdot}8\times10^{-2}$.

(iii) $1{\cdot}8\times10^{-4}$. (iv) $1{\cdot}8\times100^{\frac{3}{2}}$.

and express 73800 and 0·0000738 in this form.

F. 2.

1. The equation $s=a+bt+t^2$ is satisfied by the two pairs of values $t=3$, $s=5$ and $t=5$, $s=29$.

Find the values of a and b, if they are constants.

2. The outer dimensions of a picture frame are 16 inches by 12 inches. The width of the frame is the same all round, and the surface area of the frame alone is 75 square inches. Find the width of the frame.

3. A piece of wire 1 foot long is bent into the shape of a rectangle. If one side of the rectangle is x'', find the area enclosed in terms of x. Find also the sides of the rectangle if the area enclosed is $6\frac{3}{4}$ square inches.

4. Rewrite the following expressions without using negative or fractional indices.

(i) $(1-x)^{-2}$. (ii) klt^{-2}. (iii) $(2x)^{\frac{1}{2}}$. (iv) $2x^{\frac{1}{2}}$.

(v) $2x^{-\frac{1}{2}}$.

5. Find the value of $P^{\frac{1}{2}}H^{-\frac{3}{4}}$, when $P=144$, $H=16$.

6. Simplify:

$$\frac{1}{a-1}-\frac{a}{1-a}+\frac{2a}{2a-1}-\frac{1}{1-2a}.$$

Check your answer by putting $a=2$.

F. 3.

1. A string 14 inches long has its ends fastened to two points, A and B, which are 10 inches apart. A pencil P is pressed against the string and is moved round so that the two portions of the string AP and PB are kept taut. Find the lengths of these two portions of the string in a position in which the angle APB is a right angle.

2. For what values of x, if any, are the following statements true?

(i) $(x-1)^2=(1-x)^2$. (ii) $(x-1)=(1-x)$.

(iii) $(x-1)^2=(x-2)^2$. (iv) $(x-1)=(x-2)$.

3. The efficiency, e, of a certain machine is given by the formula $e=\frac{W(a-b)}{2P\,.\,a}$. Express a in terms of e, P, W and b.

4. Use logarithm tables to find x and y, if $x=10^{3\cdot1462}$ and $10^y=47$.

5. Write the following expressions without using fractional or negative indices.

(i) $(a^2+b^2)^{\frac{1}{2}}$. (ii) $a^{\frac{2}{3}}+b^{\frac{2}{3}}$.

(iii) $W\,.\,l\,.\,b^{-1}.\,d^{-2}$. (iv) $K\,.\,H^{1\cdot25}\,P^{-0\cdot5}$.

6. If $a=K\,.\,H^{1\cdot25}\,.\,P^{-0\cdot5}$ express P in terms of a, K and H.

F. 4.

1. Find the value when $a=2$, $b=-1$, and $n=3$ of the following expressions:

(i) $(a+b)^n$. (ii) a^n+b^n. (iii) $(a+b)^{n-1}$.

(iv) $a^{n-1}+b^{n-1}$. (v) $\frac{a^n}{b^n}\div\frac{a^{n-1}}{b^{n-1}}$.

What will be the value of (v) when $n=10$?

2. A man intends to travel by car from A to B. He has the choice of two routes, one x miles long on which he can travel y miles an hour, the other 6 miles longer on which he will be able to travel on the average 5 miles an hour faster. Prove that the extra time that he would take travelling by the longer route is given by the expression $\frac{x+6}{y+5}-\frac{x}{y}$ hours.

Simplify this expression and prove that he will save time in going by the longer route, provided $x>\frac{6y}{5}$, and that he can only save time if the journey is going to take at least 72 minutes anyhow.

3. If $x=2y+3$, express x^2-3y^2 in terms of y only.
Hence solve the equations: $\begin{cases} x=2y+3, \\ x^2-3y^2=22. \end{cases}$

4. Write down the logarithms of 1000, $\sqrt{1000}$, 1 and $(\sqrt[3]{100})^2$.

5. Use logarithms to find the value, correct to 3 significant figures, of

(i) $9{,}724{,}185 \div 45{,}620$;
(ii) $(1{\cdot}479)^5$.

6. Find the value of $\sqrt{\frac{H\,.\,D^5}{L}}$, correct to 3 figures, if $H=25{\cdot}7$, $D=2{\cdot}32$, $L=880$.

F. 5.

1. An extension ladder is formed of x separate ladders, each of length l feet, and the overlap at each junction is n feet. What is the total length of the ladder?

2. Find the value when $n=3$ of

(i) $8^{\frac{1}{n}}$; (ii) 8^{-n}; (iii) $8^{1-\frac{1}{n}}$; (iv) 16^{1-n}; (v) 16^{n-3}.

3. The perimeter of a rectangle is 34 inches, and the length of its diagonal is 13 inches. Find the lengths of its sides.

4. Two quantities s and t are connected by the formula

$$s=6t-4t^2.$$

Tabulate the values of s, when t has the values -2, -1, $-\frac{1}{2}$, 0, $\frac{1}{2}$, 1, 2. Draw a graph to show the relation between s and t, and from it read off the largest value of s, and also the value of t when $s=1$.

5. Find, correct to 3 figures, the value of:

(i) $\frac{4{\cdot}712\times 17{\cdot}95}{6{\cdot}834}$; (ii) $\sqrt{\frac{129{\cdot}1}{7{\cdot}32}}$.

6. The volume of a sphere is given by the formula $V=\frac{4}{3}\pi r^3$, where $\pi=3\cdot142$ and r is the radius.

Write the formula with r as the subject, and find the radius of a sphere whose volume is 16 cubic inches.

F. 6.

1. Prove that $\frac{n^2(n+1)^2}{4}+(n+1)^3\equiv\frac{(n+1)^2(n+2)^2}{4}$.

[Note that $(n+1)^2$ is a factor of the left-hand side of this identity.]

Write down (without multiplying out) the special cases of this identity for $n=0, 1, 2, 3, 4, 5, 6$. By adding these results together, prove that $1^3+2^3+3^3+4^3+5^3+6^3+7^3=\frac{7^2\times 8^2}{4}$. By generalising this method, find an expression for the sum of the cubes of the first n integers.

2. Make μ the subject of the optical formula $\frac{1}{f}=(\mu-1)\left(\frac{1}{r}-\frac{1}{R}\right)$.

3. Solve the equations : (i) $3x^2-2x-8=0$;
(ii) $3x^2-2x-7=0$,

giving the result of the latter correct to 3 figures.

4. Simplify the following expressions :

(i) $2x^6\div x^2$. (ii) $3x^{\frac{1}{2}}\div x^{\frac{1}{3}}$. (iii) $4x^2\times 3x^4$.
(iv) $6x^{\frac{2}{3}}\times x^{\frac{4}{3}}$. (v) $(2x^2)^3$. (vi) $(2x^{\frac{1}{3}})^3$.

5. Find the value of pv^n when $p=14\cdot21$, $v=7\cdot7$ and $n=4$, correct to 3 figures.

6. In finding the value of $\frac{1124\times 79\cdot3}{74\cdot25}$ correct to 3 significant figures, a boy obtained the answer 11,900.

Show as shortly as possible why this answer is clearly wrong, and obtain the correct answer.

F. 7.

1. $3\times 4=12$; $\therefore\ 35^2=1225$.
$4\times 5=20$; $\therefore\ 45^2=2025$.
$7\times 8=56$; $\therefore\ 75^2=5625$.
$12\times 13=156$; $\therefore\ 125^2=15625$.

What general formula covers these facts ? Prove it.

2. A bank clerk, in entering a sum of xs. yd. in a column of figures, writes it down as £x. ys. 0d. by mistake, with the result that he finds the column adds up £16. 12s. 2d. too much.

Find the sum of money which was entered wrongly.

3. A stone is thrown upwards with a velocity of 64 feet per second. Its height in feet after t seconds is given by the formula $h = 64t - 16t^2$.

Find when it is at a height of 48 feet, and explain the double answer.

4. Simplify

(i) $3a^3 \div 6a^{-2}$. (ii) $8a^{3\cdot 5} \div 2a^{5\cdot 8}$.

(iii) $\dfrac{2a^{\frac{2}{3}} \times 3a^2}{6a^{\frac{1}{6}}}$. (iv) $\dfrac{4a^{-2} \times a^3}{6a^{-4}}$.

5. Find the value, correct to 3 figures, of

(i) $\dfrac{1\cdot 462}{7\cdot 136}$; (ii) $\dfrac{10}{\sqrt{107\cdot 1}}$.

6. Find the value of $e^x + e^{-x}$, when $e = 2\cdot 718$ and $x = 2$, correct to 3 figures.

F. 8.

1. Find the value of (i) $(10^{\bar{3}})^{\frac{1}{3}}$; (ii) $(10^{\bar{3}})^{\frac{1}{4}}$; (iii) $(10^{\bar{3}\cdot 2})^{\frac{1}{4}}$.

2. Given that $x^2 - y^2 = 10$ and $x - y = 2$.
Find the value of $(x + y)$, and hence find x and y.

3. Solve the equations : (i) $6x^2 - 7x - 3 = 2$;
(ii) $6x^2 - 7x - 3 = 1$,

giving the solution of the latter correct to 3 figures.

4. In the figure, O is the centre of a circle whose radius

$$OA = OR = 5''.$$

AB is a chord at right angles to OR, and $NR = x''$. What is the length of ON in terms of x ? What is the length of AN in terms of x ?

Tabulate the values of AN when x has the values 0, 1, 2, 3, 4, 5. Show these results as a graph. What can you say about the shape the graph will take ?

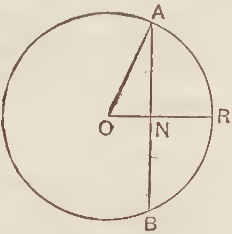


FIG. 86.

5. Factorise and find the value of $\pi(R^2 - r^2)l$, if $\pi = 3\cdot 142$, $R = 15\cdot 42$, $r = 14\cdot 42$ and $l = 250$.

6. The length of the tangent from a point on the outer of two concentric circles to the inner circle is 5 inches. The difference of their radii is 2 inches. Find their radii.

F. 9.

1. (i) Simplify $\frac{a}{a+1}-\frac{a+1}{a+2}$. Test your result when $a=1$.

(ii) Simplify $\frac{(x^2-4)(x^2-4x+3)}{(x^2-1)(x^2-5x+6)}$.

2. Express in a single formula the following facts:

$$\sqrt{6}\times\sqrt{7}=\sqrt{6\times 7}=\sqrt{42}.$$
$$\sqrt{7}\times\sqrt{10}=\sqrt{70}.$$
$$\sqrt{3}\times\sqrt{5}=\sqrt{15}.$$

Use this formula to find the value as simply as possible of

(i) $\sqrt{32}\times\sqrt{2}$;
(ii) $\sqrt{6}\times\sqrt{24}$;
(iii) $\sqrt{ab^3}\times\sqrt{a^3b}$.

3. The area of a triangle is $\sqrt{s(s-a)(s-b)(s-c)}$, where a, b, c are the sides and $s=\frac{a+b+c}{2}$. What does this formula become for an equilateral triangle, side a? Verify your result by finding a formula for the area of an equilateral triangle independently.

4. In a right-angled triangle the hypotenuse is 15″, and the difference between the other two sides is 3″. Find the other two sides.

5. The bore, d inches, of a pipe through which a pump of horse-power H can deliver G gallons per second is given by

$$d=1{\cdot}25\,.\,G^{\frac{3}{4}}H^{-\frac{1}{4}}.$$

(i) Find the bore necessary for an engine of 15 H.P. to deliver 300 gallons per minute.

(ii) Rewrite the formula, making G the subject. (C.S.C.)

6. Find the value, correct to 3 figures, of

(i) $\sqrt[3]{0{\cdot}012}$. (ii) $\sqrt[4]{0{\cdot}08}$.

F. 10.

1. In Fig. 87, AB represents a line 10″ long, and P is a point such that $AP=x''$. Squares are described on AB, AP and PB as shown.

Write down the areas of these squares and express the area of the shaded portion of the figure in terms of x.

Tabulate the values of this area as x varies from 0 to 10, and show the results on a graph. Find from your graph the position of P which makes the shaded area greatest.

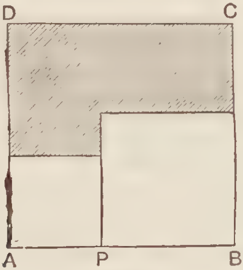


FIG. 87.

2. Find a general formula to include the following facts:

$$\frac{\sqrt{30}}{\sqrt{5}} = \sqrt{\frac{30}{5}} = \sqrt{6}. \quad \frac{\sqrt{24}}{\sqrt{12}} = \sqrt{\frac{24}{12}} = \sqrt{2}. \quad \frac{\sqrt{90}}{\sqrt{18}} = \sqrt{\frac{90}{18}} = \sqrt{5}.$$

Use this formula to find the value of

$$\text{(i)}\ \frac{\sqrt{8}}{\sqrt{2}}; \qquad \text{(ii)}\ \frac{\sqrt{72}}{\sqrt{8}}; \qquad \text{(iii)}\ \frac{\sqrt{ab^3}}{\sqrt{a^3b}}.$$

3. Solve the equations: $\begin{cases} x+y=1, \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{6}=0. \end{cases}$

4. Find, without using tables, the logarithms of

$$\sqrt[4]{10}, \quad \sqrt[4]{0{\cdot}1}, \quad \sqrt[5]{0{\cdot}001}, \quad \sqrt[3]{0{\cdot}1}.$$

5. Find the value of $\dfrac{l^2\omega}{8s}$ when $l=80$, $\omega=0{\cdot}072$, $s=276$.

6. It is stated by Professor Perry that the record times for races of all kinds by men or animals obey the law $t=k\,.\,m^{1{\cdot}175}$, where t sec. is the record for a race of m miles and k depends on the conditions of the race, but is independent of the distance. The record for the "half-mile" is 1 min. $52\frac{1}{2}$ sec. (*Whitaker's Almanack*). What should be the record for the "mile"? [Up to the present it is reported to be 4 min. $12\frac{3}{4}$ sec.]

F. 11.

1. Prove that the equations:

$$\left.\begin{aligned} 3x+y-2z&=7, \\ x-y-2z&=5, \\ 2x+y-z&=3, \end{aligned}\right\}$$

are inconsistent by obtaining two equations independent of y and comparing them.

2. Simplify $$\frac{x-y}{(x+y)^2}+\frac{1}{x+y}+\frac{x^2+y^2}{(x+y)^3}.$$

3. $(1+x)^n \simeq 1+nx$, when x is small. Use logarithms to find the error per cent. in assuming that $(1+x)^n=1+nx$, when $x=0{\cdot}025$ and $n=4$.

4. Find, correct to 3 figures the roots of the equations :

$$\text{(i)}\ \ 4x^2+5x-2=0;$$
$$\text{(ii)}\ \ 4x^2+5x-2=4.$$

5. A small object 3 mm. long when seen through a magnifying glass held d mm. from it, appears to have a length l mm. given by $l=\frac{3f}{f-d}$, where f is a constant depending on the kind of magnifying glass used.

(i) Find the value of f for a certain magnifying glass in which the apparent length l is 12 mm., when the glass is held 15 mm. away.

(ii) Find also the apparent length of the same object seen through the same glass, when the glass is held 12 mm. away.

6. The safe width of a dam at a depth of x feet below water level is $\sqrt{\left\{\frac{0{\cdot}05\,.\,x^3}{9+0{\cdot}03x}\right\}}$ feet.

How wide should a dam be at a depth of 28 feet below water level ? (Certificate.)

F. 12.

1. Two boys bicycle a distance d miles together ; the wheels of one bicycle are a'' in diameter, the wheels of the other are b'' $(a>b)$. Show that the number of revolutions made by the smaller wheels is $\frac{1760\times 36d}{\pi}\cdot\left(\frac{1}{b}-\frac{1}{a}\right)$ more than the number made by the larger. Write this in a form suitable for calculation by logarithms, and evaluate when $a=28$, $b=26$ and $d=10$.

$$[\pi=3{\cdot}142=10^{0{\cdot}4972}.]$$

2. The following equations are found to connect a force of P lb., and the distance x'' at which it acts from a fulcrum :

$$\begin{cases} P\,.\,x=88, \\ P(x+2)=120. \end{cases}$$

Find the values of P and x.

3. Find x and y if $x=\frac{1}{y}=\frac{2}{x-1}$.

4. Subtract $\frac{x^n}{(x+y)^m}$ from $\frac{x^{n-1}}{(x+y)^{m-1}}$.

5. Evaluate the following as shortly as possible, given that $\sqrt{2}=1{\cdot}414\ldots$, $\sqrt{3}=1{\cdot}732\ldots$:

(i) $\frac{4}{\sqrt{2}}$; (ii) $\frac{6}{\sqrt{2}}$; (iii) $\frac{\sqrt{6}}{\sqrt{3}}$; (iv) $\sqrt{12}$.

6. When water flows at speed v through a pipe of diameter d, the loss of pressure in a length l is equal to a head of h feet of water, where $h=\frac{0{\cdot}045}{d^{1{\cdot}16}}\times\frac{v^2 l}{2g}$.

Find h when $l=75$, $v=7{\cdot}4$, $d=2{\cdot}3$, $g=32{\cdot}2$. (C.S.C.)

F. 13.

1. Fig. 88 represents a white flag, 20 ft. by 30 ft., with a red cross; the red stripes forming the cross are of equal width, and the area of the cross is $\frac{1}{5}$ of the area of the whole flag. Find the width of the strips to the nearest $\frac{1}{10}$th of an inch. (C.S.C.)

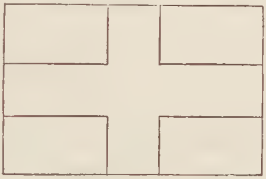


FIG. 88.

2. In a sailing-boat the direction of the wind makes an angle α with the direction of the boat's keel, and the sail makes an angle β with the boat's keel. The following table shows the best angle to choose for β when sailing into a wind:

α in degrees - -	60°	50°	40°	30°	20°	10°
β in degrees - -	$34\frac{1}{2}$°	28°	22°	16°	10°	5°

Exhibit these results on a graph, and read off from the graph the best angle at which to set the sail when the course required is due North and the wind is blowing from the North-East.

3. Find the value of $W\cdot\frac{\sqrt{2ah-h^2}}{b-h}$, when $W=112$, $b=10{\cdot}5$, $h=8{\cdot}25$ and $a=5$.

4. Find x if (i) $3^x=81$; (ii) $81^x=3$; (iii) $3^x=\frac{1}{9}$; (iv) $81^{-x}=3$.

5. Find the value of (i) 6^{20}; (ii) 2^{60}, correct to one significant figure, in the approximate form $a\times 10^b$, a being an integer less than 10.

6. Write in as simple a form as possible:

(i) $\frac{\sqrt{60}}{\sqrt{3}}$; (ii) $\frac{\sqrt{20}}{\sqrt{5}}$; (iii) $\frac{3\sqrt{5}}{\sqrt{15}}$; (iv) $\frac{4^{\frac{2}{3}}}{2^{\frac{1}{3}}}$.

F. 14.

1. A line AB, $2a$ inches long, is bisected at M. P is a point in the line between M and B, such that $MP = x''$. Prove that

$$AP^2 - PB^2 = 4ax.$$

2. Write 4^n, 8^{n-1} as powers of 2, and simplify $\dfrac{2^n \times 4^n}{8^{n-1}}$.

3. Fig. 89 represents a circle inscribed in the triangle ABC. $AR = x''$, $BP = y''$, $CQ = z''$.

Also $AB = 5''$, $AC = 6''$, $BC = 7''$.

Write down three equations connecting x, y and z.

By adding these up, show that $x + y + z = 9$, and hence find the values of x, y and z.

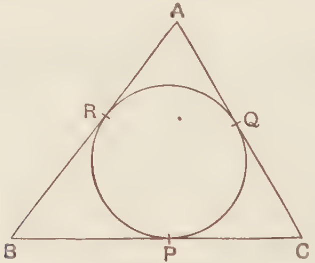


FIG. 89.

4. Find the value of $e^{\mu\theta}$ when $e = 2{\cdot}718$, $\mu = 0{\cdot}56$, $\theta = 2$.

5. Given that a sum of £P invested at r per cent. per annum, interest being reckoned half-yearly, will amount at Compound Interest in n years to a sum $£A = £P\left(1 + \dfrac{r}{200}\right)^{2n}$, find the amount on £1750 invested for 15 years at 6 per cent. per annum paid half-yearly, giving the answer in pounds, correct to 3 figures.

6. Q cubic feet of water flow over a rectangular weir per second given by $Q = 6{\cdot}78\left(b - \dfrac{h}{5}\right)h^{\frac{3}{2}}$, where b = width of weir in feet = 60 feet, h = height of water over weir in feet = $1{\cdot}25$ feet. Find how many tons of water flow over the weir in an hour, if a cubic foot of water weighs $62\frac{1}{2}$ lb.

F. 15.

1. Write 27, $\frac{1}{9}$, $3\sqrt{3}$ and $\sqrt[4]{3}$ as powers of 3, and write down their logarithms to the base 3.

2. The tendency of a plank bridge to break when a man is standing on it, is measured by the quantity $\dfrac{w}{l}(l - x)x$, where l feet is the length of the bridge, w lb. is the weight of the man and x feet is his distance from one end. Tabulate the values of this expression for a man, 150 lb. weight, standing on a bridge 20 feet long, when his distance from one end is 0, 4, 8, 12, 16, 20 feet respectively.

Draw a graph to show how the tendency to break varies, and find where the man is when the bridge is most likely to break.

3. Two quantities R and v are connected by the equation $R = a + b\sqrt{v}$, where a and b are constants.

If $R = 10$ when $v = 64$, and $R = 11 \cdot 2$ when $v = 81$, find the values of a and b.

4. There is a day-and-night motor service from A to B, the cars leaving A at each hour; also a service from B to A, the cars leaving B at the half-hour. The journey occupies x hours, where x is an integer.

How many cars will a man pass on the road when he travels from A to B in one of these cars ?

5. Evaluate

$$\text{(i)}\ \frac{1 \cdot 56 \times 10^6}{412 \cdot 7 \times 1 \cdot 175}; \qquad \text{(ii)}\ \sqrt[3]{\frac{0 \cdot 03167}{12 \cdot 58}}.$$

6. If a sum of money £P amounts to £A when invested for n years at r per cent., then $A = P\left\{1 + \frac{r}{100}\right\}^n$. Write this formula with r as the subject, and hence find the rate per cent. paid on a "War Savings Certificate," which costs 15s. 6d. and amounts to £1 in 5 years.

F. 16.

1. A man buys a number of motor-cars at £400 each. He keeps one for his own use and sells the remainder at 10 per cent. profit. As the result he finds he has got his own car for nothing. How many cars did he buy ?

2. Simplify $\left(\frac{1}{x} - \frac{1}{1+x}\right) \div \left(\frac{1}{x} + \frac{1}{1-x}\right)$.

3. Given that the volume of a sphere, radius r'', is $\frac{4}{3}\pi r^3$, find the diameter of a hemispherical bowl which will hold a gallon of water. [$\pi = 3 \cdot 142$ and 1 cubic foot $= 6\frac{1}{4}$ gallons.]

4. If $(h-a)(1-a) = (h-b)(1-b) = k$, find k in terms of a and b only. (C.S.C.)

5. A certain sheet of transparent material absorbs $\frac{1}{10}$th of the light of a certain colour falling on it. What fraction of the light of that colour gets through two such sheets ?

How many sheets must be placed one over another to reduce the light of that colour by 60 per cent. ?

6. A spherical bomb, when dropped from a height of s feet, reaches the ground with a velocity v feet per second after t seconds. Taking air-resistance into account,

$$s=\frac{L^2}{2kg}\log\left(\frac{L^2}{L^2-v^2}\right),$$

$$t=\frac{L}{2kg}\log\left(\frac{L+v}{L-v}\right),$$

where $L=400$, $k=0{\cdot}4343$, $g=32{\cdot}2$.

Find the time taken to reach the ground from a height of 1000 feet. (C.S.C.)

G.

(CHAPTERS XII.-XIV.)

G. 1.

1. Write the following expressions without using fractional or negative indices :

(i) $5x^{\frac{1}{2}}$; (ii) $(9x)^{\frac{1}{2}}$; (iii) $1{\cdot}4\times10^{-6}$; (iv) $2\left(\frac{a}{3b}\right)^{-1}$.

2. In an examination some candidates take 4 papers and some take 6 papers. There are 250 candidates, and they do 1078 papers in all. Find how many took 6 papers.

3. The following formulae connect the velocity v of a body which starts with initial velocity u, and moves with uniform acceleration f, with the time t it has been moving, and the distance s it has gone :

(i) $v=u+ft$;

(ii) $s=ut+\frac{1}{2}ft^2$.

Rewrite the first formula so as to express t in terms of u, v and f.

Substitute this expression for t in the second formula, and so obtain the formula $v^2=u^2+2fs$.

Find also a formula connecting s, v, f, t.

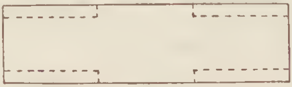


FIG. 90.

4. From a rectangular strip of paper, a'' by b'' $(a>b)$, rectangles x'' by y'' $(x>y)$ are cut out at each corner, the sides of length x'' being parallel to those of length a'' ; the remainder is formed into a box by folding the projections upwards, and the longer projections are folded again, and so just form the lid of the box.

Find (i) the relation between x, y, a; (ii) the volume of the box in terms of a, b, y.

5. In a certain Parliament the ratio of the majority to the minority is 3 : 2. Eighteen members desert the majority and join the minority, thus reducing the ratio to 5 : 4. Find the number of members. (Certificate.)

6. ABC is a triangle with a right angle at A, and AD is perpendicular to BC.

It is known that the ratio $BD : DA$ is equal to the ratio $DA : DC$. If BD is 5·4″ and DC is 9·6″, find the length of AD. Find also the length of BA

(i) from the fact that $BD : BA = BA : BC$;

(ii) by applying Pythagoras' theorem to the triangle ADB.

G. 2.

1. Find the values of $(1-x)(x+2)$ when x has integral values from -3 to $+3$. Plot a graph to illustrate these figures, and read off from it the greatest value of $(1-x)(x+2)$ for all values of x in this interval.

2. Solve the equations: $\begin{cases} 2{\cdot}7x + 0{\cdot}13y = 5{\cdot}6, \\ 3{\cdot}9x + 0{\cdot}7y = 6{\cdot}4, \end{cases}$

giving the results correct to two significant figures.

3. A man buys two horses, paying £250 for the two. He sells one at a loss of 10 per cent. and the other at a profit of 50 per cent., receiving £291 in all. What did he pay for each horse?

4. Einstein's theory of relativity gives the formula $V = \dfrac{7 \times 10^{41}}{\sqrt{\rho^3}}$, connecting V the volume of world space and ρ the mean density of matter throughout space.

(i) Write this formula with ρ as subject.

(ii) Assuming space to be spherical and the volume of a sphere to be $\frac{4}{3}\pi r^3$, find a formula for the radius r of space in the form $r = \dfrac{a \cdot 10^b}{\sqrt{\rho}}$, with b integral and a correct to one figure.

5. Two glasses are not quite full; one contains n spoonsful of water and the other n spoonsful of wine. A spoonful of wine is taken from the second glass and mixed in the first. What is the proportion of wine in the mixture? A spoonful of the mixture is now taken out of the first glass and put in the second. What is the proportion of wine to water in the second glass? Is this greater or less than the proportion of water to wine in the first glass?

6. *AD* bisects the angle *A* of a triangle *BAC* internally, and meets *BC* at *D*. It is known that the ratio $BD : DC = BA : AC$.

If the sides of the triangle are $AB = 5''$, $BC = 6''$, $AC = 7''$ and *BD* is x'', write down the equation to find x, and obtain the lengths of *BD* and *DC*.

G. 3.

1. A train in which there is seating accommodation for 600 has 84 compartments, some of which seat 6 and some seat 10. Find how many compartments there are of each kind.

2. Write the following expressions without using fractional or negative indices: (i) $a^{\frac{1}{2}}b^{\frac{3}{2}}$, (ii) $H^{\frac{3}{4}}G^{-\frac{1}{4}}$, (iii) $1{\cdot}4 \times 10^{-11}$, and write down the value of 7^{-2}, $9^{-\frac{1}{2}}$, 11^{0}.

3. Write the formula $W = 17{\cdot}6\,\dfrac{PR^2}{r}$ with *R* as the subject, and find the value of *R*, correct to 3 figures, when $W = 4480$, $r = 0{\cdot}29$, $P = 9{\cdot}6$.

4. (i) Solve the equations: $\begin{cases} x + y = 2a + 3b, \\ 4x + 3y = 7a + 5b. \end{cases}$

(ii) If $\dfrac{x+y}{x-y} = \dfrac{8}{5}$, find the ratio $x : y$.

5. The "displacements" of similar ships vary as the cubes of their lengths. Find the displacement of a ship which is half as long again as a similar ship whose displacement is 2000 tons.

6. (i) The perimeter of a triangle is 2 feet, and one side is three times another. Show that the length of the shortest side lies between 3 and 4 inches.

(ii) Show that in any triangle in which one side is three times another, the ratio of the shortest side to the perimeter lies between $\frac{1}{6}$ and $\frac{1}{8}$.

G. 4.

1. The triangle *ABC* is right-angled at *B*, and has $AB = 12$ cm., $BC = 10$ cm. *P* is a point in *AB* such that $AP = x$ cm., and *PQ* is drawn parallel to *BC* to meet *AC* at *Q*. It is known that the ratio $AP : AB$ equals the ratio $PQ : BC$. Hence express *PQ* in terms of x, and show that the area of the $\triangle APQ$ is $\dfrac{5x^2}{12}$ square cm. Tabulate the values of this area as x takes values from 0 to 12; show the results on a graph, and estimate from your graph the value of x when the area of the triangle *APQ* is exactly half the area of the triangle *ABC*.

2. Find the value of e^{-kt} when $e=2{\cdot}718$, $k=5{\cdot}6$ and $t=5$.

3. A pendulum is set swinging so that the angle it swings through is 10°. The angle of each succeeding swing is $\frac{9}{10}$th of the angle of the preceding swing. Show that after 22 swings the angle it swings through is less than 1°.

[One swing is considered to be the movement from one extreme position to another extreme position on the opposite side of the vertical.]

4. Express $x+\dfrac{(1+x)x}{1-x}+\dfrac{1-x}{1+x}$ as a single fraction.

5. Solve, correct to 3 figures, the equations:

$$\text{(i)}\quad 12x^2-43x+45=10;$$

$$\text{(ii)}\quad 12x^2-43x+44=10.$$

6. The time of oscillation of a simple pendulum varies as the square root of its length. If the time for a pendulum 2 feet long is $1\frac{1}{2}$ seconds, find the time of oscillation of a pendulum one yard long (correct to two figures).

G. 5.

1. A triangle ABC is inscribed in a rectangle as shown in Fig. 91. Find the area of the triangle ABC in the simplest possible form in terms of the lengths p, q, r, s shown in the figure.

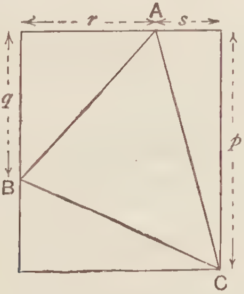


FIG. 91.

2. Simplify

(i) $a^{\frac{5}{6}}\times a^{\frac{1}{6}}$; (ii) $(a^{\frac{2}{3}})^6$; (iii) $\sqrt[3]{a^{\frac{9}{2}}}$; (iv) $\sqrt[3]{ab^2}\times\sqrt{a^2b}$.

3. Find the value of pv^n, when $p=17{\cdot}23$, $v=4{\cdot}56$ and $n=1{\cdot}05$.

4. Show that a triangle whose sides are $\sqrt{5}-1$, $\sqrt{5}+1$ and $2\sqrt{3}$ inches is right-angled, and find its area.

5. A man can scull at 6 miles an hour in still water for an indefinite time. He sculls a distance d miles upstream against a current of x miles an hour, and then sculls the same distance down with the current. Show that the ratio of the time he takes to the time he would have taken if there had been no current is $\frac{36}{36-x^2}$.

6. If a light beam is supported at each end and carries a given load at its middle point, the sag at the middle varies as the cube of the length of the beam. For a beam 6 feet long the sag is $1\frac{1}{2}$ inches. What will it be for a similar beam equally loaded 8 feet long? (Certificate.)

G. 6.

1. If a bookmaker lays odds of m to n against a horse, he will, if the horse wins, pay a man who backs it a sum which bears a ratio $m:n$ to the stake for which it was backed (in addition to returning the stake if already deposited). A man wants to back two horses for which the odds are 2 to 1 against and 7 to 4 against, so as to win exactly £5 on balance if either of the horses wins. Find how much he should put on each horse. How much does he lose if neither horse wins?

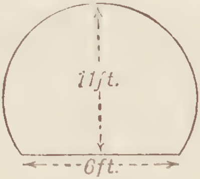


FIG. 92.

2 The section of a tube railway is a segment of a circle as shown in Fig. 92. The greatest height of the tube is 11 feet, and the width at the bottom is 6 feet. Find the radius of the tube.

3. If $\frac{a+b}{a-b}=\frac{c}{d}$, prove that $\frac{c+d}{c-d}=\frac{a}{b}$.

4. Simplify (i) $\frac{4^m \times 3^{n+2}}{2^m \times 2^{m-1} \times 9}$; (ii) $\frac{\sqrt[3]{a^4}}{\sqrt[3]{a^2}}$.

5. Find the value of $\pi a^2 - \pi b^2$ to 3 significant figures, when $\pi = 3{\cdot}142$, $a = 7{\cdot}528$, $b = 1{\cdot}461$.

6. It is found that the current required to fuse a wire of given material varies as $D^{\frac{3}{2}}$, where D is the diameter of the wire. A current of 154·1 ampères fuses a leaden wire of diameter 0·232″; find in ampères, correct to 3 figures, the current which will fuse a leaden wire of diameter 0 316″. (Certificate.)

G. 7.

1. In Fig. 93 ABC is any triangle with $BC=a''$; AN is perpendicular to BC and $AN=p''$; $PQRS$ is a square of side x''. The ratio $AS : AB = SR : BC = x : a$. Hence write down the value of the ratio $BS : BA$; use the fact that $BS : BA = SP : AN$, to obtain an equation connecting a, x and p, and show that

$$\frac{1}{x}=\frac{1}{a}+\frac{1}{p}.$$

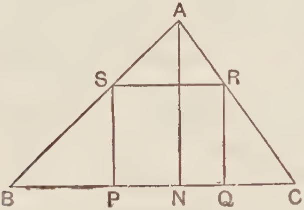


Fig. 93.

2. In the figure of Question 1, express in terms of a, x and p

(i) the area of the triangle ASR ;
(ii) the sum of the areas of the triangles BPS, CQR ;
(iii) the area of the square ;
(iv) the area of the whole triangle.

By equating the sum of the first three of these to the fourth, obtain the same equation connecting a, x, p as in Question 1.

3. (i) If $p\,.\,v^{1\cdot 4}=C$, express v in terms of C and p.

(ii) If $p\,.\,v^{r}=C$, express v in terms of C, p and r.

4. Simplify $\left\{\frac{a+b}{a-b}+\frac{a-b}{a+b}\right\}\div\left\{\frac{a+2b}{a+b}+\frac{a-2b}{a-b}\right\}$.

5. Fill in the gaps in the following:

(i) $\frac{2x}{y}=\frac{6x}{\quad}=\frac{\quad}{5y}=\frac{14x^2}{\quad}=\frac{\quad}{6y^2}$;

(ii) $\frac{a}{b}=\frac{c}{d}=\frac{a+c}{\quad}=\frac{a-c}{\quad}=\frac{\sqrt{ac}}{\quad}$.

6. The weight of the shell of a d-inch gun (*i.e.* a gun whose internal diameter is d inches) is proportional to d^3. Find the internal diameter of a gun whose shell is twice as heavy as the shell of a 4·7 inch gun.

G. 8.

1. A car will travel n miles on a gallon of petrol, which costs half-a-crown a gallon. Find the smallest value of n for which it will be more economical to make a journey by car, allowing for cost of petrol only, than to travel by train third-class at $1\frac{3}{4}$d. per mile.

2. A golf-ball is dropped from a height of 6 feet on a stone floor, and each time it bounces it rises to a height two-thirds of the previous height. Find how many times it will have bounced before the bounces are less than 2 inches.

3. Find the value of $214\sqrt{H^3+0{\cdot}035V^2H^2}$, when $H=0{\cdot}42$, $V=0{\cdot}785$.

4. Find x if (i) $10^x=4{\cdot}619$; (ii) $5^x=125$;
(iii) $\log_{10}x=1{\cdot}6125$; (iv) $\log_7 x=-2$.

5. Solve the equations :

$$\begin{cases} x-y=1{\cdot}6, \\ \dfrac{x^2}{25}+\dfrac{y^2}{16}=1. \end{cases}$$

(Certificate.)

6. It is known that the horse-power required to propel a ship of given pattern varies as $(\text{speed})^3\times(\text{displacement})^{\frac{2}{3}}$.

A ship of 1000 tons displacement is found to require a certain horse-power at 10 knots; by what factor must this horse-power be multiplied in order to propel a similar ship of 8000 tons at 12 knots? (Certificate.)

G. 9.

1. Two trains make a journey of 200 miles. One train goes 10 miles an hour faster on the average than the other and takes an hour less on the journey. Find the speed of each train.

2. Simplify (i) $\dfrac{2^{n+1}+2^n}{2^n-2^{n-1}}$; (ii) $\left(\dfrac{125}{8}\right)^{-\frac{2}{3}}$.

3. If $x:y=3:5$, find the value of

(i) $\dfrac{x+y}{x-y}$; (ii) $\dfrac{x^2+y^2}{x^2-y^2}$.

4. The following table gives values for the pressure and volume of a mass of gas kept at constant temperature :

Volume in c.c.	47·1	44·0	42·4	40·4	38·4	36·3
Pressure in atmospheres	1·47	1·57	1·63	1·71	1·80	1·90

Illustrate these figures by a graph, and read off the volume when the pressure is 1·85 atmospheres.

5. If d is the mean distance of a planet from the sun, then the time of its revolution round the sun or its "year" varies as $d^{\frac{3}{2}}$. If the mean distances of Jupiter and the Earth from the sun are roughly in the ratio 16 to 3, find the length of Jupiter's "year" in days.

6. The central load which breaks a wooden beam l feet long, b inches wide and h inches thick, supported at its ends, varies as $\frac{bh^2}{l}$. A specimen of pitch pine for which $b=1$, $h=1$, $l=2$ just broke under a load of 200 lb.

What load can be carried by a pitch-pine beam 8 feet long, 6 inches wide and 1 foot thick, if the load is not to exceed $\frac{1}{5}$ of the breaking load? (C.S.C.)

G. 10.

1. If y is known to vary inversely as x, complete the following table:

x	4	2	1	$\frac{1}{2}$			0·1	
y	3	6			15	20		100

2. The resistance of a wire varies as the length of the wire and inversely as the square of the diameter. Compare the resistances of two copper wires, one 100 feet long and of diameter $\frac{1}{16}$ inch, the other 50 feet long and diameter 0·1 inch.

3. Given $P=aW+b$. If $P=14$ when $W=6$, and $P=17{\cdot}2$ when $W=8$, find the values of a and b.

4. For what values of x, if any, are the following equations true?

(i) $(2x+1)x=(x+1)x+(x-1)x$;

(ii) $(2x+1)x=(x+1)x+1$;

(iii) $(2x+1)x=(x+2)x+(x-1)x$;

(iv) $(2x+1)(x-1)=x^2+x(x-1)$.

5. OAB are three points in order on a straight line. The distances of O from A and B are a inches and b inches respectively. P is a point in AB dividing AB so that the ratio $AP:PB=k:1$. Find the distance of O from P in terms of a, b and k. (Certificate.)

6. A clear electric lamp is found to illuminate a horizontal desk satisfactorily when it is 6 feet above it. When the glass of the lamp is frosted, its illuminating power is diminished by

25 per cent. If the illumination from a light varies inversely as the square of the distance, how much lower must the frosted lamp be hung to illuminate the desk as much as before ? (Certificate.)

H.

(CHAPTERS XII.-XVI.)

H. 1.

1. (i) For what values of x is the function $(x-5)(x-6)$ equal to 0 ?

(ii) What sort of value does this function have (*a*) when x is large and positive, say $+100$? (*b*) when x is large and negative, say -100 ?

(iii) Between what values of x is this function negative, and what can you say about its value then ?

(iv) Sketch roughly without squared paper the graph of this function.

2. (i) Find a value of x for which the function $\frac{1}{x-6}$ is

(*a*) positive and $< 0{\cdot}001$,

(*b*) negative but $> -0{\cdot}001$.

(ii) Find a value of x for which this function is

(*a*) positive and > 1000,

(*b*) negative and < -1000.

3. Solve the equation $\frac{x}{x+2}+\frac{2x}{2x+1}=1$.

4. Find the value of $a^{\frac{1}{2}}b^{-\frac{3}{4}}$, when $a=0{\cdot}523$, $b=1{\cdot}467$.

5. The weight of a column of mercury in a tube varies as the length and the square of the radius of the tube. If a column 10 cm. long in a tube, diameter 1 cm., weighs about 107 grams, find the weight of a column 45 cm. long and diameter 5 mm.

6. The ratio of a man's income to his expenditure for one year is 5 to 4. Next year his income remains unaltered, but his expenditure is increased by £100, so that the ratio then is 6 : 5. Find his income.

H. 2.

1. Make a table of values of the function $3+4x-x^2$ for values of x from -1 to 5. Draw an accurate graph of the function between these values of x, and from it read off the answers to the following questions:

(i) What is the maximum value of the function?

(ii) For what value of x is the function a maximum?

(iii) Between what values of x is the function positive?

(iv) For what values of x is the function equal to 4?

(v) Solve the equation $3+4x-x^2=6$.

(vi) Solve the equation $x^2-4x-4=0$.

2. Fig. 94 shows a circular arch, height h and base $2a$. If r is the radius of the complete circle, obtain an expression for r in terms of a and h, and find the value of r when $a=50$ ft. and $h=10$ ft.

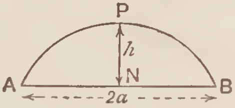


FIG. 94.

3. Find the value of x when

$$\text{(i)}\ 2^x=4^3;\qquad \text{(ii)}\ 9^x=(3^3)^{\frac{2}{5}};\qquad \text{(iii)}\ 10^x=\frac{1}{\sqrt{10}}.$$

4. Find correct to 2 significant figures two numbers whose sum is 2, such that the sum of their squares is 3.

5. If $\dfrac{x}{y}=\dfrac{7}{10}$, find the value of

$$\text{(i)}\ \frac{x+y}{x-y};\qquad \text{(ii)}\ \frac{x+2y}{2x-y}.$$

6. The force of attraction between two magnetic poles varies inversely as the square of the distance between them. If two magnetic poles attract each other with a force of 5·6 dynes, when 5 cm. apart, find the force of attraction between them when they are 7·5 cm. apart.

H. 3.

1. For what values of x is the function $\dfrac{x^2}{100}=0$? Can the function $\dfrac{x^2}{100}$ ever be negative? What is its value when x is large, *e.g.* when $x=\pm 100$? Sketch roughly the graph of this function.

2. The volume of a gas varies as the absolute temperature and inversely as the pressure. When the pressure is 15 lb. per square inch and the absolute temperature is 280°, the volume of a gas is 200 cubic inches. Find its volume when the pressure is 10 lb. per square inch and the absolute temperature 300°.

3. In Fig. 95, semi-circles are drawn on AN, NB and AB as diameters. NP is perpendicular to AB, and it is known that NP is a mean proportional between AN and NB. Prove that the area shaded in the figure is equal in area to a circle drawn on PN as diameter.

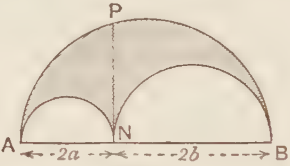


FIG. 95.

4. The ratio of the incomes of two men is 5 to 4. The income of each is raised by £150 a year and the ratio of their incomes is then 11 to 9. Find their incomes before they were raised.

5. Plot the graph of the function $\frac{x^2}{2}+\frac{1}{2x}$ from $x=3$ to $x=-3$, and from it read off the roots of the equation $\frac{x^2}{2}+\frac{1}{2x}=3$.

(Certificate.)

6. (i) Find the error in assuming $\frac{1-x}{1+x}$ to be equal to $1-2x$ if $x=0{\cdot}1$.

(ii) Find the ratio of the error to the correct value.

(iii) Find the "percentage error."

Give all answers correct to two significant figures.

H. 4.

1. Find a general statement to include the following:

the ratio 3 : 4 is greater than the ratio 2 : 3;
the ratio 6 : 7 is greater than the ratio 5 : 6;
the ratio 100 : 101 is greater than the ratio 99 : 100.

Find a value of x for which the ratio $\frac{x}{x+1}$ is greater than 0·999.

2. What is the value of the function $\frac{x}{x+1}$, (*a*) when $x \to +\infty$, (*b*) when $x \to -\infty$, (*c*) when $x \to -1$, but is < -1, (*d*) when $x \to -1$, but is > -1? For what value of x is the function zero? Draw a rough graph of this function without using squared paper.

3. Programmes are sold at a charity concert for "a shilling and anything more you like to give." A programme seller finds that everyone gives him either a shilling or a halfcrown for a programme. He sells 64 programmes and takes in all £5. 13s. 6d. Find how many people paid half a crown for their programme and how many paid a shilling.

4. For what values of x, if any, are the following true ?

(i) $(x+1)^2+(x+2)^2=2(x+1)(x-2)+1$;

(ii) $(x+1)^2+(x+2)^2=2(x+1)(x+2)+2$.

5. Find the value of $4{\cdot}75H^3d^{-4}$, when $H=1{\cdot}723$, $d=0{\cdot}1614$.

6. Sketch the time-velocity graph of a train which starts from rest and gathers speed at a uniform rate for five minutes, after which it travels steadily at 40 miles an hour for ten minutes and then slows down uniformly so as to come to rest in five minutes.

Unit for time-axis, 1″ : 5 minutes;
unit for velocity axis, 1″ : 20 miles an hour.

How far has it travelled altogether ? Suppose the graph has been accurately drawn with these units. How would a person describe the motion who thought

the time-axis unit was 1″ : 2 minutes
and the velocity-axis unit was 1″ : 5 miles an hour ;

and what would he give as the total distance travelled ?

H. 5.

1. Sketch roughly the graph of the function $x(x-2)(x-4)$.

2. Draw accurate graphs of the functions $\frac{x^2}{10}$ and $\frac{1}{x}$ on the same piece of paper and with the same scales, for values of x between 6 and -6.

For what values of x are these two functions equal ? Read off this value, (i) from the graph, (ii) using log tables.

3. If $f(x)=x^2+7x-6$, find the values of $f(1)$, $f(0)$, $f(2a)$ and $f(x+1)$.

4. Lidless rectangular tins with a square base are to be made to hold 12 cubic inches.

(i) Express the height, h'', as a function of the side, x'', of the base.

(ii) Express the surface, S square inches, as a function of h and x, and then of x only, showing that $S=x^2+\frac{48}{x}$.

Plot the values of S when x has values from 1 to 6, and hence find the value of x for which S is a minimum.

5. Simplify $\dfrac{3(x+h)^3-3x^3}{h}$, and find its value when $x=1$, $h=0{\cdot}001$, correct to 3 figures.

6. Find the error in assuming $(1+x)^{-\frac{1}{2}}=1-\frac{1}{2}x$, when $x=0{\cdot}1$. Find also the percentage error. Give both answers correct to one significant figure.

H. 6.

1. If $f(x)=x^2-\dfrac{10}{x}$, what is the value of $f(1), f(10), f(0{\cdot}1)$? Has $f(0)$ any meaning ? What are $f(-0{\cdot}1)$, $f(-0{\cdot}01)$?

2. (i) When is the function $\dfrac{x-3}{x+3}$ equal to zero ?

   (ii) Find values of x for which this function is (*a*) > 1000, (*b*) < -1000.

   (iii) To what value does this function tend

   (*a*) when $x \to +\infty$?

   (*b*) when $x \to -\infty$?

   (iv) Sketch the graph of this function.

3. The following table gives the heights in feet of certain points on a road above sea-level, and their horizontal distances in yards from the starting point:

Height h	100	112	127	139	140	138	130
Distance d	0	50	100	150	200	250	300

Show these results on a graph. Calculate the average gradient

(*a*) for the first 50 yards horizontally,

(*b*) for the first 100 yards horizontally.

By drawing a tangent, estimate the gradient at the point 100 yards distant from the starting point.

4. The strength of a rectangular beam of given material varies as the product of the breadth and the square of the depth of the cross-section. Find the breadth of a beam 4 inches deep which is twice as strong as a beam of the same material which is 4 inches broad and 3 inches deep. (Certificate.)

5. The area of a right-angled triangle is 7·5 square inches and its hypotenuse is 6·5 inches. Find the lengths of its sides.

6. Simplify and find the value, correct to 3 figures, of

(i) $\dfrac{\sqrt{32}}{2\sqrt{2}}$; (ii) $\dfrac{\sqrt[3]{81}}{\sqrt[3]{3}}$; (iii) $\dfrac{10}{\sqrt{5}}$; (iv) $\sqrt{6}\times\sqrt{30}$.

[Take $\sqrt{5}=2{\cdot}236$.]

H. 7.

1. An underground train travels from one station to the next, a distance of 1000 yards, in 80 seconds in accordance with the following table:

Time in seconds	10	20	30	40	50	60	70	80
Distance travelled in yards	50	135	260	435	610	780	900	1000

Draw a graph to illustrate these numbers.

Calculate (i) the average speed for the whole journey;
(ii) the average speed for the first 20 seconds;
(iii) the average speed for the period 20 sec. to 40 sec.

By drawing a tangent, estimate the actual speed 20 seconds after the train started. Give all results in the form feet per second.

2. Sketch roughly without squared paper the graph of the functions (i) $\pm\sqrt{x^3}$, (ii) $\pm x^{\frac{5}{2}}$.

3. If $f(x)=\frac{x^2}{2}+1+\frac{1}{2x^2}$, (i) prove that $f(x)=f\left(\frac{1}{x}\right)$, and write down another function which has this property. (ii) Also show that $f(x)=f(-x)$. What does this tell you about the graph of the function?

4. If $x^3=175t^2$ and $t^5=y$, express x in terms of y only, and find the value of x, when $y=0{\cdot}0162$, correct to 3 figures.

5. The sides of a rectangle are in the ratio 3 : 2, and its area is 54 square inches. Find its sides.

6. Find a general statement to include the following facts:

the ratio 36 : 25 is greater than the ratio 6 : 5,
the ratio 81 : 49 is greater than the ratio 9 : 7,
but the ratio 16 : 49 is less than the ratio 4 : 7,
and the ratio 9 : 25 is less than the ratio 3 : 5.

H. 8.

1. Plot values of the function $6x-x^2$ as x varies from -1 to $+7$ and draw an accurate graph.

What is the change in the function per unit change in x

(i) as x changes from 1 to 3?
(ii) as x changes from 1 to $1{\cdot}1$?
(iii) as x changes from 1 to $1+h$?

What is the limit of the last expression as $h\to 0$?

Draw a tangent to the graph at the point given by $x = 1$; estimate its gradient, and compare with the result obtained by calculation.

2. If $f(x) = x^2 + 6$, find the value of $f(1)$ and $f(-1)$.

Write down the value of $\dfrac{f(x+h) - f(x)}{h}$, and simplify this expression. To what limit does it tend, when $h \to 0$?

3. The expenses of a hotel-keeper are partly constant and partly vary directly as the number of visitors in the hotel. When there are 10 visitors, the weekly expenses are £55, and when there are 30 visitors, the weekly expenses are £135. Express the weekly expenses as a function of the number (n) of visitors in the hotel.

4. Sketch free-hand the graph of a function, the *average* gradient of which is given in the following table:

x	0	0 to 1	1 to 2	2 to 3	3 to 4	4 to 5	5 to 6	6
Average gradient	0	1	2	$\frac{1}{2}$	-1	-2	$-\frac{1}{2}$	0

The graph is to be a smooth curve.

5. A line AB, 4″ long, is produced to Q so that $AQ : BQ = 5 : 4$; find the length of AQ.

6. If $x = \dfrac{a-1}{a+1}$ and $y = \dfrac{2a-1}{2a+1}$, find a in terms of x only, and then in terms of y only, and so obtain an equation connecting x and y and not involving a.

H. 9.

1. Make a table of values of the function $y = x^2 + 2x - 3$ from $x = -4$ to $x = +2$; draw an accurate graph, and read off answers to the following questions:

- (i) What is the minimum value of the function?
- (ii) For what value of x is the function a minimum?
- (iii) Between what values of x is the function negative?
- (iv) For what values of x does the function equal 4?
- (v) Solve the equations: (*a*) $x^2 + 2x - 3 = 1$,
  (*b*) $x^2 + 2x = 2$.

2. With the function and graph of Question 1, what is the average change in x per unit change in y

- (i) When x changes from -4 to -2?
- (ii) When x changes from -3 to -2?
- (iii) When x changes from $-(2+h)$ to -2?

(iv) What is the limit of this last expression as $h \to 0$?

(v) Draw a tangent to the graph at the point given by $x = -2$, and estimate its gradient. Compare with the answer to part (iv).

3. If $y = f(x) = \dfrac{ax+b}{cx-a}$, prove that $x = f(y)$.

4. Solve the equation $4{\cdot}5x^2 + 3x - 6{\cdot}7 = 0$, giving results correct to 3 significant figures.

5. Simplify
(i) $(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})$;
(ii) $(\sqrt{5}+\sqrt{2})^2$;
(iii) $(\sqrt{5}+\sqrt{2})^{-1}$,

and give the values of these expressions correct to 3 significant figures.

6. If $f(x)$ denotes the greatest integer in x, sketch the graph of (i) $f(x)$, (ii) $x - f(x)$, when x is positive.

H. 10.

1. Make a table of values of the function $\frac{1}{10}(x^3 - 4x)$ for values of x from -4 to $+4$; draw an accurate graph of the function, and read off answers to the following questions:

(i) For what positive value of x is the function a minimum?

(ii) What is the maximum value of the function for negative values of x?

(iii) For what values of x is the function negative?

(iv) For what values of x does the function equal $0{\cdot}2$?

(v) Solve the equations
(a) $\frac{1}{10}(x^3 - 4x) = 0{\cdot}1$,
(b) $x^3 - 4x = 1$,
(c) $x^3 - 4x + 1 = 0$.

2. Find the gradient of the graph of Ex. 1 at the point given by $x = 2$,
(a) by drawing a tangent to the graph,
(b) by calculation.

3. Sketch roughly the graph of the function $(x-4)^2(x-3)$.

4. $f(x)$ is a function of x such that $f(x) \times f(y) = f(x+y)$, for all values of x and y; and $f(1) = 2$. Prove that $f(2) = 4$, $f(3) = 8$, and find the values of $f(0)$ and $f(\frac{1}{2})$.

What is the simplest form of this function of x?

5. Prove that if a, b, c are three consecutive integers, then $b^2 = ac + 1$.

6. A sum of a guinea can be made up of either a shillings and b florins, or of a sixpences and b half-crowns. Find the values of a and b.

K.

(CHAPTERS XII.-XVII.)

K. 1.

1. Soundings are made on a sea-shore at low tide to discover the shape of the sea-bottom, and the following measurements are recorded:

Distance from low water mark in yards - - - - -	50	100	150	200	250	300
Depth of water in fathoms - -	0·7	2	4	4·5	5	5·1

Draw a graph to illustrate these figures.

Estimate the average gradient of the sea-bottom

(i) For the first 300 yards out.

(ii) Between points 100 yards and 250 yards out.

(iii) Draw a tangent and find the gradient at a point 150 yards out. [1 fathom = 6 feet.]

2. Find the value of (i) $\underset{h \to 0}{\text{Lt}} \frac{1}{h}\{(x+h)^2 + 4(x+h) - x^2 - 4x\}$.

(ii) $\underset{h \to 0}{\text{Lt}} \frac{1}{h}\left\{\frac{1}{x+h-1} - \frac{1}{x-1}\right\}$.

3. Find, by any method, the gradient of the graph of the function $x^3 - 5x$.

(i) At the point where $x = 1$.

(ii) At the point where $x = a$.

4. Sketch roughly a graph of the function $(x-2)^3(x-4)$.

5. If $x = t + 1$ and $y = t^2 + t$, express (i) t as a function of x, (ii) y as a function of x only. For what values of t are x and y equal?

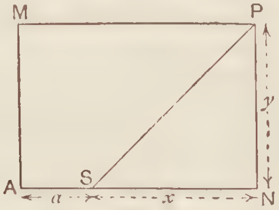


FIG. 96.

6. In Fig. 96 AS is a constant length a. P is a variable point, $PNAM$ is a rectangle such that $PS = PM$. If $PN = y$ and $NS = x$, find an equation connecting x and y, and express y as a function of x.

K. 2.

1. Draw roughly with the same axes, the graphs of

(i) $y=3-x$. (ii) $y=5-x$. (iii) $y=5$.

What are the gradients of these lines ?

2. Differentiate the following functions with respect to x.

(i) $4x-3x^2$. (ii) $3+4x-3x^2$.
(iii) $6+4x-3x^2$. (iv) $\frac{3}{2}x+\frac{2}{5}x^2$.
(v) $6x^{-1}$. (vi) $8\sqrt{x}$.

3. (i) Find the gradient of the graph $y=7+6x+3x^2$ at the point $x=2$.

(ii) At what point is the gradient equal to 1 ?

4. y is known to be a function of x of the type $mx+c$, m and c being constants. If $y=43$ when $x=17$, and $y=34$ when $x=14$, express y as a function of x.

5. Prove that, if a, b and n are all positive then the ratio $(a+n):(b+n)$ is greater or less than the ratio $a:b$ according as a is less or greater than b.

[Consider the value of $\frac{a+n}{b+n}-\frac{a}{b}$, and simplify it.]

6. With the figure and data of paper E. 2, Question 2, find the maximum sag of the rod. Find also the inclination of the rod to the horizontal at the support B.

K. 3.

1. Find $\frac{dy}{dx}$ (i) when $y=4x^2-3+\frac{1}{x}$,

(ii) when $y=16x^3-4x^2+2x-5$,

(iii) when $y=5x^{\frac{1}{3}}+x^{-\frac{1}{3}}$.

2. Find the gradient at $x=a$ of the graph

$$y=x^3-6x^2-15x+5.$$

Find for what values of x the gradient of this graph is zero. Sketch the graph roughly.

3. Sketch roughly the graph of the function $(x-3)\sqrt{x-2}$, where the positive value of $\sqrt{x-2}$ is always taken.

4. The distance, s feet, of a particle moving in a straight line from a fixed origin is given by $s=4+6t+3t^2$, where t is the number of seconds for which it has been moving. Find its velocity, (i) 2 seconds, (ii) 10 seconds after it started.

5. A particle moves in a straight line in such a way that its velocity after any time t varies inversely as its distance x from the origin. Write down a differential equation connecting $\frac{dx}{dt}$ and x.

6. If $f(x) \equiv x^3 - 5x^2 + x + 7$ and if $f(y+a)$ when expanded contains no term in y^2, find a.

K. 4.

1. Find the value of (i) $\underset{h \to 0}{\text{Lt}} \left\{ \frac{\frac{1}{(x+h)^2 - 2} - \frac{1}{x^2 - 2}}{h} \right\}$.

(ii) $\underset{n \to \infty}{\text{Lt}} \frac{n^2 + 1}{2n^2 + 1}$.

2. Differentiate the following with respect to x:

(i) $\sqrt{6x}$. (ii) $6\sqrt{x}$. (iii) $\frac{4}{x^3}$. (iv) $x^2 + \frac{1}{x^2}$.

3. For what values of x is the function $x^3 - 12x$ equal to zero? Find its derived function and find for what values of x the derived function is zero. From these data sketch a rough graph of the function.

4. There are 15 cubic feet of water in a bath. The water is allowed to run out and after t seconds θ cubic feet of water have run out. If the rate at which the water runs out is proportional to the quantity of water left in the bath, find a differential equation connecting $\frac{d\theta}{dt}$ and θ.

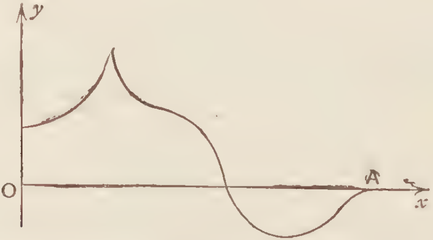


FIG. 97.

5. Fig. 97 represents the graph of $y = f(x)$, unit 1 cm. on each axis; describe in general terms the variation in the values of (i) y; (ii) $\frac{dy}{dx}$; (iii) $\frac{d^2y}{dx^2}$ as x varies from 0 to OA.

6. The force of attraction between two planets varies directly as their masses and inversely as the square of the distance between them. Compare the force with which the planets Jupiter and Mars attract the earth, at a time when Jupiter is 500 million miles away, and Mars is 150 million miles away, if the mass of Jupiter is about 340 times that of the Earth, and the mass of Mars is about one-seventh that of the Earth.

K. 5.

1. Differentiate the following functions of t with respect to t:

(i) $4t+\frac{4}{t}$. (ii) $3t^3-4t^{-1}+5t^{-2}$. (iii) $5t^{\frac{5}{2}}$.

2. Differentiate the function x^3-27x. For what values of x is its rate of change zero ? What are the maximum and minimum values of this function. Sketch its graph.

3. The weight of a certain solid is given by the formula

$$10x^2(15-2x),$$

where x is variable. Show that as x increases from 0 to 5 the solid gets steadily heavier. What happens after that ?

4. If $R=0{\cdot}0075(1-e^{-0{\cdot}2t})$, find R when $t=10$, if $e=2{\cdot}718$. Find also the value of R when $t=20$.

What is the average rate of change of R per unit change in t, as t changes from 10 to 20.

5. If $f(x)=x^2+3x$, find the value of

(i) $\frac{f(x+h)-f(x)}{h}$.

(ii) $\frac{f(x+h)-f(x)-\{f(x)-f(x-h)\}}{h^2}$.

Find the limits to which these two expressions tend, as $h \to 0$.

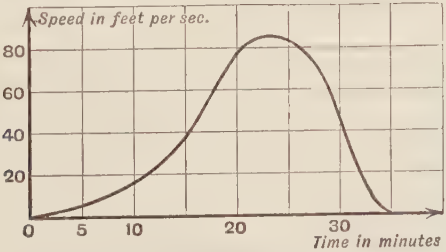


FIG. 98.

6. Fig. 98 represents the time-velocity graph of a train, units as shown on diagram.

Construct the distance-time graph and write down the distance travelled, (i) at the end of 15 minutes, (ii) in the whole journey.

K. 6.

1. Find the turning points of the function $x^3 + 3x^2 - 9x + 6$ and determine whether they are maxima or minima.

2. Show that the function $x^3 + 5x - 6$ steadily increases, as x increases from $-\infty$ to $+\infty$.

One root of the equation $x^3 + 5x - 6 = 0$ is $x = 1$ Show why it cannot have another real root.

3. A piece of wire 8 inches long is bent so as to form a rectangle. If one side of the rectangle is x'', express the area as a function of x, and find the value of x for which the area is a maximum.

4. (i) Express $5(p^2 + q^2)$ as the sum of two squares.

(ii) Express 5^4 as the sum of two squares.

5. AB is a straight line 6 inches long. M is a point on the line between A and B, and N is a point in AB produced, such that the ratio $AM : MB = AN : BN = 6 : 4$. Find the length of MN.

6. The distance, s feet, that a stone falls is proportional to the square of the time, t seconds, for which the stone has been falling. Express this fact as an equation connecting s and t, and prove that the velocity of the stone varies directly as the time for which it has been falling.

K. 7.

1. Differentiate the function $5 - 3x - 2x^3$ and show that the function steadily decreases as x varies from $-\infty$ to $+\infty$, and that it can only be zero, when $x = 1$. Draw roughly the graph of this function.

2. A particle moves along the straight line OA. B is a point on OA such that $OB = 5$ feet, and the particle starts from B and moves away from O in such a way that its velocity is proportional to its distance from O. If its initial velocity is 10 feet per second, find an equation connecting $\frac{dx}{dt}$ and x, where x feet is its distance from the starting point B, and t seconds is the time for which it has been moving.

3. The volume of a cone is $\frac{1}{3}\pi r^2 h$ and its curved surface is πrl, where r is the radius, h the vertical height and $l\ (= \sqrt{h^2 + r^2})$ the slant height of the cone. A conical tent is to be made to hold 400 cubic feet of air. Express its height as a function of r, and then express its surface S, as a function of r. Find the value of S, if $r = 7$. For what value of r is s least ?

4. Draw a rough sketch of the graph of the function $18x^2 - x^4$, and find its maximum and minimum values. (Certificate.)

5. If $x : y = 7 : 6$, find the value of

$$\text{(i)}\ \frac{3x+5y}{6x+4y}. \qquad \text{(ii)}\ \frac{x^2+y^2}{x^2-y^2}.$$

6. The tension of a wire of given material varies as the square of its length if it is to produce a given musical note. What must be the tension in a wire one foot long to produce the same note as a wire 9″ long at a tension of 11 lb.?

K. 8.

1. Tabulate the values of the function $x + \frac{4}{x}$, as x varies from -4 to $+4$. Draw an accurate graph of this function and read off the values of x for which this function is a maximum or minimum.

Verify your answers by differentiating the function, and calculating the values of x required.

2. Draw a tangent to your graph (Question 1) at the point $x = 3$, and estimate its gradient. Check your answer by calculating the value of $\frac{dy}{dx}$ when $x = 3$.

3. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$,

(i) when $y = 3x^3 - 7x$; (ii) when $y = 5x^2 - 4$;
(iii) when $y = 7x + 6$; (iv) when $y = 5$;
(v) when $y = \frac{6}{x}$.

4. Draw a rough sketch of the graph of the function $75x - 4x^3$ and find its maximum and minimum values. (Certificate.)

5. A line AB, 4 inches long, is divided at the point P, in such a way that the square on AP is 1 square inch larger than the square on PB. Find the length of AP.

6. A rectangular plate 20 inches long and $3x$ inches wide has a circular hole, radius x inches, cut out of it. Express the area of the remaining portion as a function of x, and find the value of x for which the area remaining is a maximum.

K. 9.

1. (i) Is $\frac{x^2-1}{x-1}$ always equal to $x+1$?

(ii) Can you find a value of x for which $\frac{x^2-1}{x-1}$ equals 2 ?

(iii) Find the value of $\underset{x \to 1}{\text{Lt}} \frac{x^3-1}{x-1}$.

2. A bullet fired vertically upwards rises s feet in t seconds, where $s = 500t - 16t^2$. (i) Find its height after 10 sec., 11 sec., 10·1 sec. ; (ii) What is its average speed in the intervals 10 to 11 sec. and 10 to 10·1 sec. ? (iii) What is its average speed in the interval 10 to $10+h$ sec. ? (iv) What is its velocity after 10 sec. ?

3. If $y = 2x^2$ and $x = 3z + 1$, express (i) δy in terms of δx ; (ii) δx in terms of δz ; (iii) δy in terms of δz. Hence find $\frac{dy}{dz}$.

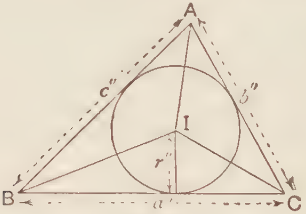


FIG. 99.

4. I is the centre and r the radius of a circle inscribed in the triangle ABC. Show by considering the areas of the triangles AIB, BIC, CIA, that the area of the whole triangle is equal to rs, where $s = \frac{a+b+c}{2}$. Hence find the radius of the circle inscribed in the right-angled triangle whose sides are 3″, 4″ and 5″.

5. If y varies as x^3, complete the table :

$x=2$	4	30
$y=5$		

6. Express $\left(\frac{1}{x} - \frac{b}{a}\right) \div \left(\frac{1}{x} + \frac{a}{b}\right)$ in a form not containing x, $a(1-x) = b(1+x)$.

K. 10.

1. If $\frac{a}{xy}=\frac{b}{x^2}=\frac{c}{yz}$, express the ratio $x:y:z$ in terms of a, b, c.

2. The sides of a triangle are 5, 5, $2x$ cm.; its area is A sq. cm. Express A as a function of x. Represent this function by a graph and find from the graph the maximum area of the triangle. (C.S.C.)

3. A hoop of radius b feet is bowled along with velocity v feet per sec. and comes to a step of height h feet. The hoop will climb the step if $v^2 > \frac{128bh}{(2b-h)^2}$. If the velocity of the hoop is 8 feet per sec., and if its radius is 18 inches, find to the nearest inch the height of the tallest step it can climb. (Certificate.)

4. Ox, Oy are horizontal and vertical lines: the graph of $y=x^2$ represents a hill side on the scale, unit for x-axis $=100$ yards, unit for y-axis $=1$ foot. What is the average slope of the hill, (i) from $x=1$ to $x=2$; (ii) from $x=1$ to $x=1\cdot1$; (iii) from $x=1$ to $x=1+h$. What is the gradient of the hill at $x=1$?

5. A, B, C are three points on Ox; APB is a semicircle above Ox and BQC a semicircle below OX; the two semicircles form part of the graph of $y=f(x)$. Describe the changes of sign of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ between A and C.

6. An open cardboard box with square ends is fitted with an overlapping lid which covers the open top, the whole of two square ends and one other side. The total area of the cardboard is 8 sq. feet. What is the maximum volume of the box?

K. 11.

1. The highest and lowest marks in an examination were 72 and 17. These were scaled so that the highest and lowest were 200 and 100. What was the mark corresponding to an original mark of 53?

2. Prove that $2^{\sqrt{3}} - 2^{-\sqrt{3}} \overset{\frown}{=} 3$.

3. A road of constant width curves through a right angle, the walls being concentric circular arcs AB, CD. The radius of the inner curve is 60 yards and it is just possible to see D from C. How wide is the road, to the nearest yard? (Certificate.)

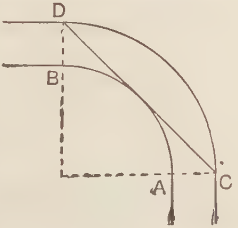


FIG. 100.

4. The electromotive force of a certain type of cell has been found to vary with the temperature as follows:

Temperature (Centigrade)	15	20	25	30
E.M.F. in volts - - -	1·4340	1·4284	1·4233	1·4188

What is the average fall in E.M.F. per degree of temperature, when the temperature rises from (i) 15° C. to 30° C.; (ii) 15° C. to 25° C.; (iii) 15° C. to 20° C.? Illustrate the table by a graph and by drawing a tangent estimate the rate of fall of E.M.F. at 20° C.

5. Find $\frac{dy}{dx}$ if (i) $y = \frac{x^3 - 1}{x}$; (ii) $y^3 = 2x^2$.

6. If a steamer travels at v knots, the cost of a certain journey is $8\left(\frac{1500}{v} + 0{\cdot}4v^2\right)$£. What is the most economical speed?

K. 12.

1. (i) If y varies inversely as the square of x, what is the effect on y of increasing x in the ratio 4 : 1?

(ii) The illuminating power of a light varies inversely as the square of the distance. Which gives the better light, a 10 candle power lamp, 3 feet away, or an 18 candle power lamp, 4 feet away?

2. Simplify the result of substituting $x = \frac{3ab}{b - a}$ in

$$\frac{1}{x - 2a} + \frac{2}{x + b} + \frac{1}{b}.$$

3. From the formula $p = 475v^{-1{\cdot}06}$, find p when $v = 0{\cdot}874$.

4. (i) Plot the function $x + (-1)^x$ for positive integral values of x.

(ii) What is the least integral value of x, for which and for all greater values of x, $x + (-1)^x > 100$?

(iii) What is $\underset{x \to \infty}{\text{Lt}} [x + (-1)^x]$?

5. If $y = ax^2 + bx$, prove that $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$.

6. ABP is a triangle, right-angled at B, having $AB > BP$; $AB = 12''$; the perpendicular bisector of AP cuts AB at Q. Find the length of AQ if triangle PBQ is of maximum area.

K. 13.

1. A regular pyramid, height h in., stands on a square base of side $2x$ in. and is used as a metal funnel to hold 48 cu. in. Assume the formulae: volume $=\frac{4}{3}hx^2$; surface of slant sides $=y=4x\sqrt{x^2+h^2}$; and find the values of h and y when $x=1, 2, 3, 4, 5$. Plot y against x and find from the graph the value of x which makes y as small as possible. (C.S.C.)

2. Use calculus methods to solve Question 1.

3. The following table gives the best wheel bases for trucks which have to travel on a curved track.

Radius of curve in feet $=r$	20	30	40	50	60
Wheel base in inches $=b$	66	69		75	78

What simple relation connects b and r? Complete the table.

4. (i) Find the circumference of a circle whose area is 3·85 sq. in.

(ii) The area of a circle is x sq. in., the circumference is y in.; express y as a function of x.

5. For what values of x is the function $108x-x^4$, (i) positive, (ii) an increasing function, (iii) a maximum? Sketch its graph.

6. A closed vessel which tapers to a point at its top and bottom is such that when the depth of liquid in it is x feet, the volume of the liquid is $12x^2-x^3$ cu. feet. Find (i) its internal height; (ii) the area of its cross-section halfway up; (iii) the total volume.

K. 14.

1. If $\frac{x}{3}=\frac{y}{5}$, prove that $\frac{x^2-y}{x-y^2}=\frac{3x-5}{3-5y}$.

2. Kepler's Third Law states that the square of the time a planet takes for one revolution round the sun varies as the cube of its mean distance from the sun. In the following table, the mean distance of the earth from the sun (*i.e.* 93,000,000 miles) is taken as the unit of distance.

	Earth	Mercury	Venus	Mars	Jupiter	Uranus
Time in years	1	0·24	0·62	1·88		
Mean distance	1	0·38		1·52	5·20	19·18

Verify Kepler's Law and complete the table.

3. (i) Solve $x\sqrt{3} - x = 4$.

(ii) Find the value of $x - \frac{1}{x}$ if $x = \sqrt{2} - 1$.

4. Use logarithm tables to evaluate $x \times \delta(\log x) \div \delta x$, when (i) $x = 5$, $\delta x = 0{\cdot}1$; (ii) $x = 7$, $\delta x = 0{\cdot}1$; (iii) $x = 9$, $\delta x = 0{\cdot}1$. What do you infer from these results ?

5. Draw freehand the graph of a function $y = f(x)$ for which

(i) over a portion AB $\frac{dy}{dx}$ is positive and $\frac{d^2y}{dx^2}$ is negative ;

(ii) over a portion CD $\frac{dy}{dx}$ is negative and $\frac{d^2y}{dx^2}$ is positive.

6. A circular electric current of radius 1 foot exerts a force P on a small magnet placed with its axis perpendicular to the plane of the wire and at a distance x feet from the centre where P varies as $\frac{x}{(1 + x^2)^{\frac{5}{2}}}$. For what value of x is P greatest ?

K. 15.

1. The weight of rails, W lb. per yard, necessary when the maximum velocity is v miles per hour and the greatest load on the driving wheel is L tons, is given by

$$W = 17\sqrt[3]{\{L + 0{\cdot}0001Lv^2\}^2}. \quad \text{Find } W \text{ if } L = 8{\cdot}3,\ v = 47.$$

2. If V varies directly as the square of x and inversely as y, complete the given table.

Values of x.

		1	2	3
Values of y	1			
	2			270
	3		80	

3. If $\frac{1}{x} + \frac{1}{y} = \frac{2}{r}$, express $\frac{1}{r - x} + \frac{1}{r - y}$ in terms of x, y.

4. From the given table evaluate $\frac{1}{10^x} \times \delta(10^x) \div \delta x$ for $10^x = 5$ and $10^x = 6$ and $10^x = 8$:

10^x	5	5·0001	6	6·0001	8	8·0001
x	0·6989700	0·6989787	0·7781513	0·7781585	0·9030900	0·9030954

What do you infer from these results ?

5. If $xy = 5$, find

(i) $\dfrac{dy}{dx}$ in terms of x; (ii) $\dfrac{dx}{dy}$ in terms of x; (iii) $\dfrac{dy}{dx} \times \dfrac{dx}{dy}$.

6. What is the greatest value of $3x - 5x^2y$ if $xy^2 = 4$ and y is positive?

K. 16.

1. (i) Express with positive indices $\dfrac{a^2b^{-3}}{2}$.

(ii) Evaluate $8^{1\frac{1}{3}}$; $16^{1\frac{1}{4}}$; $(0{\cdot}25)^{-\frac{1}{2}}$; $10^{-3} \times 10^{-1}$.

2. A man pays income tax only on part of his income; the tax is y per cent. of the portion subject to tax, which works out as equivalent to x per cent. of his total income. Find the ratio of the portion subject to tax to the portion free of tax.

3. The perimeter of a right-angled triangle is 20 cm.; the lengths of the two shorter sides are a cm., b cm., and the area is A sq. cm. (i) Prove that $(20-a)(20-b) = 200$; (ii) express A in terms of a, and draw a graph showing the relation between A and a as a varies from 0 to 10 cm.; (iii) find from the graph the maximum area of the triangle. (C.S.C.)

4. If $f(x) = x^2 - 3x + 5$, (i) express $\dfrac{f(x+h) - f(x)}{h}$ in terms of x, h; (ii) find the limit of this expression when $h \to 0$; (iii) for what value of x is this limit zero; (iv) interpret the results of (i), (ii), (iii) geometrically.

5. If $y = \dfrac{1}{z}$ and $z = 1 - x^2$, (i) express δy in terms of z, δz; (ii) express δz in terms of x, δx; (iii) express δy in terms of x, δx; (iv) find $\dfrac{d}{dx}\left(\dfrac{1}{1-x^2}\right)$.

6. The function $y = 3x + \dfrac{a}{x}$ decreases as x increases from 0 to 5, and increases as x increases from 5 to 10. (i) What is the value of a? (ii) what is the value of $\dfrac{d^2y}{dx^2}$ when $x = 5$ and $x = -5$? (iii) find maximum and minimum values of the function, distinguishing between them; (iv) explain your results by a sketch of the function.

L.

(CHAPTERS XII.-XVIII.)

L. 1.

1. A beam AB supported at its ends A, B carries a load at P. The greatest load the beam can stand varies directly as its breadth, its length and the square of its depth, and inversely as AP and PB. The breaking load for a chestnut beam 20 feet long, 8 in. wide, 5 in. deep at 2 feet from its mid-point is $2\frac{3}{4}$ tons. Obtain a general formula for any chestnut beam.

2. In what ratio is the expression $\dfrac{0{\cdot}043x^3y}{x+y}$ altered if x and y are each altered in the ratio 2 : 3 ?

3. (i) Describe the changes of sign in the function $\dfrac{3(x-2)(x-4)}{(x-1)(x-5)}$ when x increases from -1 to 6.

   (ii) Find correct to one significant figure its value when $x = 1{\cdot}0001$.

   (iii) Can you find a value of x for which the function equals 3 ?

   (iv) Find by logarithms the value of the function when $x = 1000$.

   (v) Simplify $\dfrac{3(x-2)(x-4)}{(x-1)(x-5)} - 3$, and evaluate

$$\operatorname*{Lt}_{x\to\infty} \frac{3(x-2)(x-4)}{(x-1)(x-5)}.$$

   (vi) Sketch the graph of the function for positive values of x.

4. Find correct to two significant figures the value of x for which $\sqrt{x} - x^{1{\cdot}06}$ is a maximum.

5. (i) Solve $\dfrac{dy}{dx} = x - \sqrt{x}$, given $y = 1$ when $x = 1$.

   (ii) Solve $\dfrac{d^2y}{dx^2} = \dfrac{1}{x^3}$, given $y = 4$ when $x = \frac{1}{2}$ and $y = -2$ when $x = -\frac{1}{2}$.

6. Find the values of (i) $\int\left(5x^2 + \dfrac{4}{x^2}\right)dx$; (ii) $\int\left(x^{1{\cdot}2} + \dfrac{1}{x^{1{\cdot}2}}\right)dx$.

L. 2.

1. Fig. 101 represents the end-view of a circular cylinder on a table between two rectangular blocks, distant d in. apart; the smaller is of height h in. and the radius of the cylinder is r in. Find r in terms of d, h.

(C.S.C.)

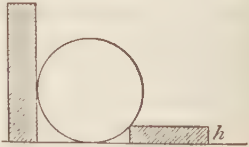


FIG. 101.

2. In what ratio must x be altered to become

$$\text{(i) } y; \qquad \text{(ii) } x+xy; \qquad \text{(iii) } \frac{x}{y}?$$

3. Find a and b, if x^2 is a factor of $(1+x)(1+bx)^2-(1+ax)^2$.

4. Two points A, B at the same level are 200 feet apart. A wire fixed at A, B carries a heavy weight at its mid-point O; the portions AO, OB may be taken as straight, but the length of the wire varies with the temperature. When the wire is $2s$ feet long, the depth of O below AB is y feet; when the length is $2(s+h)$ feet, the depth of O is $y+k$ feet. (i) Find a relation between s, y; (ii) find a relation between h, k, s, y; (iii) find a simple form of this relation when h, k are small; (iv) find an expression for $\frac{dy}{ds}$ in terms of s; (v) if the wire stretches from 210 to 210·1 feet, find approximately the increase in depth of O. (C.S.C.)

5. Interpret geometrically $\int_1^4 (2x+3)dx$; evaluate it (i) by direct calculation and (ii) geometrically.

6. The volume of a bowl of depth x cut from a sphere of radius a is $\pi\int_{a-x}^{a}(a^2-x^2)dx$. Compare the volumes of two bowls of depths $\frac{a}{2}$ and $\frac{a}{3}$.

L. 3.

1. (i) If $x\sqrt{x}=x^n$, find n; (ii) what is the value of $(-1)^{-\frac{1}{3}}$? (iii) simplify $(1000)^x \div 10^{2x}$.

2. Two cylindrical glasses contain equal amounts of liquid; the depths of liquid in the glasses are in the ratio $a:b$. What is the ratio of their diameters?

3. (i) One solution of the equations $x^2+y^2=25$, $\frac{x^2}{9}+\frac{y^2}{16}=2$ is $x=3$, $y=4$; write down the other three pairs of solutions.

(ii) If $x^2+y^2=25$, what are the greatest and least values of x, if y is real?

(iii) If $\frac{x^2}{9}+\frac{y^2}{16}=2$, what are the greatest and least values of y if x is real?

(iv) *Sketch* the graphs of $x^2+y^2=25$ and of $\frac{x^2}{9}+\frac{y^2}{16}=2$.

4. Four rods each of length 10 cm. are jointed together at their ends to form a rhombus; if the length of one diagonal is x cm., express the area of the rhombus as a function of x, and find (i) the value of x for which the area is greatest, and (ii) the maximum area.

5. The distance of the centre of gravity of a circular cone of height h in. and base-radius r in. from the vertex is

$$\frac{\int_0^h \pi x y^2 dx}{\int_0^h \pi y^2 dx} \text{ in.,}$$

where $y = \frac{r}{h}x$. Evaluate this expression.

6. A body moves along a line OA *towards* O in such a way that its velocity at any point is proportional to its distance from O. Its velocity at A, where OA is 30 feet, is 12 feet per sec., and t sec. after passing A, its distance from O is x feet. Find a differential equation connecting x and t, and express $\frac{d^2x}{dt^2}$ in terms of x.

L. 4.

1. (i) Simplify the inequality $\frac{x-1}{2} > 7$.

(ii) Within what limits must x lie if $x^2 < 3x$?

2. The air resistance to a shell varies as the square of its diameter and the cube of its velocity. The resistance to a shell 5 in. in diameter when travelling 1200 feet per sec. is 150 lb.; find the resistance to a shell 4·5″ in diameter when travelling 900 feet per sec. Find also a general formula.

3. (i) Sketch the graph of $y = (1-x)(x-5)$. Calculate the gradient of the tangents at the points where it crosses the x-axis, and find where the gradient is zero.

(C.S.C.)

(ii) Find the area of the portion of the curve which lies on the positive side of the x-axis.

4. If a force of P lb. acts on a body for t sec., the impulse given to the body is $\int_0^t P \, . dt$ lb.-sec. units. Find the impulse given by a force which acts for t sec. and is such that $P = 6 - 2t$.

5. If the graph of $x^2 = 5y$ is rotated about the x-axis, find the volume of the portion between $x = 1$ and $x = 4$.

6. In Fig. 102 the dotted line is a time-velocity graph and the continuous line is a time-distance graph. If the units are properly chosen, can the two graphs both refer to the same motion? Describe the motion in general terms, the time shown being 35 minutes in each case and the maximum velocity 30 miles per hour. Find the distance gone after 14, 28, 35 minutes.

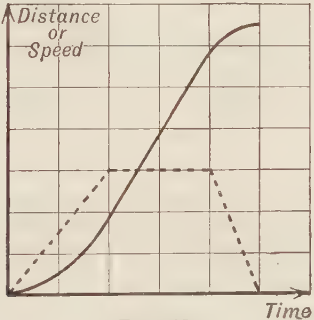


FIG. 102.

L. 5.

1. Express $(x-2)(x-3)$ in the form $a(x-1)^2+b(x-1)+c$.

2. $ABCD$ is a rectangle; P is a point on CD; AP meets BD at Q; $AB=12$, $AD=8$, $PD=x$ cm.; the area of the triangle DPQ is y sq. cm.; express y as a function of x, and find the length of DP when the area is 16 sq. cm.

3. Find the value of $a \cdot e^{pt}$ when $a=32{\cdot}6$, $e=2{\cdot}718$, $p=-0{\cdot}37$, $t=4{\cdot}5$.

4. (i) For what range of values of x is $2x^3-9x^2$ a decreasing function?

(ii) Find a minimum value of this function.

5. If a triangle, base b in., height h in., is immersed in water and held in a vertical plane with its base in the surface, the depth of the centre of pressure is given by $\int_0^h by^2\left(\frac{h-y}{h}\right)dy \div \frac{1}{6}bh^2$. Simplify this expression.

6. Fig. 103 is a perspective view of a piece of steel formed by two cylinders of radius a, whose axes are horizontal and cut at right angles. A horizontal plane is drawn at height x above the axes; what is the shape of the portion of the plane which lies inside both cylinders ? Prove that its area is $4(a^2 - x^2)$. Hence prove that the volume common to the two cylinders is $\frac{16}{3}a^3$.

(Trin. Coll.,

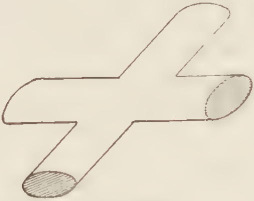


FIG. 103.

L. 6.

1. The rate at which a man can dig a trench is constant for the first 15 minutes, and afterwards, without any abrupt change, varies inversely as the square root of the total number of minutes he has been at work. At the end of 2 hours he is excavating at the rate of 22 cu. feet per hour. Find the rate of work (i) at the end of a 4-hour spell, (ii) at starting. (C.S.C.)

2. How is the value of $\frac{x^3 + 7x^2y}{3xy^2 - y^3}$ affected if x and y are each altered in the ratio $a : b$?

3. If $x = y^2 = z^3$ and $xy = t^2$, express z in terms of t.

4. Find the radius of a cylinder which is cut from a cone of height h in. and base-radius r in. if (i) the volume of the cylinder is a maximum, (ii) the *total* surface is a maximum, given $h > 2r$.

5. The portion of the graph of $y = \frac{x^3}{6} + \frac{1}{2x}$ lying between $x = 1$ and $x = 2$ is rotated about the x-axis so as to form a surface. The area of this surface is $\int_1^2 2\pi y \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} \cdot dx$. Evaluate this expression.

6. The differential equation of a graph is $\frac{d^2y}{dx^2} = 4x - 6x^2$; the graph cuts the x-axis at the origin and the point (2, 0) ; find the value of y when $x = 1$ and the slope at the origin.

L. 7.

1. (i) Draw the graph of $x + 2y = 3$.

(ii) How can you represent graphically all pairs of values of x, y such that $x + 2y > 3$?

(iii) Represent graphically the pairs of values of x and y for which simultaneously $x + 2y > 3$, $x - y < 1$, $5y - 2x < 10$.

(iv) Hence obtain all the integral values of x and y which satisfy the inequalities in (iii).

2. Two glasses contain equal amounts of spirits and water respectively: $\frac{1}{n}$th of the first glass is transferred to the second and stirred up, and then an equal amount of the mixture is transferred back again. Is the ratio of spirits to water in the first glass greater than, equal to or less than the ratio of water to spirits in the second glass ?

3. Given that $2=10^{0\cdot3010}$ and $3=10^{0\cdot4771}$, express as powers of 10 without using tables, (i) 1·2; (ii) 250; (iii) 0·015; (iv) 6·4.

4. (i) Sketch the graph of $y=x(x-1)(x-2)$ for values of x from -1 to $+3$.

(ii) Prove that $\int_0^2 x(x-1)(x-2)\,dx=0$, and interpret this result geometrically.

(iii) Find the minimum and maximum values of y. (C.S.C.)

5. If a force of P lb. acting on a body moves it s feet in the direction of the force, then the "work done" by the force is $\int_0^s P\,.\,ds$ ft.-lb. Find the work done by a variable force $P=3s+2$, which moves a body 10 feet.

6. A train runs from stop to stop in 10 minutes, and its speed is as follows :

Time in minutes ·	0	1	2	3	4	5	6	7	8	9	10
Speed in miles per hr.	0	12	24	36	44	49·5	51	50	45	31	0

Use (i) Dufton's rule, (ii) Simpson's rule to find the total length of run. (C.S.C.)

L. 8.

1. Sketch the graph of the function $y=\frac{700x}{x+540}$ as x varies from 160 to 10,000. It has been suggested [see Edgeworth, *Levy on Capital*] that the following is a fair income-tax scheme. For an income of £x ($x>160$), an abatement of £y is allowed, where $y=\frac{700x}{x+540}$, and the tax is £T, where $T=\frac{x-y}{10}$: for $x =$ or < 160, there is no tax. Express T in terms of x, and show that the percentage of a man's income taken as tax increases steadily with x, but never exceeds 10 per cent. Find the percentage for the following incomes: (i) £160; (ii) £460; (iii) £10,000; (iv) £100,000.

2. The volume of a gas varies directly as its absolute temperature and inversely as its pressure. At an absolute temperature of 300°, the volume is 900 cu. cm. under a pressure of 420 mm. (mercury). Find the pressure if the volume becomes 800 cu. cm. and the absolute temperature 280°.

3. The cubical elasticity of a gas whose volume is v and pressure p is measured by $-v\dfrac{dp}{dv}$. Find the cubical elasticity of a gas for which $p \, . \, v^{1 \cdot 408} = c$, where c is constant.

4. The portion of the curve $y = x^2 + 1$ between $x = 0$ and $x = 2$ is rotated about the x-axis to form a solid of revolution. The x-coordinate of the centre of gravity of this solid is

$$\int_0^2 (x^2+1)^2 x \, dx \div \int_0^2 (x^2+1)^2 \, dx.$$

Evaluate this expression.

5. If a horizontal circular disc of radius 4″ is under the influence of a unit charge of electricity at a point 5″ from and vertically above the centre, the density of the induced distribution at a point on the disc x in. from the centre is $\dfrac{3}{2\pi^2} \cdot \dfrac{1}{(x^2+25)\sqrt{16-x^2}}$ units per sq. in. For what value of x is this a minimum ?

6. Given $x^4 \dfrac{d^2y}{dx^2} = 1$ and that when $x = 1$, $\dfrac{dy}{dx} = 1$ and $y = 2$, express y in terms of x.

L. 9.

1. (i) How is x limited if $\dfrac{3x-2}{2}$ is greater than $x + 1$? Illustrate the result graphically.

(ii) Is it possible to have $a > b$ and $ax < bx$ simultaneously ?

2. $ABCD$ is a rectangle ; $AB = p$, $BC = q$ in. $(p > q)$; H, K are points on AB such that $AH = KB$, and $CDHK$ is a quadrilateral in which a circle can be inscribed. Find the length of AH in terms of p, q.

3. The rate of flow in a circular sewer of diameter D inches is $\dfrac{9 \cdot 4 \sqrt[3]{D^2}}{\sqrt{i}}$ feet per sec., where i = length of sewer ÷ fall.

If the diameter is decreased in the ratio 5 : 6, and if the fall is increased from 5 feet in a mile to 6 feet in a mile, find the ratio in which the velocity is altered.

4. The distance of the centre of gravity of a uniform hemisphere of radius a from the centre of the hemisphere is

$$\int_0^a \pi(a^2-x^2)x\,dx \div \int_0^a \pi(a^2-x^2)\,dx.$$

Evaluate this expression.

5. Sketch the graph of $y=\frac{1}{x^2}$; mark the point $P(2, \frac{1}{4})$ on it, and draw the tangent at P cutting the x-axis, Ox, at T. Find (i) the gradient of PT, (ii) the length of OT.

6. Interpret geometrically the expressions,

$$\text{(i)} \int_1^2 \frac{1}{x^2}\,dx; \qquad \text{(ii)} \int_1^2 \pi\cdot\frac{1}{x^4}\,dx,$$

and evaluate them.

L. 10.

1. Fig. 104 represents a bridge whose span AB is a feet, supported on an arch in the form of a circular arc of radius r feet; the heights of the roadway above C, D and the top of the arch E are b, b, c feet; express r in terms of a, b, c. Find r if $a=80$, $b=16$, $c=1$. (C.S.C.)

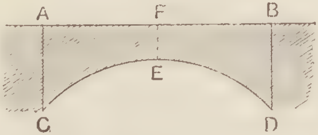


FIG. 104.

2. A_1, A_2, A_3, A_4, A_5, ... are points on a line such that the lengths of A_1A_2, A_1A_3, A_1A_4, A_1A_5, ... are log 2, log 3, log 4, log 5, ... feet. Find the lengths of

(i) $A_1A_2 - A_2A_4$;
(ii) $A_1A_8 - 3 \,.\, A_1A_2$;
(iii) $A_1A_{10} - A_1A_2 - A_1A_5$.

What facts do these results illustrate?

3. If z varies directly as x and inversely as y, complete the table:

Values of x.

		1	2	3	4	5
Values of y.	1					
	2			90		
	3					100

4. If a rectangle of height a in. is immersed in water with its plane vertical and its upper edge h in. below the surface, then the depth of the centre of pressure below the surface is

$$\int_0^a (x+h)^2\,dx \div \int_0^a (x+h)\,dx.$$

Simplify this expression.

5. A glass is so shaped that when the depth of liquid in it is x inches, the volume of the liquid is $\frac{1}{10}x^4$ cu. in.; water is poured into it at the rate of 2 cu. in. per sec.; at what rate is the level rising after 3 seconds?

6. Given $\dfrac{d^2y}{dx^2}=1-x^3$ and $x=0$ when $y=0$, and $x=2$ when $y=2$, find $\dfrac{dy}{dx}$ when $x=0$, and express y in terms of x.

M.

(CHAPTERS XII.-XIV. and XIX.)

M. 1.

1. Write down the 1st and 10th terms of sequences whose nth term is

(i) $5n-4$; (ii) $(n+1)(n+2)$; (iii) n^2-1.

2. Fill in the gaps in the following sequence: 2, 5, 8, , 14, 17, , 23, , , 32 and write down the value (i) of the 15th term, (ii) of the nth term.

3. How many terms are there in the sequence 1, 5, 9, 13,... 101? Find the sum of these terms.

4. A line AB, 4 inches long, is divided at P so that the square on AP is twice the square on PB. Find the length of AP correct to 3 figures.

5. Show that a triangle with sides $\sqrt{2}-1$, $\sqrt{2}+1$ and $\sqrt{6}$ will be a right-angled triangle, and find its area.

6. Two ships are similar (in the mathematical sense), but are of tonnage 8000 and 27,000 respectively. Assuming that the tonnage of similar ships varies as the cube of the linear dimensions, and that it costs £A to paint the smaller ship, what will it cost to paint the larger ship? (Certificate.)

M. 2.

1. Find the sum of all the numbers up to and including 100 which are divisible by 5.

2. A car starts from rest and is steadily accelerated in such a way that its speed after 1 second is 1 mile per hour, after 2 seconds, 3 miles per hour, after 3 seconds 5 miles an hour, and so on. Find what its speed will be according to the same law after 8 seconds, and find when it will attain a speed of 20 miles an hour.

3. Solve the equations: $\begin{cases} 4{\cdot}5x + 7{\cdot}2y = 1{\cdot}4, \\ 3{\cdot}6x + 2{\cdot}4y = 0{\cdot}49. \end{cases}$

4. If $x : y = 10 : 9$, find the value of (i) $\dfrac{x+y}{x-y}$, (ii) $\dfrac{x^2+y^2}{x^2-y^2}$.

5. The triangle ABC in Fig. 105 is right-angled at B, and has $AB = 8''$, $BC = 6''$.

The side AB is divided into 8 equal parts and rectangles are constructed as in the figure. If MN is the side of any one of these, then the ratio $MN : BC = AM : AB$. Hence find the length of MN and the area of the rectangle which has MN as its side. Find the sum of the areas of all these rectangles, and find the difference between their sum and the area of the $\triangle ABC$.

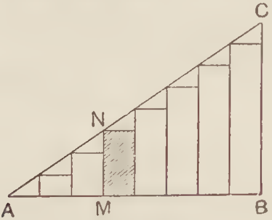


FIG. 105.

6. Show that if in Question 5 the side AB is divided into n equal parts, so that $(n-1)$ rectangles are formed, each with base $\dfrac{8}{n}$ inches, then the smallest rectangle formed has area $\dfrac{48}{n^2}$ square inches, and the largest $\dfrac{48(n-1)}{n^2}$ square inches, and that the sum of all of them differs from the area of the triangle by $\dfrac{24}{n}$ square inches.

M. 3.

1. A solid sphere of radius $4''$ weighs 32 lb. Find the weight of a shell of the same material, whose internal and external radii are $3''$ and $5''$. (Certificate.)

2. If $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$, express $x^2 + xy + y^2$ in terms of t only.

3. The nth term of a series is $5n - 7$. What is the 1st term? what is the $(n+1)$th term? Find the sum of $(n+1)$ terms.

4. Draw with the same axes and scales the graphs of $x^2 - 7$ and $\frac{4}{x}$ for values of x between the limits ± 3. Show that the points of intersection of the graphs will give the solution of the equation $x^3 - 7x - 4 = 0$, and hence determine how many real roots this equation has, and their approximate values. (Certificate.)

5. An equilateral triangle ABC has all its sides divided into 12 equal parts. Corresponding points are joined as in Fig. 106 (6 parts only are shown), and a number of smaller equilateral triangles thus formed. How many small triangles are there (i) in the fifth row (PQ) ? (ii) in the 12th row ? How many small triangles are there in all ?

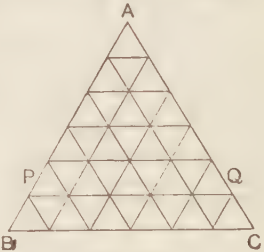


FIG. 106.

6. Generalise Question 5, taking any triangle ABC and dividing each side into n equal parts. By joining corresponding points, a number of similar triangles are formed as in Fig. 106. How many are there in the rth row ? how many are there in all ? What geometrical proposition does this result illustrate ?

M. 4.

1. The horse-power of a certain type of water-wheel is given by the formula H.P. $= 0{\cdot}00113QH$, where Q is quantity of water flowing in cubic feet per min. and H is the head of water in feet. Find the horse-power of a wheel of this type driven by 850 cubic feet of water per minute from a height of 2·6 feet.

2. Convert the relation $a = 2{\cdot}5c^{\frac{1}{2}}d^{\frac{3}{4}}$ into a relation of similar type, giving c in terms of a and d. (Certificate.)

3. Solve the equation $x^2 + 3x - 5{\cdot}23 = 0$, giving the values of x correct to 3 significant figures.

4. What is the nth term of the sequence $2 + 1$; $2^2 + 2$; $2^3 + 3$; $2^4 + 4$; ... ?

If the nth term of this sequence is a, the $(n + 1)$th term b and the $(n + 2)$th term c, prove that $3b = 2a + c + 1$.

5. Find the nth term and the sum of n terms of the sequence 2, 6, 18, 54, ...

6. A clerk's commencing salary is £100 a year, and he is offered a choice between a yearly rise of £5 and a rise of £22 every 4 years. Calculate the total sum that he will receive in the course of 33 years under each arrangement. (Certificate.)

M. 5.

1. An engine exerting a pull of F tons draws a train weighing W tons (including the engine) up an incline of 1 in m on a road whose resistance is r lb. per ton. It can be proved that in t seconds the train will have moved from rest through s feet, where

$$s=\tfrac{1}{2}\left(\frac{F}{W}-\frac{1}{m}-\frac{r}{2240}\right)gt^2.$$

Express F in terms of the remaining letters. (Certificate.)

2. If $x=\dfrac{4y-5}{5y+1}$ and $y=\dfrac{4z-5}{5z+1}$, find the values of y and z, when $x=1$.

3. Find the value of $\dfrac{8132h^{2\cdot5}}{n^2}$, when $h=4\cdot65$ and $n=325$.

4. Find x from the following equations:

(i) $a^x \times a^5 = a^9$;
(ii) $(a^x)^6 = a^9$;
(iii) $10^x = 417$ to 3 significant figures;
(iv) $3^x = 2$ to 3 significant figures.

5. Prove that $\dfrac{101+103+105+\ldots+199}{1+3+5+\ldots+99}=3$.

6. Take any positive number x_1; take its positive square root, add one and square the result, call the resulting number x_2. Repeat the process, starting with x_2 instead of x_1 and call the result x_3; repeat with x_3 and call the result x_4, and so on. Find x_{10} in terms of x_1. Also find x_n in terms of x_1.

M. 6.

1. A certain engineering formula is of the type

$$H=\frac{ml}{d^x}\cdot\frac{V^n}{2g}.$$

Rewrite this formula, expressing V in terms of the other letters.

2. If $U=0\cdot4343\,.\,P\,.\,V\,.\log\dfrac{p_1}{p_2}$, find U when $P=16$, $V=7$, $p_1=15\cdot8$ and $p_2=7\cdot4$.

3. The proprietor of a tea-shop finds that the average expenditure of each of ninety customers is 5d.; but that while each man spends on the average 7d., each woman spends on the average 4d. How many of his customers are men? (Certificate.)

4. A sequence is formed whose nth term is $6n-4$; write down the first three terms, and show that the difference between two successive terms is always 6.

5. What is the nth term of the sequence 27, 9, 3, 1, ... ? Find its value approximately when $n = 20$, using logs.

6. A long strip of paper, thickness 0·005 inch, is wound on to a cylindrical bobbin of diameter $3\frac{1}{2}$ inches, and the outer diameter of the roll thus made is 14 inches. How many layers of paper are there in the roll ? Assuming that the lengths of successive layers increase in Arithmetic Progression, and that each length is a complete circle, find in yards the total length of the strip. (Certificate.)

M. 7.

1. According to a certain scheme for paying income tax a man paid at the rate of 2s. 6d. in the £ on "earned" income and 4s. in the £ on "unearned" income (*i.e.* on dividends from investments, etc.). His income was £800, and his total income tax was £115. How much of his income was "earned" ?

2. If $W = \sqrt{\dfrac{w(p_1 - p_2)d^5}{L}}$, find the value of W when $w = 0{\cdot}0012$, $p_1 = 18{\cdot}4$, $p_2 = 17{\cdot}8$, $d = 2{\cdot}5$ and $L = 1760$.

3. Find the 16th term and the sum of 16 terms of the sequence $1, \frac{4}{3}, \frac{5}{3}, 2, \ldots$.

4. What would appear to be the form of the nth term of a sequence whose first three terms are formed as follows ?

(i) 3×2; 3×2^2; 3×2^3;
(ii) $3 + 2$; $4 + 2^2$; $5 + 2^3$.

5. Find the sum of 9 terms of a G.P. whose first two terms are 1 and $\frac{3}{5}$, using logs to get the result correct to 3 significant figures.

6. A window is in the shape of a rectangle surmounted by the semi-circle drawn on the upper side of the rectangle, this side being one of the shorter sides.

If h feet is the height of the window, measured from the bottom side to the highest point of the semi-circle, and the width of the window is a feet, show that its area is $(ah - \frac{3}{28}a^2)$ square feet. [Take the area of a circle, radius r feet, to be $\frac{22}{7}r^2$ sq. feet.]

If the area is $9\frac{1}{2}$ square feet and the height is 5 feet, find the width of the window to the nearest half-inch. (Certificate.)

M. 8.

1. Solve, by a graph if possible, the following problem :

A party, climbing a mountain 4000 feet high by the regular track, climbs at the rate of 1000 feet per hour, measured vertically ; it starts at 11.5 a.m. and halts half-an-hour for lunch at 1.30 p.m. Another party, whose rate of descent is 1800 feet per hour, measured vertically, leaves the summit at 1.45 p.m., following the regular track. At what time will it meet the first party ?

(Certificate.)

2. Simplify $\frac{3x}{x-1}-\left(\frac{x}{x+1}\div\frac{1}{1-x}\right)$.

3. If $T=\frac{0{\cdot}196(D^4-d^4)}{D}$, find T when $D=3{\cdot}12$, $d=1{\cdot}74$.

4. Show that the recurring decimal 0·7777... is equal to the "limiting sum" of a G.P., and find its value as a fraction.

5. If $x:y=13:4$ and $2x+3y=19$, find the values of x and y.

6. The area cut off from the surface of a cone between two circular sections, the distances of whose edges from the vertex measured along a slant side of the cone are a'' and b'', is $c(b^2-a^2)$ sq. inches, where c is a constant depending on the angle of the cone. A cone, for which the constant c is 0·8, and whose slant side is 30″, is painted in alternate strips of red and blue, the red starting at the vertex, the width of each strip being 1″. Show that the areas of the consecutive strips of each colour form two arithmetic progressions, and find the total areas painted in the two colours. (Certificate.)

M. 9.

1. The following formula connects the penetration t inches into steel-armour plates, with the velocity v ft. per sec. of the shell:

$$t=\frac{15v}{6200-v}.$$

Find the velocity of the shell that penetrates 7″. (Certificate.)

2. OAB are three points in order on a straight line, such that $OA=a''$ and $OB=b''$. A point P divides AB so that the ratio $AP:PB=5:3$. Find the length of OP in terms of a and b.

3. If $H=\frac{G^2L}{(3d)^5}$, express d in terms of H, b, and L, using fractional and negative indices.

4. Find the value of $T\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}}$ correct to 3 significant figures, when $T=147$, $p_1=16{\cdot}5$, $p_2=13{\cdot}8$ and $n=1{\cdot}6$.

5. An estimate for boring a well is 4s. for the first yard, 4s. 2d. for the second yard, and so on; each yard costing 2d. more than the preceding yard. If the sum to be spent on boring is £7. 2s., how deep is the well to be? (Certificate.)

6. After heavy rain the quantity of water passing a point on a small stream in hilly country is found to be 4000 cubic feet in one hour, 2000 cubic feet in the next, 1000 cubic feet in the next, and so on. Show that the stream will have practically dried up in twelve hours, if this rate of decrease continues, and find how much water will have passed the point then.

M. 10.

1. In any triangle ABC, AD is perpendicular to BC and meets BC at D. $AD=3''$, $BC=6''$.

Denoting the length of BD by x'', express AB^2 and AC^2 in terms of x. If $AC^2=k\,.\,AB^2$, form an equation to find x. What is the meaning of the roots of this equation, (i) when $k=1$, (ii) when $k=5$?

2. If $L=6{\cdot}6d^{\frac{2}{3}}$, express d in terms of L, and find the value of d, when $L=118$.

3. Find the first term and the common difference of an arithmetic progression whose 3rd term is 15 and 10th term 43.

4. What is the smallest number larger than 100 which is divisible by 9 ? What is the largest number less than 200 and divisible by 9 ? Find the sum of all the numbers between 100 and 200 which are divisible by 9.

5. Find the sum to n terms of the series $1-\frac{1}{4}+\frac{1}{16}-\dots$.
What is the "limiting sum" of this series ?

6. A War Savings Certificate costs 15s. 6d., and amounts to 20s. at compound interest in 5 years. Find the number of years in which a sum invested at this rate of compound interest will amount to double the original principal; in other words, solve the equation $\left(\frac{20}{15{\cdot}5}\right)^{\frac{x}{5}}=2$, calculating x correct to one place of decimals.

(Certificate.)

N.

(CHAPTERS XII.-XIV. and XX., XXI.)

N. 1.

1. The senior and junior marks obtained in an examination are 137 and 29. These are scaled from 150 to 0. If x marks, when scaled, becomes y marks, express y in terms of x. What mark is unaltered by the scaling ?

2. Give the general rule which includes the following:

$$\text{(i)}\ {}^7P_5=\frac{\underline{|7}}{\underline{|2}};\qquad \text{(ii)}\ {}^{10}P_7=\frac{\underline{|10}}{\underline{|3}};\qquad \text{(iii)}\ {}^{11}P_6=\frac{\underline{|11}}{\underline{|5}},$$

and evaluate ${}^{20}P_2$.

3. (i) What is the coefficient of x^2 in the expansion of

$$(x+a+b)(x+c)(x+d)\ ?$$

(ii) What are the terms in the expansion of

$$(a+b+c)(a-b+d)(a+c-d)(b-c-d),$$

which have b^3 as a factor ?

4. If $p \, . \, v^{1\cdot 4} = 28 \cdot 7$, find v when $p = 17 \cdot 3$.

5. The distance a spring stretches (*i.e.* its extension) when a weight is suspended from it varies (within certain limits) directly as the weight. The following measurements were made in an experiment, state which (if any) of them are probably incorrect, and find the probable equation between the extension l cm. and the weight w grams which produces it:

Total length of spring in cm., l -	21·0	22·7	26	27·7	34·2	36	39·5	41
Weight attached in gr., w - -	0	5	15	20	35	45	50	60

6. (i) There are n circles in a plane, each lying outside the rest; what is the greatest number of lines that can be drawn, each of which touches two circles ?

(ii) Two sets of 3 concentric circles are drawn in a plane, and no two circles intersect. How many lines can be drawn, each of which touches two circles ?

N. 2.

1. The resistance R lb. to a parachute of area A sq. ft. moving v ft. per sec. varies as A if v is constant, and as v^2 if A is constant. Complete the table for R.

How does v vary if R is constant ?

Values of A.

		4	8	10
Values of v.	5			
	10		1	
	15			

2. The following formulae occur in the mensuration of a circular cone: $s = \pi r l$; $l^2 = r^2 + h^2$; $V = \frac{1}{3}\pi r^2 h$.

If $\frac{r}{3} = \frac{h}{4}$, evaluate $\frac{s^3}{V^2}$.

3. In the Morse code, letters are represented by dots or dashes or combinations of both ; how many letters can be represented by using not more than four symbols for any one letter ?

4. (i) Simplify $(x+y)^3 - 3y(x+y)^2 + 3y^2(x+y) - y^3$.
   (ii) What is the coefficient of x^3 in $(1+2x+x^2)^5$?

5. (i) Find the roots of $x^3 - 100x = 0$.
   (ii) One root of $x^3 - 100x = 1$ is obviously nearly equal to 10; by putting $x = 10 + h$, where h is small, and neglecting h^2 and h^3, find a closer approximation to this root.

6. If the diameter of a circular cylinder increases 2 per cent. and the height increases 1 per cent., find the approximate percentage increase in the volume.

N. 3.

1. Fig. 107 represents the graph of $y = (x-1)(x-2)$ on a certain scale. Sketch, with the same units, the graphs of

(i) $y = 1 + (x-1)(x-2)$;
(ii) $y = (x+1)(x+2)$;
(iii) $y = (x-2)(x-3)$;
(iv) $y = \dfrac{1}{(x-1)(x-2)}$.

Sketch also the graph of

$$y = (x-1)(x-2),$$

if the x-unit is halved and the y-unit is doubled.

2. In an ordinary box of safety matches the ratio of the length, the breadth and the height is 4 : 3 : 1. If the area of the matchwood used in making the box is A sq. inches and the volume of the box is V cu. inches, find an equation between A and V.

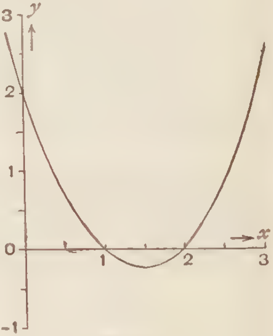


FIG. 107.

3. A tea-pot with a spout 2″ long holds a pint; how much will a tea-pot of the same shape with a spout 3″ long hold ?

4. A carriage holds 4 people inside and 2 outside; in how many ways can 6 people be divided between the inside and outside places, the arrangement inside and outside being immaterial ?

5. A pendulum of length l feet vibrates $\dfrac{108{\cdot}4}{\sqrt{l}}$ times a minute. What is the reduction in the number of vibrations per hour if the length is increased, (i) from 9 to 9·1 feet, (ii) from l to $l+h$ feet, where h is small compared with l ?

If a grandfather-clock gains, would you screw the weight at the end of the pendulum up or down in regulating it ? And why ?

6. What is the coefficient of x^3 in

(i) $(1-x^2)^2(1+x)^5$; (ii) $(1-x)^2(1+x)^6$?

N. 4.

1. If the length of a pendulum is l feet and the time of vibration is t sec., it is found by observation that when $l=1$, $t=1{\cdot}11$, and when $l=2$, $t=1{\cdot}58$, and when $l=3$, $t=1{\cdot}93$. If these obey a law of the form $t=al^n$, find a and n.

2. The side BA of the triangle ABC is produced to E; if the bisector of $\angle EAC$ cuts BC produced at P, it can be proved that $\frac{BP}{CP}=\frac{BA}{AC}$. (i) If $AB=7''$, $AC=3''$, $BC=5''$, calculate CP; (ii) if $AB=z$, $BC=x$, $CA=y$ in., calculate CP.

3. In signalling by semaphore, a single arm can be shown in any one of 7 positions, exclusive of the position of rest. These positions denote the first seven letters of the alphabet. The remaining letters can be signalled by showing two arms in any pair of these seven positions. How many letters can be signalled altogether ?

4. If $a=b+h$, where h is small compared with b, show that $\sqrt{\frac{a}{b}}$ and $\frac{3a+b}{a+3b}$ have nearly the same values. What is approximately the value of each when $a=11$ and $b=10$?

5. (i) What is the result of putting $x=2y-1$ in the expression

$$x^4+4x^3-14x^2-36x+45 \ ?$$

(ii) Solve the equation $x^4+4x^3-14x^2-36x+45=0$.

6. A police-court witness is asked to identify 2 suspected persons from a group of 12. If he really knows nothing about them, what is the chance that he picks out the two suspects ?

N. 5.

1. Draw the graph of $y=x^2-x$. With the same axes and without any further calculation, sketch the graphs of

(i) $x=y^2-y$; (ii) $y=x^2+x$; (iii) $y=x^2-x+1$.

2. When the horizontal course of an aeroplane is altered, there is a tendency for the axis of rotation of the propeller to turn up or down measured by $\frac{\pi R^2 MN}{32{\cdot}2T}$ foot-lb. Find the turning tendency for a "Gnome" motor, where R = radius of gyration $=0{\cdot}742$ feet, M = weight $=247$ lb., N = number of revolutions per sec. $=20$, T = number of seconds aeroplane takes to turn through a complete circle $=$ **19·5.**

3. Solve $$\frac{xy}{4}=\frac{yz}{9}=\frac{zx}{6}=\frac{y+z}{2{\cdot}5}.$$

4. A, B, C, D are 4 points in a plane, no three of which are collinear; how many different angles at A, B, C, D can be formed by lines joining these points, but not produced beyond them?

5. How many arrangements can be made of the letters of the word *referee*?

6. (i) If x is small, prove that $\sqrt{x^2+16}-\sqrt{x^2+9}\simeq 1-\frac{x^2}{24}$.

(ii) If x is large, prove that $\sqrt{x^2+16}-\sqrt{x^2+9}\simeq\frac{7}{2x}$.

(iii) Hence sketch the graph of $\sqrt{x^2+16}-\sqrt{x^2+9}$.

N. 6.

1. (i) If $\log_a x=p$, express x in terms of a, p.

(ii) If $\log_b x=q$, find an equation between a, b, q, p,

(iii) Hence prove that $\frac{\log_a x}{\log_b x}$ is independent of x.

(iv) What is the value of $\frac{\log_8 x}{\log_2 x}$?

(v) Given the graph of $\log_2 x$, how would you deduce the graph of $\log_8 x$?

2. If the lengths of the sides of a cyclic quadrilateral taken in order are a, b, c, d in., the area is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s is its semi-perimeter.

(i) If the sides of the quadrilateral touch a circle, it can be proved that $a+c=b+d$; simplify the expression for the area.

(ii) Interpret geometrically the result of putting $d=0$ in the given formula.

(iii) What is the result of putting $a=b=c$ and $d=0$ in the given formula, and what is the geometrical interpretation?

3. In how many ways can 8 people be seated at two round tables, each of which holds four? [Arrangements in which at least one person has a different right-hand neighbour are to be counted as different arrangements.]

4. The area of the wing surface of an aeroplane with load W lb. for a speed of v miles an hour varies as $\frac{W}{v^2}$. How must the area be altered approximately if W is increased by 3 per cent. and v is reduced by 2 per cent.?

5. Find (without using logarithms) the approximate value of the cube root of (i) 125·5; (ii) 127.

6. (i) In the game of Crown and Anchor, two dice are used, the faces of each of which are marked with a crown, an anchor, a spade, a heart, a diamond, a club. The gambler chooses one of these on which to stake. The dice are then thrown. If both dice show the figure he has chosen, the Banker pays him 4 times his stake and returns his stake; if one of the dice shows his figure, the Banker pays him twice his stake and returns his stake. If his choice does not turn up at all, the Banker takes his stake. If during a session the total stakes amount to £600, how much profit will the Banker expect to make?

(ii) If three dice are used, the Banker pays, besides returning the stake, four times the stake for a triple success, twice the stake for a double success and a sum equal to the stake for a single success; in the event of failure he keeps the stake. What fraction of the total sum staked does the Banker expect to win?

N. 7.

1. The horse-power H required to drive an electric fan of diameter d feet to deliver V cu. feet of air per second is given by $V = 14d^2\sqrt{H}$.

(i) Express this as a formula in which the units are inches and cu. inches for the diameter and volume.

(ii) What is the effect on the horse-power if the volume and diameter are each doubled?

2. In a game of chess, the first player (white) can make any one of twenty moves. Black can then also make any one of twenty moves. White can play his second move in about twenty ways and Black has the same choice for his second move. Find approximately the number of ways in which the first two moves on each side can be played.

3. (i) If h is small compared with x, find an approximation for $\sqrt{x^2+h^2}-x$.

(ii) The height, h in., of an isosceles triangle ABC is small compared with the base BC; the excess of $AB+AC$ over BC is d in.; prove that the area of the triangle is approximately $\frac{h^3}{d}$.

4. A tank, fitted with a tap enabling $\frac{1}{N}$th of the contents to be drawn off, is full of a certain liquid. As much as possible is drawn off, and the tank is filled up with water. What fraction of the original liquid remains after the process has been repeated, (i) twice, (ii) p times?

If $N=10$, how many times must the process be repeated in order that less than half the original liquid may remain? (C.S.C.)

5. A policeman on a motor bicycle, pursuing a thief who has stolen a car, comes to a cross-roads; he turns down a road at random, and then comes to a point at which the road branches into two; he again chooses at random. What is the chance he is still on the right road?

6. It can be proved that if water stands in a spherical vessel of radius a in. to a depth of d in., the volume of the water is

$$f(d) \equiv \pi d^2\left(a - \frac{d}{3}\right).$$

(i) Use this formula to obtain the volume of a sphere; (ii) evaluate $f(d) + f(2a - d)$.

N. 8.

1. The electrical resistance of a copper wire varies directly as its length and inversely as the square of the diameter of the cross-section. How is the resistance altered if the length is doubled and the diameter is halved?

2. What would the graph of $(-1)^x$ drawn for all positive values of x look like? What is its chief peculiarity?

3. How many different signals can be sent, using one or more of 4 flags of different colours, (i) if different orders count as distinct signals, (ii) if the order does not matter?

4. (i) If y is so small that y^3 can be neglected, express $\frac{1}{1+y}$ in the form $a + by + cy^2$.

(ii) If x is so small that x^3 can be neglected, express

$$\frac{2n + (n+1)x}{2n + (n-1)x}$$

in the form $a + bx + cx^2$. Express also $\sqrt[n]{1+x}$ in the same form.

(iii) What can you deduce from the results of (ii)?

5. (i) In how many ways can 12 competitors in a race be divided into two heats of six, if the order of the heats is immaterial?

(ii) What is the chance that the two best runners are in the same heat?

6. The volume of a circular cone, base-radius r in. and height h in., is $\frac{1}{3}\pi r^2 h$. A frustum of a cone is the portion between two planes perpendicular to the axis. If the areas of the two plane faces of the frustum are A, B sq. inches, and if the distance between them is h in., prove that the volume of the frustum is $\frac{1}{3}h(A + B + \sqrt{AB})$.

N. 9.

1. If a disc of diameter d feet revolves at the rate of n revolutions per minute, the power wastage due to the air is P ft.-lb. per sec., where $P^2 = 10^{-22}d^{11}n^7$.

Express this formula with P as its subject for a disc of diameter x inches revolving m times a second.

2. Motor-cars are numbered with either one or two letters (*e.g.* P; AA; LH) followed by any number from 1 to 9999. How many cars can be numbered differently if all letters in the alphabet are used ?

3. If water to a depth of d in. stands in a hemispherical bowl of radius a in., the volume of water is $\pi d^2\left(a - \frac{d}{3}\right)$ cu. in.

If the contents form $\frac{1}{n}$th of the bowl, where n is large, find an approximate value of d in terms of a, n.

4. Four people wish to settle how to play a foursome ; three of them spin coins, and the odd man out plays with the fourth. What is the chance that the first spin is decisive ?

5. (i) There are 4 teams competing against each other on the "knock-out" system. In how many ways can the draw be made ?
- (ii) What is the number of ways if there are 8 teams ?
- (iii) What is the number of ways if there are 16 teams ?

6. (i) A framework of rods has 5 corners, no four of which are coplanar ; what is the greatest number of rods in the framework ?
- (ii) If the rods are jointed freely at their extremities, how many can be removed without affecting the rigidity of the framework ?
- (iii) A framework of rods has n corners, no four of which are coplanar ; what is the greatest number of rods in the framework ?
- (iv) If the rods are jointed freely at their extremities, what is the greatest number that can be removed without affecting the rigidity of the framework ?

N. 10.

1. The acceleration g feet per sec. of a falling body near the surface of a planet varies directly as the mass of the planet and inversely as the square of the radius. The time of oscillation of a pendulum varies directly as the square root of its length and inversely as the square root of g. Find how the mass of a planet varies in terms of the time of oscillation of a pendulum of given length and the radius of the planet ?

2. (i) How many terms are there in the expansion of

$$(1+2+2^2+2^3)(1+5+5^2+5^3+5^4)\ ?$$

(ii) In how many ways can 5000 (*i.e.* $2^3 \times 5^4$) be expressed as the product of two factors ?

(iii) In how many ways can 10,000 be expressed as the product of two factors ?

3. A domino consists of two halves, each of which may have any number of pips marked on it from 0 to 6 ; how many different kinds of dominoes are there ?

4. What is the coefficient of t^3 in the expansion of

(i) $(1+at+a^2t^2+a^3t^3+a^4t^4)(1+bt+b^2t^2+b^3t^3+b^4t^4)$
$\times(1+ct+c^2t^2+c^3t^3+c^4t^4)$;

(ii) $(1+t+t^2+t^3+t^4)^3$?

5. $f(x)$ is a function of x such that

(i) $f(x) \equiv x-2n$ if $2n < x \leqslant 2n+1$,

and (ii) $f(x) \equiv 2n-x$ if $2n+1 < x \leqslant 2n+2$, where n is zero or any positive integer. Sketch the graph of $f(x)$ for positive values of x.

6. A, B play a match of the best three sets at tennis. Either is equally likely to win any set. A wins the first set ; what is his chance then of winning the match ?

P.

(CHAPTERS XII.-XVIII. and XXII., XXIII.)

P. 1.

1. In a certain formula ah is taken as an approximation for $\dfrac{a^2h}{a-h}$; find the error as a percentage of the true value : and evaluate it for the special case $h=2$, $a=4000$. (Certificate.)

2. Two men measure independently the length of a fence ; the first makes it a yards, and his error is less than x per cent. ; the second makes it b yards, and his error is less than y per cent. Find two inequalities connecting a, b, x and y.

3. An experiment to determine the force P lb. required to lift a weight of W lb. by a system of pulleys gave the following results :

W	5	10	15	20	30	40
P	1·23	2·15	3·06	3·98	5·82	7·64

Find the best fit equation for P in terms of W.

4. Describe in general terms the value of the function $\frac{x-4}{x-3}$, when x is (i) large and positive; (ii) slightly greater than 4; (iii) slightly less than 4; (iv) slightly greater than 3; (v) slightly less than 3; (vi) less than -1000.

For what range of values of x is this function negative? Sketch its graph.

5. Simplify (i) $\frac{d}{dx}(x+2)(x-3)$; (ii) $\frac{dy}{dx}$ if $xy=5$;

(iii) $\int\left(x^2-\frac{1}{x^2}\right)dx$; (iv) $\int_{-1}^{+1}(x+1)^2\,dx$.

6. (i) If a sphere of radius a in. contains water to a depth of x in., the volume is $\int_0^x \pi(2ax-x^2)\,dx$. Simplify this expression.

(ii) Water is poured into a spherical bowl of radius 5 in. at the rate of 4 cu. in. per sec. At what rate is the water-level rising when the depth is 3 inches?

P. 2.

1. Given $\log x = 3{\cdot}2 \log t + 7{\cdot}4$, express x in terms of t without using logarithmic notation.

2. I is an index-number representing the cost of living compared with a fixed pre-war standard. A workman receives a fixed wage of 40s. a week plus a bonus directly proportional to the cost of living. The purchasing power P of his wages is measured by $W \div I$, where W shillings is the weekly wage. When $I=250$, he receives 90s. a week. Express W and P as functions of I. Prove that, when the wages increase, their purchasing power decreases.

3. The following readings connect the candle-power and voltage of an incandescent lamp:

Candle-power -	20·68	23·24	26·00	28·96
Voltage - -	94	98	102	106

If the candle-power varies as the nth power of the voltage, find n.

4. The horizontal cross-sections of the crater of a volcano are circles with their centres on a vertical line, the radii of the circles at different heights are as follows (measurements in feet):

Height -	0	10	20	30	40	50	60	70	80	90	100
Radius -	0	17	24	27	30	35	41	52	65	78	94

Find its volume in cu. ft. to two significant figures, (i) by Dufton's rule, (ii) by Simpson's rule.

5. The work done in stretching a spring of natural length a inches to a length b inches is $\frac{1}{30}\int_a^b (x-a)\,dx$ foot-lb. Find the work done in stretching the spring from its natural length, 10 inches, to a length of 14 inches.

6. A particle projected in a resisting medium finally comes to rest. It travels s feet in t sec., where $s = 50t - \frac{1}{6}t^3$. How far does it go before stopping?

P. 3.

1. Simplify (i) $\log(x^3) - \log x$; (ii) $\frac{\log(x^3)}{\log x}$;

(iii) $\log(10x) + \log\left(\frac{1}{x}\right)$; (iv) $\log x$ if $x^2 = 100$.

2. (i) If $xyz = a$, $xzw = b$, $xyw = c$, express the ratios $y : z : w$ in terms of a, b, c.

(ii) Divide $a^2 - b^2$ into two parts in the ratio $a : b$.

3. The following table gives the distance, d yards, in which a train running V miles per hour can be stopped:

V	30	40	45	50	60
d	100	170	224	271	400

Plot d against V^2, and then express d in terms of V.

4. The moment of inertia of a spherical segment, radius r in., height h in., about its axis is $\frac{\pi d}{2}\int_{r-h}^{r} (r^2 - x^2)^2 . dx$, where d is its density. A sphere of radius r is cut into two segments by a plane dividing a diameter in the ratio 3 : 1. Compare the moments of inertia of the two segments about their axes.

5. A piece of wire two feet long is cut into two parts, one of which is bent to form a square and the other an equilateral triangle. If the sum of the areas is a minimum, find the side of the equilateral triangle.

6. A weight hangs at the end O of a spiral spring: it is pulled down a certain distance and released, and then oscillates backwards and forwards. If its velocity is v inches per sec. when its distance from O is x inches, the differential equation of the motion is $\frac{d}{dx}(v^2) = -\frac{x}{10}$. (i) If it is pulled down 5″ and released, find its velocity when passing O; (ii) how far must it be pulled down to pass O with a velocity of 1 inch per sec.? (iii) with the data of (i), find an equation expressing δt in terms of x, δx, and write down the integral which gives the time of a complete oscillation.

P. 4.

1. If $pv^n = 329$, find n when $p = 11{\cdot}6$, $v = 3{\cdot}82$.

2. The connection between the speed v feet per sec. of a column of water produced by a pressure of head h feet is shown in the following table:

h	4	16	25	36	64
v	15·7	32·7	39·3	49·1	63·3

Plot v against $\sqrt{h}$ and find the most probable relation expressing v in terms of h.

3. A circular cylinder of radius a in., height l in., is closed at one end with a flat cover, and at the other with a spherical cap of height h in. Find the total surface. [The area of a spherical cap of height h which is part of a sphere of radius r is $2\pi rh$.]

4. Given $y = (x-1)^3$: (i) find the value of $\dfrac{dy}{dx}$ when $x = 1$; (ii) find the sign of $\dfrac{dy}{dx}$ when $x = 0{\cdot}9$ and when $x = 1{\cdot}1$; (iii) is $x = 1$ a turning point of the function $(x-1)^3$? (iv) find the sign of $\dfrac{d^2y}{dx^2}$ when $x = 0{\cdot}9$ and when $x = 1{\cdot}1$; (v) sketch the graph of $(x-1)^3$ from $x = 0$ to $x = 2$, and explain the geometrical interpretation of results (i) to (iv).

5. The gradient of a curve which passes through the origin is given by the equation $\dfrac{dy}{dx} = x^2(1-x)$; find the area between the curve, the x-axis and the ordinate $x = 1$.

6. If a gas occupies v cu. feet under a pressure of p lb. per sq. inch, where $p \,.\, v^{1{\cdot}42} = 300$, the work done in expanding from v_1 cu. ft. to v_2 cu. ft. is $\int_{v_1}^{v_2} p \,.\, dv$ ft.-lb.; find the work done in expanding from 5 to 8 cu. ft.

P. 5.

1. If a exceeds b by x per cent., by how much per cent. does $2a$ exceed b? In what ratio must b be increased to give a?

2. The following figures are given for torsional resistance (steel):

Diameter of shaft in inches, d	1	2	2·5	3	3·5
Torsional resistance in tons, m	1·28	10·2	19·9	34·5	54·7

Plot $\log m$ against $\log d$; hence find the best fit equation expressing m in terms of d.

3. Construct a simple line chart connecting Centigrade and Fahrenheit temperatures from 0° to 50° C., given 0° C. = 32° F. and 100° C. = 212° F. Read off the equivalents of 18° C., 98° F.

4. If fuel contains C per cent. carbon, H per cent. hydrogen, N per cent. oxygen, the calorific value of 1 lb. of fuel in British thermal units is V, where $V = 145C + 620H - 77{\cdot}5N$. Construct a nomogram for $70 < C < 90$, $2 < H < 6$, $4 < N < 8$. Read off from it the value of V when $C = 75$, $H = 3$, $N = 6$.

5. If $z = \dfrac{1}{\sqrt{y}}$ and $y = 49 - x^2$, express δz in terms of y, δy, and express δy in terms of x, δx; hence find δz in terms of x, δx, and deduce the value of $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{49 - x^2}}\right)$.

6. Sketch the graph of $(x-2)(5-x)$, and find the area of the portion of it which lies above the x-axis.

P. 6.

1. If $t = 2\pi\sqrt{\dfrac{l}{g}}$ is the same formula as $l = kt^2$, find k correct to 3 significant figures, given $\pi = 3{\cdot}142$, $g = 32{\cdot}2$.

2. If a jet of water is projected from a $\frac{3}{4}$-inch nozzle under a pressure of P lb. per sq. in., the effective height, h feet, of the jet is connected with P as follows:

P	40	50	60	70	80	90	100
h	61	67	72	76	79	81	83

Plot $\dfrac{P}{h}$ against P, and hence find an approximate formula for h in terms of P.

3. Represent by a straight-line graph the formula $l = 39t^2$ for $0{\cdot}5 < t < 1{\cdot}5$.

4. For ordinary wrought-iron shafting, if $D=$ diameter of shaft in inches, $H=$ indicated horse-power transmitted, $N=$ number of revolutions per minute, $D=\sqrt[3]{\left\{\frac{65H}{N}\right\}}$; construct a nomogram for $10<H<100$, $10<N<300$. Read off from it the value of D when $H=80$, $N=100$.

5. Sketch the graph of the function $y=f(x)$ from $x=0$ to $x=8$, from the data in the following table :

x	0	0 to 1	1	1 to 2	2 to 4	4	4 to 6	6	6 to 8
y	0			+	−			−	
$\frac{dy}{dx}$	+	+	0	−	−	0	+	0	−

6. Interpret graphically $\int_0^1 \sqrt{1-x^2}\,dx$.

By using (i) Dufton's rule, (ii) Simpson's rule, find an approximate value of this definite integral.

P. 7.

1. If a 1 candle-power lamp is at A, and if $AB=1$ foot, the amount of illumination at B of a plane perpendicular to AB is called the unit of illumination. The amount of illumination varies inversely as the square of the distance from the light and directly as the candle-power. (i) How many units of illumination are given by an x candle-power lamp at a distance of r feet ? (ii) $AC=15$ feet, a 20 candle-power lamp is at A and a 40 candle-power lamp at C ; a screen perpendicular to AC cuts AC at P ; find the position of P if both sides of the screen are equally illuminated.

2. Given that 1 litre = 1·76 pints, construct a simple line chart connecting pints and litres, up to 1 gallon.

3. The formula given for staying flat surfaces in boilers is $d=t\sqrt{\left(\frac{16000}{P}\right)}$, where $d=$ pitch of stays, $t=$ thickness of plate in inches and $P=$ pressure of steam in lb. per sq. inch. Construct a nomogram for $0<t<1$, $0<P<200$.

4. The following swimming records are taken from *Whitaker's Almanack* :

Distance in yards, d -	150	220	300	440	500
Time in seconds, t - -	92·4	145·4	210	323	367·2

Plot log t against log d, and express t in terms of d. The record for 300 metres is 230·2 sec. Is this what your formula would lead you to expect? [1 metre = 1·094 yards.]

5. (i) At what point on the graph of $y = x^2 - 2x$ is the gradient 1, if the unit for each axis is 1 inch?

(ii) How is the answer to (i) affected if the unit for the x-axis is 1 inch and the unit for the y-axis 2 inches?

6. If the graph of $y = f(x)$ passes through the origin and has a maximum at $x = 1$ and a minimum at $x = 2$, but no other turning points, and if the slope at the origin is 1, find a value of $f(x)$.

P. 8.

1. $ABCD$ is a rectangular sheet of paper; $AB = a$ in., $BC = b$ in. $(a > b)$; a rectangle $APQD$ is cut off the ends just large enough to allow two equal circles to be cut from it so that when the remainder $PBCQ$ is rolled into a circular cylinder, these two circles will form its ends. (i) Express the volume of this cylinder in terms of a, b. (ii) If the area of the paper is 20 sq. in., for what value of b is the volume of this cylinder greatest?

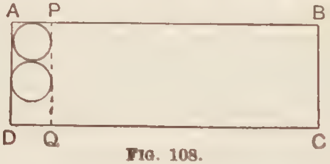


FIG. 108.

2. What equation can be solved from the graphs of $xy = 4$ and $y = x^2 - x$? By *sketching* their graphs state how many real roots this equation has. Check your answer by an algebraic solution, showing that $x = 2$ is one answer.

3. The following table gives the diameters of Cornish boilers in inches for various indicated horse-powers:

Horse-power, H	10	20	30	40	60
Diameter d,	36	51	62·5	72·2	88·5

Plot log d against log H, and hence express d in terms of H.

4. Represent by a nomogram the formula (p. 300),

$$F = \frac{Wv^2}{30d} \text{ for } 150 < W < 200,\ 40 < v < 60,\ 150 < d < 200,$$

and read off the value of F for $W = 180$, $v = 50$, $d = 160$.

5. (i) Sketch the graph of $xy=4+x^2$.

(ii) At what points on this graph is the tangent parallel to the x-axis ?

6. The weight per inch length of a rod AB varies as the cube of its distance from A ; the rod is 20 inches long and the weight at B is 5 lb. per inch length. Find (i) the total weight of the rod, (ii) the length of AC if the portions AC, CB are of equal weight.

Q.

(CHAPTER XXIV.)

Q. 1.

1. If $a \neq b$, prove that $x(x+a)+b$ and $x(x+b)+a$ have a common factor if, and only if, $a+b=-1$.

2. If a number and its square root contain respectively p and q digits, prove that $2p-4q+1=(-1)^p$.

3. Write $5(a+2b)(3a-4b)$ as the difference of two squares.

4. What digit must the symbol x represent if the product of the two numbers $x5$ and $3x$ is the number $2xx5$?

5. In the sequence $u_1, u_2, u_3, \ldots$, if $u_n=2u_{n-1}+a^n$ and $u_1=a$, write down the value of u_2, u_3, and express u_n as a function of a and n.

6. Solve the equations: $\begin{cases} xy=72, \\ yz=60, \\ x+z=11. \end{cases}$

Q. 2.

1. What is the condition that the values of x and y found from the equations $ax+by=p+qx=p+qy$ may be, (i) inconsistent, (ii) indeterminate ?

2. Prove that the fifth power of any whole number has the same last digit as the whole number.

3. If $ab+bc+ca=0$, prove that $\sqrt{\dfrac{a+c}{b+c}}=\pm\dfrac{a}{b}$.

4. Prove that the relation $y=\dfrac{3(2x-5)}{x-2}$ can be written in the form

$$\frac{y-p}{y-q}=\frac{3(x-p)}{x-q},$$

and find the numerical values of p and q.

5. If a is the nth root of x and if y is nearly equal to x, then the nth root of y is approximately $\frac{y+(n-1)x}{nx}a$. Use this result to find the cube root of 1020, and compare your answer with that obtained by logarithms.

6. A company advances £95 to one of its employees for the purchase of £100 War Loan, and it is to be repaid by 10 equal annual instalments, the first to be paid in one year's time. Allowing compound interest at 5 per cent., find the amount of each instalment.

Q. 3.

1. If $a+\frac{1}{b}=c$, $b+\frac{1}{c}=d$, $c+\frac{1}{d}=a$, prove that $cd=\frac{1}{2}$, $bc=-\frac{1}{2}$, and find the numerical value of ab.

2. If $x(a-b)=y^2$ and $y(a+b)=x^2$, express a^2-b^2 and $\frac{a}{b}$ in terms of x and y.

3. Solve the equations : $\begin{cases} x+y=z=2xy, \\ \frac{1}{x}-\frac{1}{y}=1. \end{cases}$

4. If $(x+1)^2$ is greater than $5x-1$ and less than $7x-3$, find the only possible *integral* value of x.

5. Solve the equation $\frac{x^2-5}{x^2-1}+3+\frac{2}{x-1}=0$,

and explain the appearance of the root $x=1$, which does not solve the original equation.

6. If the result of substituting $\frac{X+Y}{\sqrt{2}}$ for x and $\frac{X-Y}{\sqrt{2}}$ for y in $ax^2+2hxy+by^2$ is $AX^2+2HXY+BY^2$, prove that

(i) $A+B=a+b$; (ii) $AB-H^2=ab-h^2$.

Q. 4.

1. Rewrite the formula $T_1=T_2\left(\frac{p_1}{p_2}\right)^{\frac{n-1}{n}}$ with n as the subject.

2. Solve the equations : $\begin{cases} x(1+y)=b(a-1), \\ y(1+x)=a(b-1). \end{cases}$

3. Prove that $x=\frac{1}{1-a}$ is a root of the equation :

$$x+\frac{1}{1-x}+\frac{x-1}{x}=a+\frac{1}{1-a}+\frac{a-1}{a}.$$

What are the other two roots ?

4. If
$$\frac{x^2-yz}{a}=\frac{y^2-zx}{b}=\frac{z^2-xy}{c},$$
prove that
$$\frac{a^2-bc}{x}=\frac{b^2-ca}{y}=\frac{c^2-ab}{z}.$$

5. A betting-man learns that only three horses are going to start in a certain race, and finds that the odds* quoted against them by a bookmaker are 3 to 2 against, 3 to 1 against and 2 to 1 against. Show that the man by adjusting his bets properly can make certain of winning £a, whichever horse wins, and find how much he must bet on each horse to do this.

6. Generalise Question 5 by considering the case in which three horses only are running and the odds quoted against them are p_1 to 1, p_2 to 1 and p_3 to 1; show that a man betting on them can be sure of winning £a, whichever horse wins, provided
$$p_1p_2p_3-p_1-p_2-p_3-2 \text{ is positive.}$$
Examine the case when $p_1=5$, $p_2=2$ and $p_3=1$.

Q. 5.

1. If b^2 is an arithmetic mean between a^2 and c^2, prove that $\frac{1}{a+c}$ is an arithmetic mean between $\frac{1}{a+b}$ and $\frac{1}{b+c}$.

2. If $x+y=1$, prove that $(x^2+y)^2=(y^2+x)(1-xy)$.

3. If $p=ny-mz$, $q=lz-nx$, $r=mx-ly$ and $x+y+z=0$, prove that $\frac{qrx+rpy+pqz}{xyz}$ can be expressed in terms of l, m, n only.

4. A circle of radius r is divided into 12 equal sections by 6 diameters. Through the extremity A of one of these diameters a perpendicular AN_1 is drawn to an adjacent diameter; then N_1N_2 is drawn perpendicular to the next diameter, and so on. Find an expression for the distance of N_n from the centre of the circle, and show that the total length of the spiral so formed cannot be greater than $r(\sqrt{3}+2)$.

[Use the fact that $\cos 30°=\frac{\sqrt{3}}{2}$, $\sin 30°=\frac{1}{2}$.]

5. Prove that $\sum_{a,b,c}(b-c)(x-b)(x-c)\equiv-(a-b)(b-c)(c-a)$.

6. (i) Find a symmetrical, homogeneous expression of the third degree in x and y, which has $3x-y$ as a factor, and is equal to 8 when $x=y=1$.

(ii) Find the most general rational integral algebraic function of x, of the fourth degree, which leaves remainder 1, when divided by $(x-1)$, $(x-2)$, $(x-3)$ or $(x-4)$.

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* For the meaning of this, see paper G. 6, Question 1.

Q. 6.

1. If $\frac{1}{a+b}+\frac{1}{c}=\frac{1}{a}+\frac{1}{b+c}$, prove that $\frac{1}{(a+b)^3}+\frac{1}{c^3}=\frac{1}{a^3}+\frac{1}{(b+c)^3}$.

2. If
$$f(x, y) \equiv 2xy+x+y+2=0,$$
and if
$$X=\frac{x^3-3x-1}{3x(x+1)}, \quad Y=\frac{y^3-3y-1}{3y(y+1)}$$
prove that
$$f(X, Y)=0.$$

3. If
$$\begin{cases} x^2=y^2+z^2-2lyz, \\ y^2=z^2+x^2-2mzx, \\ z^2=x^2+y^2-2nxy, \end{cases}$$
prove that
$$\frac{x^2}{1-l^2}=\frac{y^2}{1-m^2}=\frac{z^2}{1-n^2}.$$

4. Find the conditions that $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ may be the square of an expression of the type $lx+my+nz$.

5. Find the factors of (i) $\sum_{x,y,z}(y-z)^3$;

(ii) $\sum_{a,b,c} a(b-c)^3$.

6. Prove that if x is real and a and b are positive quantities, such that $b > a$, then $\frac{x^2-ax+\frac{1}{4}b^2}{x^2+ax+\frac{1}{4}b^2}$ cannot be less than $\frac{b-a}{a+b}$ or greater than $\frac{a+b}{b-a}$.

Q. 7.

1. If $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$ and $xy+yz+zx=0$, prove that $x^2y^2z^2=\frac{a^2b^2c^2}{(a+b+c)^3}$.

2. Prove that the values of x, y which satisfy the equations $(2x+y)^2=2x-3y+1$ and $x+2y=1$ must also satisfy the equation $x^2-xy+3y^2=0$.

3. If $x^{\frac{1}{2}}+y^{\frac{1}{3}}=1$, prove that $y^2-2y(1+3x)+(1-x)^3=0$.

4. Solve the equations : (i) $(x+1)(x+3)(x+5)(x+7)=384$;

(ii) $\sqrt[3]{1-x}+\sqrt[3]{2-x}=\sqrt[3]{3-2x}$.

5. Find the limits between which m must lie if both roots of the equation $x^2-(m-1)x+(m-1)(m-2)=0$ are real.

6. (*a*) What is the equation whose roots are α, β, γ ?

(*b*) If α, β, γ are the roots of the equation $x^3+px+q=0$, prove that $\alpha+\beta+\gamma=0$; and express in terms of p, q the values of (i) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$;

(ii) $\Sigma(\alpha^2)$;

(iii) $\Sigma(\alpha^3)$.

Q. 8.

1. (i) If $$\frac{ab}{b+x}-\frac{cd}{d+y}=\frac{bc}{x}-\frac{ad}{y}=z,$$
prove that either $\frac{x}{b}+\frac{y}{d}+1=0$ or $z=a-c$.

(ii) Given $ax^2=\frac{1}{y}+\frac{1}{z}$, $by^2=\frac{1}{z}-\frac{1}{x}$, $cz^2=\frac{1}{x}+\frac{1}{y}$,
prove that $$abcx^2y^2z^2=b+c-a.$$

2. If y is small, an approximate value for $\sqrt{(1+4y)}$ is $\frac{1+3y}{1+y}$; if $y=z+z^2$, find in terms of z the error in the approximation.

3. If $\Sigma x=0$, prove that
$$(x+y-z)(y+z-x)(z+x-y)=-8xyz.$$

4. Determine the values of a, b, c and d so that
$$a+b(x-1)+c(x-1)(x-2)+d(x-1)(x-2)(x-3)$$
may be identically equal to $4x^3-5x^2+6x-7$.

5. The following transformation is used in the theory of Relativity, $x_1=\beta(x-vt)$; $t_1=\beta\left(t-\frac{vx}{c^2}\right)$, where $\beta=\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. Express x and t in terms of x_1, t_1 etc., in as simple a form as possible. What do you notice about the forms of x and t so obtained?

6. Solve the equations: (i) $\sqrt{a+x}+\sqrt{b+x}=\sqrt{a+b+2x}$;
(ii) $2x^4-7x^3+9x^2-7x+2=0$.

Q. 9.

1. If $x=m^2-n^2$ and $y=(2m+n)n$, prove that x^2+xy+y^2 is a perfect square.

2. If $x^3=2$, prove that $(l+mx+nx^2)(3-x+x^2)$
$$\equiv(3l+2m-2n)+(-l+3m+2n)x+(l-m+3n)x^2.$$

3. Find an equation whose roots are those of $2x^2-x-5=0$, each diminished by c, and find the value of c in order that in the new equation the sum of the roots may be zero.

4. Find the factors of: (i) $(a+b)(b+c)(c+a)+abc$;
(ii) $(a+b+c)^3-\Sigma(b+c-a)^3$.

5. Solve the equations : $\begin{cases} x^2 + 2yz = y^2 + 2zx = 3, \\ z^2 + 2xy = 19. \end{cases}$

6. (i) Write $\dfrac{7-3x}{(2-x)^2(3-x)}$ in the form

$$\frac{A}{(2-x)^2} + \frac{B}{(2-x)} - \frac{C}{3-x}.$$

(ii) When $f(x)$ is divided by $x-1$, the quotient is $\phi(x)$, and the remainder is 2 ; when $\phi(x)$ is divided by $x-3$, the quotient is $\psi(x)$ and the remainder is 4. What is the remainder when $f(x)$ is divided by $(x-1)(x-3)$?

Q. 10.

1. If $\dfrac{a^2-d^2}{b^2c^2-d^4} = \dfrac{a}{bcd}$, prove that either $\dfrac{a}{b} = \dfrac{c}{d}$ or $d^3 = -abc$.

2. Prove that $a + b(1-a) + c(1-a)(1-b) + d(1-a)(1-b)(1-c)$

$$\equiv 1 - (1-a)(1-b)(1-c)(1-d).$$

3. If $\dfrac{ax-b}{1} = \dfrac{bx-c}{y} = \dfrac{cx+a-1}{y^2}$ and $b^2 \neq ac$, prove that x, y are the roots of a certain quadratic equation.

4. The following is a method in use amongst Russian peasants for multiplying by numbers larger than 2.

To multiply 34 × 43, halve the left-hand number and double the right-hand number, ignoring fractions, and continue the process until the left-hand number is 1, thus : writing each line of working immediately under the preceding figures.

Cross out all the lines in which the numbers in left-hand column are even, and add up remaining numbers in right-hand column.

~~34 × 43~~
17 × 86
~~8 × 172~~
~~4 × 344~~
~~2 × 688~~
1 × 1376

1462

Then $34 \times 43 = 1462.$

Show that any number can be written in the form

$$a \,.\, 2^x + b \,.\, 2^{x-1} + c \,.\, 2^{x-2} + \ldots + k,$$

where $a, b, c, \ldots k$ are either 1 or 0, and hence justify this method of multiplication.

5. A number of cannon-balls are arranged in the form of a triangular pyramid, so that the sides of the successive layers are made up of 1, 2, 3, 4, ... balls, while the layers contain 1, 3, 6, 10, ... balls. It is known that when there are n layers, the number of balls in the pyramid is given by an expression of the form

$$an + bn^2 + cn^3,$$

where a. b and c are independent of n. Find their values.

(Army C.)

6. Prove that $(\sum_{x,\,y,\,z,} x)^3 - 2(\Sigma x)(\Sigma x^2) - 8xyz$ has a factor $x+y-z$, and hence find all its factors.

7. If $x=\phi(a)$ is one root of the equation

$$f(x)=f(a),$$

prove that $x=\phi[\phi(a)]$ is also a root.

Use this result to find the remaining roots of the equation in Paper Q. 4, No. 3.

SYMBOLS

$\bumpeq$ is approximately equal to.

$a : b$ equals $\frac{a}{b}$. See p. 284.

$a : b : c : d : e : \ldots.$ See p. 288.

$\propto$ varies as.

$\leq$ is less than or equal to.

$\geq$ is greater than or equal to.

$\sqrt[n]{x}$. See p. 255.

$\log_a x$. See p. 280.

$f(x)$ a function of x.

Σ. See p. 454.

$\rightarrow$ tends to.

$\rightarrow +0$ tends to zero through positive values.

$\rightarrow -0$ tends to zero through negative values.

∞ infinity. See p. xv, 311, 324.

$\left.\begin{array}{c}|a-b|\\ a \sim b\end{array}\right\}$ the positive value of $\pm(a-b)$ if a, b are real.

$\underset{x \to a}{\mathrm{Lt}}\, f(x)$ the limit of $f(x)$ when x tends to a. See p. xv, 324.

δx a small change in x. See p. xvi, 339.

$\frac{d}{dx}$. See p. 340.

$\int f(x)\,dx$, $\int_a^b f(x)\,dx$. See pp. 368, 370.

${}^nP_r$, ${}^nC_r$. See p. 398.

$\underline{|n}$. See p. 395.

GLOSSARY AND INDEX

References in Roman **numerals are to pages of the Introduction in the** Teacher's Edition.

Abscissa. The x-coordinate of a point. p. 325.

Approximation. By differentiation. p. 356.
By binomial theorem. p. 408.

Arbitrary constant. A constant ($q.v.$) whose value is as yet undetermined.

Areas. Calculation of. p. 368.
Approximate calculation of. p. 376.

Arithmetic mean (A.M.). p. 382.

Arithmetical progression (A.P.). p. 382.

Best-fit line. p. 417, p. xxi.

Binomial. The sum or difference of two single terms such as $(a+b)$ or $(x-a)$.

Binomial theorem. p. 407, p. xx.

Characteristic. The integral part of a logarithm. p. 269.

Combination. p. 398.

Common difference of an A.P. p. 382.

Common ratio of a G.P. p. 385.

Constant. A number whose value does not change.

Cross-multiplication. p. 451.

Decreasing function. p. 346.

Definite integral. p. 370.

Dependent variable. A variable number whose value depends upon the value of one or more independent variables.

Derived function. p. 340.

Determinant. p. 451.

Differential coefficient. p. 340.

Differential equation. p. 364.

Differentiation. p. 339, p. xvi.

Discriminant. p. 441.

Dufton's rule. p. 376, p. xviii.

Elimination. p. 461.

Empirical formula. A formula based upon numerical results found by experiment. p. 415, p. xx.

Equation of a straight line. p. 423.

Factorial. Factorial n, where n is an integer is written $\underline{|n}$ or $n!$ and means the product of the numbers $n(n-1)(n-2) \ldots 3 . 2 . 1$.

Function. A number, or quantity, is said to be a function of one or more variables, if its value can be determined when the values of these variables are known. p. 306, p. xiv.

Functional notation. p. 320.

Geometrical mean (G.M.). p. 385.

Geometrical progression (G.P.). p. 385.

Gradient. A measure of the steepness of a graph. pp. 333, 338, p. xv.

Graphical solutions. p. 314.

Growth function. p. 392.

Homogeneous function. p. 454.

Incommensurable. Two numbers whose ratio cannot be expressed in the form $p : q$, where p and q are integers, are said to be incommensurable. p. xii.

Increasing function. p. 346.

Indefinite integral. p. 370.

Independent variable. A number which varies unconditionally and independently of other variable numbers.

Indices. Fractional. p. 260, p. ix.
Integral. p. 253.
Negative. p. 260.

Infinity. pp. 311, 324, p. xv.

Integer. A whole number.

Integral. p. 370.

Integration. p. 364.

Interpolation. When the value of a function is known for some values of the variables within a certain range, approximate values of the function may be determined from these for other values of the variables within this range: this process is known as interpolation. p. xx. See Exx. 20, 21, p. 290, and Exx. 18,19, p. 422.

Irrational number. A number which cannot be expressed in the form $\frac{p}{q}$, where p and q are integers. p. 261, p. x.

Limit. p. 324, p. xv.

Limiting sum. The number to which the sum of a series tends as the number of terms of the series tends to infinity. p. 388.

Line chart. p. 424.

Logarithms. pp. 267, 269, 275, p. x.

Logarithmic graduation. p. 428.

Logarithmic notation. p. 280.

Logarithm paper. p. 429, p. xxii.

Mantissa. The decimal fraction of a logarithm. p. 270.

Maximum. The value of a function which is greater than all other values of the function in the immediate neighbourhood. p. 347.

Mean. The average of two or more quantities. See also Arithmetic mean and Geometric mean.

Minimum. The value of a function which is less than all other values of the function in the immediate neighbourhood. p. 347.

Nomogram.
Nomography. The graphical representation of equations and formulae containing more than two variables. pp. 424, 431 *et seq.*, p. xxii.

Odds. p. 400, p. 488, Ex. 1.

Ordinate. The y-coordinate of a point. p. 325.

Permutation. p. 394.

Probability. p. 400, p. xix.

Proportion. pp. 288, 450.

Proportional. Third, Fourth, Mean. p. 288.

Quadratic equations. Theory of. p. 441.

Rate of change. pp. 328, 332, 334, 360, p. xv.

Ratio. p. 284, p. xi.

Rationalise. pp. 261, 262.

Reference line. p. 432.

Remainder theorem. p. 457.

Root. An n^{th} root, p. 255. Of an equation, p. 441.

Sequence. A succession of numbers which may or may not obey some definite law. p. 380.

Series. A succession of positive or negative numbers obeying some definite law and regarded as a single group. p. 380.

Sigma notation ($\sum$). p. 454.

Simpson's rule. p. 377.

Slide rule. p. x.

Sum to infinity (see Limiting sum). p. 388.

Support line. p. 432.

Symmetrical function. p. 454.

Turning point. p. 346.

Turning value. p. 347.

Variable. A number whose value changes.
See Dependent variable and Independent variable.

Variation. Simple. pp. 291, 299, p. xii.
Joint. p. 299.

Volumes. Calculation of. See Ex. 8, p. 372.

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

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LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

π may be taken to be 3·1416.

log π may be taken to be 0·4971.

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	3	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	4	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

ANSWERS.

PART I.

EXERCISE I. a. (p. 3.)

1. $n+n=$ twice n. **2.** $n-n=0$. **3.** $x \times x = x^2$.

4. $a+b=b+a$. **5.** $2n-1$ is an odd number.

6. $a \times \frac{1}{a} = 1$. **7.** $\frac{3 \times x}{5 \times x} = \frac{3}{5}$. **8.** $x^2 \times x = x^3$.

9. What must be added to 3 to obtain x? $x-3$.

10. By what must 7 be multiplied to obtain x? $\frac{x}{7}$.

11. $n-1$, n, $n+1$ are three consecutive whole numbers.

12. If the angle ABC is x degrees, the angle ABD is $180-x$ degrees.

13. The number of shillings in x guineas is $x \times 21$ (or $21x$).

14. There are $\frac{n}{2}$ even numbers up to and including n (where n is an even number).

15. $0 \times x = 0$. **16.** 3 times $x + x = 4$ times x.

17. Since $a-b=c$, $\therefore\ a=b+c$.

18. x per cent. of a number is $\frac{x}{100}$ of that number.

19. $20-7$; $20-3$, $b-a$. **20.** $x+2$; $x+1$.

21. 28×3; $a \times 3$ (or $3a$); $\frac{b}{3}$. **22.** $\frac{x}{100} \times 85$; $\frac{x}{100} \times y$.

23. 5^2+5; n^2+n.

I. b. (p. 6.)

1. (i) Multiply 2 by a (or a by 2). (ii) Multiply 3 by a (or a by 3) and add b. (iii) $x \times x \times x$. (iv) Add x to $x \times x \times x$. (v) Multiply 2 by a and then take the square root. (vi) Add 1 to the square root of x. (vii) $4 \times a \times b \times b$. (viii) Multiply a by b and subtract b.

2. x^5; x^{13}. **3.** $2 \times x \times x$; $2x^2$.

4. (i) 18. (ii) 11. (iii) 11. (iv) 81. (v) 36. (vi) 3.

5. (i) 5. (ii) 5. (iii) 6. (iv) 6. (v) 16. (vi) 0. (vii) 16. (viii) 60. (ix) 12. (x) 18. (xi) 5. (xii) $\frac{3}{4}$. (xiii) 4.

6. (i) 8. (ii) 1. (iii) 20. (iv) 20. (v) 7. (vi) 17.

7. (i) $\frac{1}{2}$. (ii) $\frac{1}{4}$. (iii) 1. (iv) $2\frac{1}{2}$. (v) 2. (vi) $\frac{1}{4}$.

8. (i) 30. (ii) 90. (iii) 120. (iv) 90. (v) 38. (vi) 10. (vii) 0.

9. (i) 8. (ii) 14. (iii) 54. (iv) $\frac{3}{8}$. (v) 0. (vi) $\frac{1}{4}$. (vii) 8.

10. (i) $\frac{4}{15}$. (ii) $\frac{2}{15}$. (iii) $\frac{3}{8}$. (iv) $\frac{5}{3}$. (v) $\frac{4}{9}$. (vi) 5.

11. (i) 1. (ii) 0. (iii) 2. (iv) 0. (v) 0. (vi) 0. (vii) 4.

12. 6. 13. 8. 14. 21. 15. 6. 16. 12. 17. 7. 18. 432

I. c. (p. 9.)

1. If $x=1\frac{1}{2}$, $3x=4\frac{1}{2}$, $4x=6$ and $1\frac{1}{2}+4\frac{1}{2}=6$; and so on.

11. $9\times3=12+15$; $5\times3=0+15$. 12. $5\times\frac{12}{5}=10$.

13. 7 is the square root of $36+12+1$. 14. $77-21=56$.

15. $9^2-8^2=17$. 16. 36. 17. 59° F. 18. 35 sq. in.

19. 160 Km. 20. $1+2+3+4+5+6=18+3$.

21. $10=3\times4-2$; $22=3\times8-2$. 22. (i) 4. (ii) 12 right angles.

23. 7 o'clock. 24. 79 sq. in. 25. $4\frac{1}{2}$d.

II. a. (p. 11.)

1. 5×4. 2. $4a$. 3. $3x^2$. 4. $2xy$. 5. $6x$. 6. $30a$.

7. 8×17. 8. $8x$. 9. 5×9. 10. $5x$. 11. 6×9. 12. $6b$.

13. $5ab$. 14. $5x^2$. 15. $7a$. 16. x^2. 17. $21xy$. 18. $5a^3$.

II. b. (p. 12.)

1. $2a+b$. 2. $3x+2y$. 3. $3b+7$. 4. $3x+2y$.

5. $3x+8$. 6. $a+2b+6$. 7. $a+b$. 8. $5x+y$.

9. $5a+b$. 10. $x+2y$. 11. b. 12. $2a+2b+2c$.

13. $3x+3y$. 14. $2x$. 15. $4x$. 16. $2a+2c$.

17. $2x+y+8$. 18. $a+b+1$. 19. $2xy+2yz$. 20. $x^2+y^2+z^2$.

21. $x^2+xy+xz$. 22. $4xy+x+y$. 23. $10a^2+10b^2-5c^2$. 24. $6yz-6xy$.

II. c. (p. 13.)

1. $2.10^2+6.10+5$; $2x^2+6x+5$; no.

2. $4.10^3+6.10^2+9$; $4x^3+6x^2+9$; no. 3. No. 4. No.

5. $2x^2+3x$. 6. x^2-1. 7. x^3-x^2. 8. 129. 9. $2x^2$.

10. x^3+x. **11.** a^3+2a^2+3a. **12.** $2x^3+2$. **13.** x^2+8x-8
14. x^2+x-2. **15.** $3a^4$. **16.** a^2+4. **17.** x^4+4x^3+1.
18. $3-6a$. **19.** a^2+2a^2+2a+1. **20.** $5a^3+5a^2+5a+5$.
21. (i) $7x^4$. (ii) 2. (iii) 5. **22.** (i) $5x^3$. (ii) 1. (iii) 2.
23. (i) $9x^4+x^3+7x^2+2x+2$. (ii) $2+2x+7x^2+x^3+9x^4$.
24. (i) $x^8+4x^6+3x^2+x+1$. (ii) $1+x+3x^2+4x^6+x^8$.
25. $4+2x+x^3$. **26.** $6x^2+2$.

II d. (p. 14.)

7. 6^5. **8.** a^5. **9.** y^7. **10.** $2 . 5^3$. **11.** $2a^3$. **12.** $6c^5$.
13. $2a^2b$. **14.** $3x^2yz$. **15.** $9a^6$. **16.** 5. **17.** a. **18.** 11^2.
19. x^2. **20.** $2a$. **21.** $6a^2$. **22.** $3x^4$. **23.** 3. **24.** y.
25. $4a$. **26.** $12a^3b^2c$. **27.** $3a^2b^2c$. **28.** $2a^2bc^2$. **29.** a. **30.** pqr.

II. e. (p. 15.)

7. $9x^2$. **8.** a^6. **9.** a^2b^4. **10.** $25x^4y^2$. **11.** $16a^2b^4c^2$.
12. $4a^4b^6c^{10}$. **13.** 3^2. **14.** a^8. **15.** $2x^2$. **16.** $3y^4$.
7. abc. **18.** $4xy^2$. **19.** (i) $8x^3$. (ii) $27a^3b^6$. **20.** $4x^9$.

II. f. (p. 16.)

1. $4x^3, 8x^4y, 2x^2y^2$. **2.** $3x, 2x^2$. **3.** (i) and (iii).
4. Yes. **5.** No. **6.** No. **7.** Yes.
8. (i) $2, 2x, x$. (ii) $3x, x, xy$. (iii) x. (iv) $4x^2, 12x^3y^3, 6x^2y^2, 6x^2y, 18x^4, 6x^2z$. (v) $12x^3y^3, 6x^2y^2, 6x^2y, 6xy, 3x^2y, 6xyz^2$. (vi) $12x^3y^3, 6x^2y^2, 6x^2y$.
9. (i) b, a, bc. (ii) $6, 3a^2, a, 2a^3$. **10.** $2a$. **11.** $6ab$.
12. $4x$. **13.** bc. **14.** $6a^2b$. **15.** ab. **16.** $6a^2b^3$.
17. $10a^3bc$; $10bc, 2a^2, 5abc$. **18.** $12x^4yz^3$; $3x^2z^3, 2xy, 4x^3z, 6yz^3$.
19. H.C.F. $5x^2$; L.C.M. $30x^3$. **20.** H.C.F. 3; L.C.M. $24xyz$.

II. g. (p. 18.)

1. $\frac{x}{y}$. **2.** $\frac{1}{a}$. **3.** x. **4.** $\frac{2}{3}$. **5.** $\frac{b}{c}$. **6.** $3b$.
7. $\frac{x}{y}$. **8.** 1. **9.** $\frac{a}{3b^2}$. **10.** $\frac{1}{b}$. **11.** $\frac{xy}{10z^2}$. **12.** $\frac{1}{bc}$.
13. $\frac{1}{b}$. **14.** $\frac{a}{b}$. **15.** x^3. **16** $\frac{3b}{4c}$. **17.** 1. **18.** $\frac{2ad^2}{3bc}$.
19. $\frac{x^2z^2}{y}$ **20.** $\frac{1}{ab}$. **21.** 1. **22.** $\frac{3b^2c}{2a}$. **23.** $\frac{2}{ac}$. **24.** $x^5y^4z^3$

II. h. (p. 19.)

1. (i) $\frac{4}{6}$. (ii) $\frac{20}{30}$. (iii) $\frac{28}{42}$. 2. (i) $\frac{2x}{2y}$. (ii) $\frac{ax}{ay}$. (iii) $\frac{xy}{y^2}$.

3. (i) $\frac{4x}{10y}$. (ii) $\frac{2xz}{5yz}$. (iii) $\frac{6xy}{15y^2}$. 4. $\frac{a^2}{ab}$, $\frac{b^2}{ab}$. 5. $\frac{15xy^2}{6x^2y^3}$, $\frac{x^3}{6x^2y^3}$, $\frac{2y^3}{6x^2y^3}$

6. (i) $\frac{6xy}{3y^2}$. (ii) $\frac{6a^3b}{12a^2b^2}$, $\frac{2a}{4b}$. (iii) $\frac{a^3b^2}{b^2}$. (iv) $\frac{y^2}{xy^2}$. (v) $\frac{a^3b^3}{ab^3x}$.

7. (i) $\frac{5}{7}$. (ii) $\frac{5}{a}$. (iii) $\frac{1}{a}$. 8. (i) $\frac{3}{2\times 7}$. (ii) $\frac{3}{2x}$. (iii) $\frac{8y}{5x}$.

9. (i) $\frac{1}{2}$. (ii) $\frac{5}{2a}$. (iii) $\frac{5b}{2a}$. 10. $\frac{5}{6x}$. 11. $\frac{x^2+y^2}{xy}$. 12. $\frac{5}{3}$.

13. $\frac{a+b}{b}$. 14. $\frac{1+3x}{x}$. 15. $\frac{1+2b}{ab}$. 16. (i) $\frac{3}{7}$. (ii) $\frac{3}{a}$. (iii) $\frac{x-y}{2a}$.

17. (i) $\frac{2}{3\times 7}$. (ii) $\frac{2}{3x}$. (iii) $\frac{y}{3x}$. 18. (i) $\frac{3}{10}$. (ii) $\frac{3}{2a}$. (iii) $\frac{3bc-b}{3a}$.

19. (i) $\frac{5}{7}$. (ii) $\frac{b-a}{b}$. (iii) $\frac{xy-1}{xy}$.

20. (i) $\frac{2}{3\times 11}$. (ii) $\frac{11-x^2}{11x}$. (iii) $\frac{z-x^2y^2}{xyz}$.

21. (i) $\frac{33}{5}$. (ii) $\frac{7x-2}{x}$. (iii) $\frac{abd-c}{d}$. 22. $\frac{a^2}{6}$. 23. $\frac{x+y}{y}$.

24. $\frac{a+c}{b}$. 25. $\frac{1}{6x}$. 26. $\frac{c-b}{a}$. 27. $\frac{2-x^2}{2x}$.

28. (i) $\frac{2\times 3}{7}$. (ii) $\frac{3a}{b}$. (iii) $\frac{xz}{y}$. 29. (i) $\frac{2}{3\times 7}$. (ii) $\frac{a}{3b}$. (iii) $\frac{x}{yz}$.

30. (i) $\frac{3\times 2}{7}$. (ii) $\frac{3a}{b}$. (iii) $\frac{xz}{y}$. 31. (i) $\frac{2\times 5}{3\times 7}$. (ii) $\frac{2a}{3b}$. (iii) $\frac{2a}{3b}$.

32. (i) $\frac{2\times 7}{3\times 5}$. (ii) $\frac{2b}{3a}$. (iii) $\frac{3a}{2b}$. 33. (i) $\frac{3\times 7}{2}$. (ii) $\frac{3b}{a}$. (iii) $\frac{yz}{x}$.

34. (i) $\frac{ac}{b}$. (ii) $\frac{a}{bc}$. (iii) $\frac{ad}{bc}$. 35. (i) $\frac{c}{b}$. (ii) $\frac{z}{y}$. (iii) $\frac{ab}{c^2}$.

36. (i) y. (ii) z^2. (iii) 1. 37. (i) xy^2. (ii) $\frac{a^2}{bc}$. (iii) abc.

38. (i) $\frac{15}{b^2}$. (ii) $\frac{c}{5a}$. (iii) $\frac{3x}{z}$. 39. (i) $\frac{3a}{4}$. (ii) $\frac{93x}{100}$. (iii) $\frac{11y}{9}$.

40. (i) $\frac{27x}{25}$. (ii) $\frac{x}{40}$. (iii) $\frac{xy+x}{y}$.

41. (i) $\frac{ab+ac}{bc}$. (ii) $\frac{x^2+xy+y^2}{xy}$. (iii) $\frac{a^2+b^2}{a}$.

42. (i) $\frac{y+1}{xy}$. (ii) $\frac{3}{a}$. (iii) $\frac{p+r}{q}$. 43. $\frac{x}{y}$. (i) $\frac{3}{5}$. (ii) $\frac{7}{2}$. No.

44. (i) $36x$. (ii) $\frac{x}{36}$. 45. (i) $12a+b$. (ii) $\frac{12a+b}{12}$. 46. $\frac{x}{12y}$.

47. $\frac{1}{4}$. **48.** $\frac{100-x}{100}$. **49.** $\frac{x-1}{x}$. **50.** No; it always $=\frac{3}{4}$.
51. No; it always $=2$. **52.** x. **53.** $3a-4b$.
54. $\frac{5}{8}$. **55.** 1. **56.** (i) $\frac{a}{a+b}$. (ii) $\frac{a+b}{b}$.

II. k. (p. 22.)

1. Subtract b from a; multiply the result by 3.
2. Multiply a by 3 (or 3 by a); from the result subtract b.
3. Add z to y; multiply by 2; add the result to x.
4. Add $2y$ to x; add z to the result.
5. Add 2 to x; multiply by y; add z.
6. Add 2 to x; multiply by y; add z.
7. Add 2 to x; add z to y; multiply the two results.
8. Add b to a; from the result subtract c.
9. Add b to a; subtract the result from c.
10. Add b to a; subtract the result from c.
11. From c subtract a; add b.
12. Add b to a; add c to a; subtract the second from the first result.
13. Add c to b; multiply by 2; subtract the result from a.
14. Add c to $2b$; subtract the result from a.
15. Subtract 2 from a; add c to b; multiply the two results.
16. Subtract $2b$ from a; add c.
17. Add b to a; take the square root of the result.
18. Subtract y from x; multiply by 2; take the square root of the result.
19. $2a-(b+c)$. **20.** $(a-b)-(c-d)$. **21.** $2(x-y)$.
22. $2x(x-y)$. **23.** $\frac{1}{2}(b+c)$. **24.** $a\div(x+y)$.
25. $(x+y)^2$. **26.** $(x+y)^2+(x-y)^2$. **27.** $\{2(x+y)\}^2$.
28. $\frac{1}{2}(x+y)+\frac{3}{4}(x-y)$. **29.** $\sqrt{x-y}$. **30.** $\sqrt{x}-\sqrt{y}$.
31. $1-(x+y)$. **32.** $(x-2)(a+b)$. **33.** $a(a+1)(a+2)$.
34. $12(x+y)$. **35.** $a-(b+c)$ in. **36.** $(a+b)-(x+y)$ mi.
37. $15-\left(x-\frac{x}{4}\right)$ sh. **38.** $(a+b)c$ sq. ft. **39.** $\sqrt{x^2+xy}$ ft.
40. $871(100-1)$. **41.** $59(2-\frac{1}{7})$. **42.** $350-(x+y)$ pounds
43. $(x+y)+\frac{1}{10}(x+y)$ pounds. **44.** $35\div(x+y)$ shillings.
45. $50(p+q)+r$ lb. **46.** $100-(x-y)$. **47.** $x+\frac{1}{2}(y-x)$ lb.

II. l. (p. 26.)

1. (i) $b+c$. (ii) $a-(b+c)$. (iii) $a-b$.
(iv) $a-b-c$; $a-(b+c)=a-b-c$ hours.

2. (i) $y-z$. (ii) $x+(y-z)$. (iii) $x+y$. (iv) $x+y-z$; $x+(y-z)=x+y-z$

3. (i) $11-4$. (ii) $11-4$ or 7. (iii) $11-6+2$ or 7 inches.

4. (i) OR. (ii) PR. (iii) PQ. (iv) PR. (v) $a-(b-c)$; $a-b+c$ inches

6. $2+10+5$; $2+(10+5)$ shillings. 7. $a-(b+c)$; $a-b-c$ inches.

8. Floor, $3+6+4$; $3+(6+4)$.

9. $2155+4425-4325$; $2155+(4425-4325)$ ft.

10. $x-y+z$; $x-(y-z)$ pounds. 16. $x+y+2z$.

17. $x+2y-z$. 18. $2x+y$. 19. y. 20. $2x-y$.

21. $2x$. 22. 1. 23. x^2-2x+1. 24. $2-x$.

25. 0. 26. $a-b-2c$. 27. $a-2b+c$. 28. $a-b$.

29. $2a$. 30. $2b$. 31. $2x^2$.

32. $a+6b$. 33. x^3+x^2-x-1. 34. x^3-x^2+x-1.

35. $1-3x+3x^2-x^3$. 36. $a^3-a^2b+ab^2-b^3$. 37. $x^2-2xy+y^2$.

38. $4xy+2y^2$. 39. x^2-3x+5. 40. x^3-3x^2+3x-1.

41. $x-y$. 42. 0. 43. $3a-3b$.

44. x^2+2z^2. 45. $x^2+y^2+z^2-2yz+2zx+2xy$.

II. m. (p. 31.)

1. $xz+yz$. 2. $xz+yz$. 3. $xz-yz$. 4. $xz-yz$.

5. $2ab-2ac$. 6. $4x^3-2xy$. 7. $2x^3+3x^2$. 8. x^3-2x^2+x.

9. $x+y$. 10. $a-b$. 11. $x-2$. 12. $2x^2-3$.

13. x^2+x-2. 14. $a-b-c$. 15. $2x^2y-2xy^2$. 16. $a-1$.

17. ab. 18. a^2. 19. x^2+x. 20. $2x^2+x-2$.

21. x^2+y^2. 22. $4x-23y$. 23. $2x^2-3xy$. 24. $4xy+3$.

25. x^2+11xy. 26. $7a-9b+38c$. 27. $2a^3+3b^3$. 28. $x-1$.

29. $xy-1$. 30. $b+a$.

II. n. (p. 32.)

1. Subtract the sum of c and d from b and add the result to a.
2. Multiply the sum of c and d by b, add e, and multiply the result by a.
3. Multiply the sum of b and c by a, subtract d, and divide the result by e.
4. Subtract c from b, subtract d from the remainder, and divide a by this result.

6. (i) $\{(3\times 20+7)4+1\}\div 80$. (ii) $\{4(20x+y)+z\}\div 80$.

7. $\{a(b+c)-x^2\}\div 144$. **8.** $2b+c$. **9.** $4x$. **10.** $6x+6y$.

11. $a+b$. **12.** $3b-c$. **13.** 0. **14.** $4x-2y$. **15.** $a-b+c-d-e$.

16. $4x^2-26xy+y^2$. **17.** xy. **18.** $a-4b+7ac$. **19.** $x-4y$.

20. 0. **21.** $a+b-a^2+ab$. **22.** $2b-2a$. **23.** $6b+c$.

24. $xy+y^2$. **25.** $1-ab+ac+d$. **26.** $2x^2-2y^2$.

II. p. (p. 33.)

1. $6x$ sh. **2.** $x+1$; $x-3$ years. **3.** $2x^2$. **4.** $15x$ miles; $3\frac{1}{3}$ hr.

5. $7x-4$; 3. **6.** $a+b$ miles an hour; $x(a+b)$ mi. **7.** $a+2b+c$.

8. $5x$. **9.** $12a+b$; $24a+2b$. **10.** $10x$ in. **11.** $3ab$.

12. (i) $12y$. (ii) $30z$. **13.** $6x$, 12, xy; $6x$, 12.

14. (i) $7x^2$. (ii) $3x^2+4x$. (iii) $12x^3$. **15.** $2x+y$ years.

16. 47, 62, 39. x followed by y. **17.** (i) $25xyz$. (ii) $9x^6y^6$. (iii) $\frac{2x}{5}$.

18. $2x+3y$ sh.; $ax+by$ sh. **19.** $a-b+c$. **20.** (i) 14. (ii) 0

21. $2z$; $\frac{4ay}{5bx}$. **22.** $3a+b$. **23.** $240a+12b$. **24.** $19x$ in. **25.** $2a+b-c$

26. (i) $\frac{1}{a}$. (ii) bx. (iii) $\frac{d}{c}$. **27.** $8x$. **28.** (i) $\frac{2}{5}$. (ii) $\frac{5}{9}$. (iii) $\frac{n-m}{n}$.

29. $y+xz$ lb. **30.** $24-3x$ lb.; $8-x$ lb.; $a(8-x)$ sh. **31.** x in.

32. $5(m-n)-3(m+n)$. **33.** (i) 8; (ii) 5. **34.** $\frac{1}{12}(b+2c)$.

35. $3\frac{1}{3}+2\frac{3}{4}-4\frac{1}{4}$; 34. **36.** (i) $\frac{10a}{11}$. (ii) $\frac{a}{2}$. (iii) $\frac{1}{22a}$. **37.** $4p^2q$; $\frac{2p+q}{4p^2q}$.

38. $120-\left(a+\frac{b}{2}\right)$ pence. **39.** $24a^2b-2ab^2$; $2ab$. **40.** 2.

41. (i) $\frac{b+c+d}{a}$. (ii) $\frac{p}{r}$. (iii) $\frac{1}{2}$. **42.** (i) x^3-2x^2+3x-2. (ii) $a-b$.

43. x^2-4z^2. **44.** $4a^2-\{2x(x-y)+y^2\}$. **45.** $20(x+1)-17$.

46. $(2x-y)^2$; $2\frac{1}{4}$. **47.** $12-2x$ miles. **48.** $120x+3000$ gallons

49. (i) $25a^2b^6$. (ii) $3x^2y^2z^2$. **50.** (i) $\frac{a^2}{c}$. (ii) $\frac{b-1}{ab}$. (iii) $\frac{2x}{3}$.

51. $(20x+y)12+z$. **52.** $67950-1359=66591$.

53. (i) $4ax$. (ii) $\frac{3x}{5y}$; $3x+5y$. **54.** (i) $16a^3b^3c^3$. (ii) $2ac$. (iii) mn.

55. (i) $4a-6$ ft. (ii) a^2-3a sq. ft. **56.** (i) $2xy$. (ii) $\frac{b^2}{c^2}$.

57. $a^m\times a^n=a^{m+n}$. **58.** (i) $\frac{10}{x}$. (ii) $\frac{99x}{100}$. **59.** b. **60.** $a-b$.

61. $2(bc+ca+ab)$ sq. in.; $6a^2$ sq. in. **62.** $\sqrt{a^2+ab}$. **64.** $\frac{a+b}{ac+bc}$,

65. $\frac{c}{ac+bc}$. **66.** $\frac{\frac{xz}{y}}{z}$; $\frac{66\frac{2}{3}}{100}$. **67.** $2\{2(2a+b)+b\}+b$.

III. a. (p. 39.)

1 x pence; $2x$ pence. **3.** ab square feet. 4. x degrees; $x+3$ degrees

5. x pence per lb. 6. x pence. **8.** x^2 square inches

9. x shillings; y shillings; *either* $(x-y)$ shillings $=3$ shillings *or* $x-y=3$

10. x miles per hour.

III. b. (p. 41.)

1. $10x$ lb. **2.** $3x$ sh **3.** $\frac{y}{3}$ yd. 4. $\frac{30x}{a}$ sh. **5.** $\frac{ay}{b}$ yd.

6. $3y$ sh. **7.** $120x$. **8.** $\frac{3x}{5}$ oz. 9. (i) 20 min. (ii) $20x$ min.

10 $\frac{y}{5}$ miles. **11.** $\frac{1}{x}$. 12. $\frac{2240}{x}$. 13. $\frac{3x}{2}$. 14 $\frac{5x}{2}$.

6. $\frac{125}{2x}$. 16. $1600(x+y)$. **17.** 1 day; $\frac{5x}{y}$ days. **18.** $\frac{y^2}{x}$ sh.

19. $\frac{ab}{x}$ days. 20. $\frac{nd}{88}$ mi. per hr.; 40 mi. per hr. 21. £$\frac{20x}{19}$.

22. $\frac{2240}{x}$ tons. **23.** $\frac{3}{7}(20x+y)$ lb. **24.** x; £$\frac{40x}{31}$. **25.** $\frac{9x+10y}{2x+2y}$ d.

III. c. (p. 43.)

1. $24x$ sq. ft. **2.** $\frac{4x}{3}$ sq. yd. **3.** $\frac{144}{x^2}$. **4.** £$\frac{ab}{4}$. **5.** $\frac{a}{7}$ ft.

6. $\frac{x^3}{24}$. **7.** $\frac{7x}{64}$ (1 sheet gives 2 pages). **8.** xyz cu. in.

9. $\frac{x}{ab}$ ft. 10. $24h$. **11.** $100A$. **12.** $72xy$. **13.** $18hx$ sq. ft.

14. $\frac{l}{12}$. **15.** xy. **16.** $\frac{3hxy}{320}$ sh. **17.** $\frac{5l}{18}$. **18.** $\frac{7n}{5}$. **19.** πd in

20. $\frac{\pi d^2}{4}$ sq. in. **21.** $1728x$ in. ($48x$ yd.). **22.** $x(x-y)$ sq. in.

23. $2(ah+bh+ab)$ sq. ft. **24.** $\frac{2\pi r}{3}$ ft. **25.** πr^2h; $\frac{25\pi r^2h}{4}$.

III. d. (p. 45.)

1. (i) $\frac{80}{10}$. (ii) $\frac{x}{10}$. **2.** (i) $60\times\frac{6}{5}$. (ii) $\frac{6x}{5}$. **3.** $\frac{100a}{b}$.

4. (i) $\frac{1}{2}\times 100$. (ii) $\frac{4}{5}\times 100$. (iii) $\frac{100}{x}$. (iv) $\frac{100x}{y}$. **5.** £$6x$.

6. £$\frac{3P}{100}$. **7.** £$\frac{3Pr}{100}$. **8.** $\frac{21n}{20}$. **9.** $100-x$ %. **10.** $\frac{100}{x}$ hr.

11. $\frac{n}{4}$. **13.** $10a$ sh. **14.** $\frac{5(12a+b)}{12}$. **15.** $\frac{100a}{a+b}$. **16.** 5 sh.; 2½ %

17. £$\frac{3x}{2}$. **18.** £$\frac{21x}{20}$. **19.** £nr; £$(100+nr)$. **20.** $\frac{500-x}{5}$. **21.** p.

22. $\frac{mn}{100}$. **23.** $\frac{100x}{100+x}$. **24.** $2a+\frac{3b}{2}$ sh. **25.** $\frac{6000}{100+x}$ sh. per qr.

III. e. (p. 48.)

1. $A=xy$. **2.** $V=lbh$. **3.** $m=\frac{5k}{8}$; 12½ mi. **4.** $W=\frac{5P}{4}$.

5. $p=\frac{5x}{6}\left(or\ p=x-\frac{x}{6}\right)$; 15s. **6.** $p=2(l+b)$. **7.** $f=\frac{3x}{7}$; 9 f.

8. $y=x\left(1+\frac{R}{100}\right)$, *or* $\frac{x(100+R)}{100}$; 27s. per week. **9.** $V=\frac{11d^3}{21}$.

10 $C=180-(A+B)$. **11.** $A=\frac{1}{4}(x_1+x_2+x_3+x_4)$. **12.** $A=6a^2$.

13. $A=2h(l+b)$. **14.** $l=24-2t$; 14 hr. **15.** $h=900(30-x)$

16. $v=\frac{r}{3}$; $18\frac{2}{3}$ mi. per hour. **17.** £$\frac{125(n-2)}{6}$. **18.** $DAC=90+\frac{x}{2}$.

19. $10\frac{3}{4}+\frac{n-1}{15}$ hr., *or* 10 hr. $(41+4n)$ min. **20.** 6 hr. + $4l$ min.

III. f. (p. 50.)

1. (i) 30°. (ii) 30° × 5. (iii) $30x$°. **2.** $50(y-x)$ sh., *or* £$\frac{5(y-x)}{2}$.

3. $\frac{x}{12}$ ft. **4.** $15n$ years. **5.** £$(a+4b)$. **6.** £$\frac{x}{10}$. **7.** $\frac{4x}{5}$.

8. $\frac{x}{40}$ days. **9.** $\frac{5x}{4}$. **10.** $\frac{a^2}{4}$ sq. in. **11.** $10x$ Kg. per sq. m.

13. $2a^2$ sq. ft. **14.** $\frac{144}{x}$; $\frac{144}{x+1}$. **15.** $5x$ %. **16.** $\frac{12l}{\pi}$.

17. $s-\frac{7x}{2}$ miles. **18.** $I=\frac{PRT}{100}$; 8s. **21.** $16+n$. **22.** $\frac{3x}{2}$.

23. $\frac{xy}{36}$. **24.** $\frac{12w}{11}$ grains. **25.** Error $< 0{\cdot}002d^3$.

26. $x-2$ o'clock. **27.** 10 %; 10 %. **28.** $5x-10$ pence.

29. n greater than 100. **30.** $\frac{64x}{63}\left(or\ x+\frac{x}{63}\right)$; $\frac{63x}{64}\left(or\ x-\frac{x}{64}\right)$.

31. Income of A is £$\frac{1}{2}I$; income of C is £$\frac{3}{4}I$. **32.** $\frac{bc}{x}$ days.

33. $\dfrac{120}{x+y}$. **34.** $\dfrac{9}{7}(x-2000)$; $2000+\dfrac{7y}{9}$. **35.** $4x-3y$.

36. £$(6a+4b)$. **37.** 5 % ; $4\frac{16}{21}$ %. **38.** 6. **39.** $4x-\dfrac{y}{6}$ sh

40. $\dfrac{2w}{3}$ ft. **41.** $\dfrac{3a+3b}{x}$. **42.** $\dfrac{315+x}{21}$ yr., *or* $15+\dfrac{x}{21}$ yr. **43.** £$\dfrac{2yz}{x}$.

44. $\dfrac{5x}{2}$. **45.** $11a=9b$. **46.** £$\dfrac{2a+b}{3}$. **48.** $2h+(\pi+2)r$; 81·4 ft.

IV. a. (p. 54.)

1. 5, 18, 13·5, 28·1, 22·5, 25·8. **2.** 0·5, 1·8, 1·35, 2·81, 2·25, 2·58 in.

3. 1, 3·6, 2·7, 5·62, 4·5, 5·16 lb. **4.** 2·5, 9, 6·75, 14·05, 11·25, 12·9 sec.

5. (5, 5) ; (18, 4) ; (13·5, 11) ; (28·1, 17) ; (22·5, 9·7) ; (25·8, 0·8).

6. Divide each coordinate in Ex. 5 by 10.

7. (1, 5) ; (3·6, 4) ; (2·7, 11) ; (5·62, 17) ; (4·5, 9·7) ; (5·16, 0·8).

8. Same as Ex. 7, units 1 in. 1d. **9.** P_1, P_3, P_6.

IV. b. (p. 56)

2. (i) 1904. (ii) 1910, 1911. (iii) 1907. (iv) 1905.

3. (i) 1841. (ii) 1811 to 1841. (iii) 1841 to 1911.

4. (i) 1906. (ii) 1902, 1908 to 1911. (iii) 1902 to 1906.

IV. c. (p. 62.)

3. 10·7. **4.** 7·1 ft. per sec. **6.** 1s. 1d. ; 30 lb. **7.** (i) 1·85. (ii) 1·73.

8. 4·1 sec. **9.** 47·3° F. **11.** (i) 2·6 (2·7). (ii) 0·21.

12. 32·2 sq. in. ; 3·56 in. **13.** (iii) 57·1. (iv) 2·71. (v) 1·71.

15. From 0 to 2 ; 2 ; 4. **16.** 4·25 mi. **17.** 14·4 mi.

V. a. (p. 66.)

1. (i) 10, 9, 8, 7. (ii) 2, 1, 0, (–1). (iii) (–2), (–3), (–4), (–5).

2. (i) 3, 4, 5, 6. (ii) (–6), (–5), (–4), (–3). (iii) (–2), (–1), 0, 1.

3. (–7)° C. **4.** (i) 1°. (ii) 0° C. **5.** 8. **6.** 10.

7. 10. **8.** (i) 25. (ii) 20. (iii) 0. (iv) (–20) ft.

9. (i) (½). (ii) (–½) mile. **10.** (i) 30. (ii) 0. (iii) (–10) miles.

11. £20, (–6), (–20), 0, 30 ; £(–20), 6, 20, 0, (–30).

12. 5, (–5), (–8), 1 min. **13.** 40, (–60), (–10), 20 yards.

14 0·7, 1, (–1), (–5), (–5·8) sh. **18** (–2) mi. **19.** (–50) ft.

20. 5 mi. **21.** 1300 ft. **22.** 3900 ft.
23. (-2600) ft. **24.** £(-2); £2. **25.** (-36), (0), 33 ft.
26. (i) 4. (ii) 0. (iii) (-4) ft. **27.** (-3); 2. **28.** 2; 9.
29. (i) 30. (ii) (-50) yd. 50 yd. E.
30. (i) 4. (ii) (-28) ft. per sec. It is coming down with a velocity 28 ft. per sec.

V. b. (p. 69.)

10. A gain of £5 + £(-3) = a gain of £(-3) + £5.
11. A rise of $3^\circ + (-7)^\circ$ = a rise of $(-7)^\circ + 3^\circ$.
12. (i) A meaning has been given to the number (-3), but not to the process $+(-3)$, *i.e.* add (-3).
13. (i) We assume that we add 5 to a negative number as we do to a positive, by counting up 5. (ii) We assume that $a+b=b+a$, whether a and b are positive or negative.
14. (i) The number after (-3) in the scale 0, (-1), (-2), (-3). (ii) The result of adding 1 to (-4). (iii) The result of adding (-4) to 1.
15. $(-2)+(-4)$.
16. *Either* a fall of $2^\circ + 4^\circ$ = a fall of 6°, *or* a rise of $(-2)^\circ + (-4)^\circ$ = a ri of $(-6)^\circ$.

V. f. (p. 75.)

1. $x+x+x+x+x$; $(-2)+(-2)+(-2)+(-2)+(-2)$.
2. (i) 12. (ii) (-12). **3.** $a \times b = b \times a$.
4. (i) $6 \times 9 = 9 \times 6$. (ii) $(-4) \times 3 = 3 \times (-4)$. (iii) $5 \times (-2) = (-2) \times 5$.
5. (i) (-12). (ii) (-12). **6.** (-35).
7. (i) 4×2, *or* 8. (ii) $4(5-3)$, *or* 8. (iii) $4(5-3)$, *or* 8.
9. (i) $3(9-5) = 3 \times 9 - 3 \times 5$. (ii) $3(5-7) = 3 \times 5 - 3 \times 7$, *i.e.* $3(-2) = (-6)$. (iii) $5(0-4) = 5 \times 0 - 5 \times 4$, *i.e.* $5(-4) = (-20)$.
10. $(-4)(3)$, *i.e.* (-12); (-20); 8, since $(-12) = (-20) + 8$.
13. 24. **14.** 12. **15.** $(a)(-b) = -(a)(b)$.
16. $(-a)(b) = -(a)(b)$. **17.** $(-a)(-b) = +(a)(b)$.
18. 24. **19.** 30. **20.** (-6). **21.** 3; 3.
22. (-5); (-5). **23.** (-3); (-3). **27.** (i) 0. (ii) 0.

V. g. (p. 77.)

1. (-30). **2.** (-35). **3.** 30. **4.** (-8). **5.** (-9).
6. 16. **7.** 3. **8.** (-2). **9.** 4. **10.** (-4).
11. 4. **12.** (-3). **13.** 1. **14.** 9. **15.** 1.

16. (-8). 17. (-2). 18. 4. 19. (-9). 20. 4.
21. (i) 0. (ii) 0. 22. 0. 23. (-100).
24. £6000; 10 years ago he had £6000. 25. £40; £(-40).
26. (i) 160 ft. per sec. (ii) (-160) ft. per sec. (iii) (-80) ft. per sec.
27. 9; 9; 3, (-3); 3. 28. $5a$, $(-5a)$; $5a$ if a is positive.

V. k. (p. 82.)

1. $5x$. 2. $-a$. 3. $-7b$. 4. 0. 5. $-7y$. 6. $-11a$.
7. a^2. 8. $-x$. 9. $5ab$. 10. $-6x^2$. 11. $-4a$. 12. $-2a^2$.
13. $-5a$. 14. $6x$. 15. $-2c$. 16. $2x$. 17. $10b$. 18. $-6x$.
19. $12y$. 20. $-2x^2$. 21. $-ab$. 22. a^2-b^2. 23. $x+2y$.
24. $-2a-1$. 25. $-6a^2$. 26. $-6a^2$. 27. $-xy$. 28. $-15x$.
29. $-6x^2$. 30. $-ab$. 31. $6ab$. 32. $-24a^3$. 33. $-3xyz$.
34. $4abc$. 35. x^3y^3. 36. 0. 37. $-3a$. 38. $-5x$.
39. $-2x$. 40. $4xy$. 41. $-4x^2$. 42. a. 43. -3.
44. $12x^3$. 45. $-2ab$. 46. $-3xy^2$. 47. -1. 48. 0.
49. $-a$. 50. $-2x$. 51. $-2x$. 52. $6x$. 53. $5y$.
54. $-2x^2$. 55. $-4a^2$. 56. $-6xy$. 57. $-4x^3$. 58. $-8ab$.
59. $7y$. 60. a^2+2a. 61. $-30x^2$. 62. -4. 63. $-16x^4$.
64. $2a^2$. 65. $-a^3$. 66. $2a^2b^2$. 67. $-2a^2$. 68. $-xy$.
69. x^3. 70. $9a^2b^2$. 71. $-4x^2y^2$. 72. $-8a^3$. 73. $-\frac{x}{y}$.
74. $-\frac{x}{y}$. 75. $\frac{x}{y}$. 76. $-2a$. 77. $\frac{1}{y}$. 78. $-\frac{1}{xy}$.
79. $-\frac{a}{b}$. 80. $-a^2$. 81. $\frac{a}{b}$. 82. $-a$. 83. x^2.
84. $-\frac{1}{2}$. 85. $-2x$. 86. x^2-4x. 87. $-3x$. 88. $2a-b-c$.
89. $-3a+3b+3c$. 90. $x^2-2xy+y^2$. 91. $x-y$.
92. $-2x$. 93. $2a^3-4a^2+2a$. 94. $a^2-2ab-b^2$. 95. $-x^2+x+2$.
96. x^2+3x-5. 97. $2y-2z$. 98. 0. 99. x^2+2x-4.
100. $2y^2-xy$. 101. $x+y+z$. 102. $-x^2+x-1$. 103. $-x^2+x-1$.
104. x^3. 105. $-5a-4$. 106. $2a^3+a+2$. 107. 0. 108. $a^3+a^2b-ab^2$

V. l. (p. 84.)

1. 4. 2. -2. 3. -4. 4. 4. 5. 0. 6. 1.
7. 3. 8. 5. 9. 8. 10. -5. 11. -8. 12. -1.
13. -4. 14. -9. 15. -2. 16. -12. 17. 9. 18. 0.

19. -6. **20.** 3. **21.** 12 **22.** 0. **23.** 18. **24.** 12.
25. -6. **26.** -1. **27.** -6. **28.** 2. **29.** $-1\frac{1}{2}$. **30.** -1.
31. $\frac{1}{2}$. **32.** $-1\frac{1}{3}$. **33.** 2. **34.** 4. **35.** 16. **36.** 4.

V. m. (p. 86.)

1. $3a-3b$. **2.** $2a$. **3.** $2x-y$. **4.** $x-y$.
5. $3x-2$. **6.** $-x-4$. **7.** $a-b$. **8.** $x-1$.
9. $2x^2+2$. **10.** $2x^2+x-1$. **11.** x^2+6x+3. **12.** $2y$.
13. $2a+4c$. **14.** $-3x+3$. **15.** $3-3x^2$. **16.** 0.
17. $6-3x$. **18.** $-x-y-z$. **19.** $-4x+3$. **20.** $-x^2-8x-2$
21. $a-2b$. **22.** $x-2y$, **23.** $-x+2y$. **24.** $a+b$.
25. $2y+2z$. **26.** $x-1$. **27.** $3x-2$. **28.** $2x+2$.
29. x^2+2x-3. **30.** $-x^2-1$. **31.** $-x^2-6x+2$. **32.** $9x-1$.
33. $a+b+c$. **34.** $-x^2+x-1$. **35.** $7x-7$. **36.** $3x-3$.
37. $-4x+13$. **38.** $3+2x-x^2$. **39.** $a^2-4ab+2b^2$. **40.** $2a-2b$.
41. $ax+bx$. **42.** $-ax-bx$. **43.** $-ax+bx$. **44.** $-a^2+ab$.
45. x^2-x. **46.** $-x^2+x$. **47.** $2y-2xy$. **48.** $-b+2ab$.
49. $-2ax+6x^2$. **50.** $-a^2b-ab^2$. **51.** $-2x^2+4x$. **52.** $-10x^2+15x$
53. $-2x^3+x^2-3x$. **54.** $-x+x^2-3x^3$. **55.** $-8a^2-2a^3$.
56. $-2x^3+2x^5$. **57.** $-x^2yz+xy^2z+xyz^2$. **58.** $-6a^2+3ab+9ac$.
59. $-18x^4+12x^3-30x^2$. **60.** $-6a^3b-2a^2b^2-4ab^3$.
61. $b+c$. **62.** $-b-c$. **63.** $-b+c$. **64.** $x-2$.
65. $-x^3-x$. **66.** $-2a+b$. **67.** x^2-3x+4. **68.** $-x^2+2x-6$.
69. $-4-x+3x^2$. **70.** $-1+x$. **71.** $a-4b$. **72.** $xy-2$.
73. a^2+4b^2. **74.** $x-y-z$. **75.** $-x^2+1$. **76.** $1-2x+x^2$.
77. a^2-c^2. **78.** $-x^2+x+1$. **79.** $x+y+z$. **80.** $-a^4+b^4$.

V. n. (p. 87.)

1. 7; -7. **2.** $-x$. **3.** 300 ft. **4.** $4x^2$. **5.** 8. **6.** -3; 2; 8.
7. $-1{\cdot}1$; $-0{\cdot}3$; $1{\cdot}7$; $-0{\cdot}5$. **8.** $3x^2$. **9.** $a-b$. **10.** $-x+y$.
11. 1500; 2000; -500; 1000; -5000; -1000 ft. **12.** $x+y$; $-(x+y)$.
13. $-1\frac{1}{3}$. **14.** 9; 9; 3, -3. **15.** $\frac{-x-1}{5}$; -1.
16. -11; $-10{\cdot}1$; $-7{\cdot}7$; $-2{\cdot}3$; $3{\cdot}5$; $9{\cdot}8$; $13{\cdot}1$; 12; $7{\cdot}6$; $0{\cdot}4$; $-6{\cdot}1$; $-9{\cdot}7$.
17. -7; -4; 0 ft. **18.** A fall of 3° C.; -1° C. **19.** $(x-y)$ mi.; 5 mi.
20. $100-x$ degrees; -10 degrees. **21.** Each $=8$. **23.** (i) -1. (ii) -1.

24. (i) -8. (ii) $-x$. (iii) $-(m-n)$, *or* $n-m$.
25. $-yz+zx-xy$; $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$. 26. $-\frac{x+y}{xy}$.
27. -15 ft.; $(a-30)$ ft.; $(30-a)$ ft. 29. (i) $2x^2-2x$. (ii) x^3+x-1.
30. (i) 1. (ii) 4. (iii) $\frac{a-b}{2}$. 31. $a^3-b^3=(a-b)(a^2+ab+b^2)$.
32. $a^2+b^2+c^2+2bc-2ca-2ab$. 33. (i) 1. (ii) -1. 34. 5.
35. $(-1, -\frac{1}{2})$. 36. (i) 1. (ii) 0. 37. One or three. 38. Each $=16$.
39. (ii) Any value between 0 and $1\frac{1}{2}$. 40. $\frac{a}{b}$. 41. (i) 0. (ii) We cannot say

VI. b. (p. 97.)

1. 4. 2. 2. 3. 4. 4. -4. 5. 7. 6. 6.
7. -3. 8. 30. 9. $\frac{3}{8}$. 10. 21. 11. 16. 12. 72.
13. 2. 14. -8. 15. $8\frac{1}{2}$. 16. 4. 17. 15. 18. 0.
19. -1. 20. 24. 21. 8. 22. 5. 23. $3\frac{1}{2}$. 24. $8\frac{1}{2}$.
25. 8. 26. 8. 27. $1\frac{5}{7}$. 28. $2\frac{7}{9}$. 29. $-\frac{1}{8}$. 30. 0.
31. 5. 32. 2. 33. 9. 34. 13. 35. 2. 36. -5.
37. 2. 38. 1. 39. $3\frac{2}{5}$. 40. 9. 41. $1\frac{1}{2}$. 42. $2\frac{2}{3}$.
43. $2\frac{1}{3}$. 44. $3\frac{1}{3}$. 45. 1·9. 46. 1·4. 47. 10. 48. 5.
49. 27. 50. 1·6. 51. 28. 52. 15. 53. 20. 54. -3.
55. 35. 56. 6. 57. $4\frac{1}{2}$. 58. 3. 59. 0. 60. 5.
61. 13. 62. 19. 63. 4. 64. 6. 65. 17. 66. $1\frac{1}{2}$.
67. 9. 68. 5. 69. 2. 70. -19. 71. 13. 72. 19.
73. 4. 74. 5. 75. 17. 76. 8. 77. $4\frac{2}{7}$. 78. 3.
79. $\frac{9}{10}$. 80. -13. 81. 5. 82. 6. 83. $37\frac{1}{2}$. 84. 60.
85. 2·414... ($2\frac{12}{29}$). 86. 60. 87. $8\frac{1}{3}$. 88. 50. 89. $1\frac{1}{2}$.
90. 4. 91. 4. 92. 17. 93. -2. 94. 2. 95. 13. 96. $3\frac{1}{2}$
97. 3. 98. $4\frac{2}{7}$. 99. 3. 100. 8. 101. 7. 102. 2·30... ($2\frac{10}{33}$).

VI. c. (p. 100.)

1. 9. 2. 18. 3. 18. 4. 3. 5. 5.
6. 15. 7. 14. 8. 36. 9. 19, 20. 10. 36.

VI. d. (p. 104.)

1. 120°, 60°. 2. 135°, 45°. 3. 80 yd. 4. 120 ft. 5. 47s. 3d. per qr.
6. 72°, 72°, 36°. 7. 25° E. of N. 8. 16 ft., *or* 24 ft. from an end.
9. 5s. 6d.; 4s. 6d. 10. 7. 11. 152 girls, 114 boys. 12. In 22 years.

13. 14 gallons. **14.** £400. **15.** 60°, 30°. **16.** 6. **17.** 35.

18. $2\frac{1}{2}$d. per gallon. **19.** 250. **20.** $\frac{2n-4}{n}$ right angles; 8.

21. Day $13\frac{1}{2}$ hr., night $10\frac{1}{2}$ hr. **22.** $42\frac{1}{2}$° N. of E.

23. $11\frac{1}{4}$ from A, or $3\frac{3}{4}$ from B. **24.** 100 lb.

25. 16s. **26.** 3s. 4d., 1s. 8d. **27.** 4s. $4\frac{1}{2}$d., 2s. 6d.

28. £160. **29.** 30s. per ton. **30.** 100 yd.; 200 yd. **31.** 24 mi.

32. 40. **33.** First kind, 300; second kind, 500. **34.** 2.

35. 16 ft. **36.** 6 mi. **37.** 400. **38.** £400. **39.** 120, 180.

40. $42\frac{1}{2}$, $25\frac{1}{2}$. **41.** 120. **42.** 10 of 12 in.; 8 of 15 in.

43. $4\frac{1}{2}$ in. **44.** 25 gallons. **45.** 18, 19.

46. 50 min. after second train starts; $33\frac{1}{3}$ mi. from the start.

47. After 20 years. **48.** £500. **49.** 20 lb.

50. 84 bushels. **51.** After 6 yr. **52.** $13\frac{1}{3}$ lb.

53. $11\frac{2}{3}$ mi. from Croydon, $28\frac{1}{3}$ mi. from Brighton.

54. 4. **55.** £280. **56.** More than 1750 were sold.

VI. e. (p. 109.)

1 (i) 1·67 ($1\frac{2}{3}$); (ii) 1·67; (iii) 1·67. **2.** $x=20$.

3. (i) x greater than 0·4. (ii) x less than 0·4. (iii) $x=0·4$.

4. 1·37. **5.** x less than 1·47. **6.** (i) 10·5 f.s. (ii) 8·3 sec

7. £12. 16s. 3d.; £$\frac{1}{16}(x-160)$ *or* £$\left(\frac{x}{16}-10\right)$.

8. No; 1 lb. coffee to 0·77 lb. chicory. **9.** 2·38 yr.

10. About 71 mi. from the station.

VII. a. (p. 112.)

1. $3x+15y=39$, $3x+7y=23$. **2.** $12x+3y=39$, $12x-10y=26$.

3. $30x+3y=9$, $30x-2y=4$. **4.** $12x+2y=40$, $5x+2y=19$.

5. $4x-14y=-10$, $49x+14y=63$. **6.** $3x+24y=15$, $20x-24y=8$.

7. $9y=23$. **8.** $x=5$. **9.** $5x=17$.

10. $3(y-2)=2(y+1)$. **11** $x=10$, $y=7$. **12.** 4, 3.

13. 4, 1. **14.** 4, 1. **15.** 5, 2. **16.** 3, 7. **17.** 3, 7.

18. 2, 1. **19.** $6\frac{2}{7}$, $3\frac{1}{7}$. **20.** $1\frac{1}{7}$, $\frac{3}{7}$. **21.** $\frac{1}{2}$, -1. **22.** 11, 5.

23. 1, $\frac{1}{4}$. **24.** 6, 4. **25.** $-10\frac{1}{2}$, $5\frac{1}{2}$. **26.** $\frac{1}{2}$, $\frac{1}{4}$. **27.** 1, 2.

28. 7, 5. **29.** 8, 8. **30.** 0·5, 0·3. **31.** 11, 6. **32.** 4, 6.

33. 2·51..., 0·26.... **34.** 4·89..., 15·45.... **35.** 13, 15.

36. 6, 3. **37.** 5, 2. **38.** 4, 1. **39.** 2, 2. **40.** 7, 3.

41. $x+2y=6$; $3x+y=12$. **42.** $3x-2y=4$; $6x-y=1\frac{1}{2}$.

43. $5x+6y=70$; $15x-y=13\frac{1}{3}$. **44.** $6x-7y=28$; $5x+9y=38\frac{1}{6}$.

45. $2x-y=3\frac{3}{17}$; $x-8y=0$. **46.** $4x+3y=38\frac{2}{5}$; $3x+4y=33\frac{1}{5}$.

47. 14, 1. **48.** $\frac{1}{3}$, 1. **49.** 6, 0. **50.** 14, -1. **51.** 4, $\frac{1}{6}$.

52. $-2\frac{1}{6}$, $-4\frac{4}{5}$. **53.** $2\frac{97}{168}$, $\frac{53}{168}$. **54.** 6, 6. **55.** 4, 6.

56. $\frac{13}{115}$, $-\frac{73}{230}$. **57.** $6\frac{4}{5}$, $10\frac{1}{5}$. **58.** 12, 16. **59.** 29, 47. **60.** 30, 28.

VII. b. (p. 115.)

1. $a=\frac{2}{7}$, $b=16$; 56 lb. **2.** $a=\frac{5}{9}$, $b=-17\frac{7}{9}$. **3.** $a=\frac{1}{8}$, $b=-15$.

4. Yes ; $a=\frac{2}{3}$, $b=-2$. **5.** $a=\frac{4}{5}$; 3s. 6d. **6.** £80.

7. $a=15$, $b=0{\cdot}6$. **8.** $b(n-a)$ shillings ; $\frac{5}{6}(n-500)$ shillings.

9. 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$.

VII. c. (p. 116.)

1. 14, 11. **2.** 54 sq. in. **3.** Classical 240, Modern 210.

4. Penny $\frac{1}{3}$ oz., half-penny $\frac{1}{5}$ oz. **5.** 63, 99. **6.** £6.

7. 147, 168. **8.** 5 mi. **9.** 280 shillings, 80 half-crown tickets

10. £800. **11.** P, $7\frac{1}{2}$; Q, $12\frac{1}{2}$. **12.** 32, 12. **14.** 53.

15. $x=30$, $y=35$. **16.** (i) $CA=4$, $AB=4\frac{2}{3}$. (ii) $TB=18\frac{2}{11}''$; $TC=21\frac{9}{11}''$.

17. 9. **18.** 20 Kg. **19.** $4\frac{1}{2}$ pages of large type, $5\frac{1}{2}$ of small.

20. 8s. 3d. **21.** 72 sq. in. **22.** $3x-y=180$; $67\frac{1}{2}°$, $45°$.

23. $4x+y=180$; $x=24$, $y=84$. **24.** £1500. **25.** 160 tons.

26. 75 mi. **27.** $AB=41{\cdot}5\ldots$, $BC=46{\cdot}5\ldots$ ft.

VIII. a. (p. 121.)

1. (i) 10 multiplied by $x+3y$ (*or* $x+3y$ multiplied by 10). (ii) $10x+30y$ (iii) 10 and $x+3y$.

2. $x+y$; 7. **3.** $x+y$. **4.** $a(x-3y)$. **5.** $x+1$; c.

6. $a(y-1)$. **7.** $x(x-1)$. **8.** $x-3y$. **9.** $3a^2(a^2-5b^2)$.

10. (i) $x(y+z)$. (ii) x. **11.** (i) $3bc(b-2c)$. (ii) $b-2c$. (iii) $3bc$.

12. No. **13.** $2a^2(a-3b)$. **14.** $5(2-3x^2)$. **15.** $xy(y-x)$.

16. $x(x^2+x+1)$. **17.** $c(c^2+cd+d^2)$. **18.** $3ab(ab-2c+5a)$.

19. $x^2(x+y+x^2)$. **20.** $a^2(a+2b)$. **21.** $a^2(5-a^3)$.

22. $7x^3y^3(x-5y^2-2x^2y)$. **23.** x. **24.** $a+b$. **25.** $6(a+b)$.

26. (i) a. (ii) $x+y$; $x-y$. **27.** (i) $a(x+y)(x-y)$. (ii) $x-y$; $x+y$.

28. (i) $xy(x+1)$. (ii) y ; x. **29.** $6x^2y(3x-y)$. **30.** $a(a+b)$.

31. $30ab(a-b)$. **32.** x^2+1. **33.** (i) $\frac{2}{3}$. (ii) $\frac{4}{5}$. (iii) $\frac{3}{4}$. (iv) $\frac{5}{6}$.

34. (i) $\frac{2}{x+y}$. (ii) $\frac{1}{x+y}$. **35.** (i) $\frac{15}{50}$. (ii) $\frac{ac}{bc}$. (iii) $\frac{a(x+y)}{b(x+y)}$.

36. $\frac{x}{x(x+y)}$. **37.** $\frac{x+y}{x(x+y)}$. **38.** $\frac{ab}{b(a-b)}$; $\frac{a(a-b)}{b(a-b)}$.

39. $\frac{2x+y}{x(x+y)}$. **40.** $\frac{2a}{(a+b)(a-b)}$. **41.** H.C.F. $x-y$; L.C.M. $x(x-y)$

42. (i) $\frac{x^2y}{xy}$; $\frac{a(a+b)}{a+b}$. (ii) $\frac{x^3}{x-y}$. **43.** $\frac{1}{x+y}$. **44.** $\frac{1}{6(a-b)}$.

45. $\frac{a^2}{a+b}$. **46.** $\frac{1}{x^2(x+1)}$. **47.** $\frac{x-3y}{6(x-y)}$. **48.** $\frac{a^2+ab+b^2}{ab(a+b)}$

49. $\frac{2x-y}{y(x+y)}$. **50.** $\frac{3x^2-2y^2}{x(x+y)}$. **51.** $\frac{a}{(a+b)^2}$. **52.** $-\frac{b^2}{a-b}$.

53. 9. **54.** 0. **55.** -12. **56.** 6.

57. 2. **58.** -2. **59.** 2. **60.** $1\frac{1}{4}$.

VIII. b. (p. 124.)

1. $x^2+7x+12$. **2.** $6x^2+19x+10$. **3.** $x^2-5x-14$.

4. $2x^2-x-6$. **5.** $6x^2-5x+1$. **6.** $2x^2+7xy-15y^2$.

7. a^2-4b^2. **8.** $25x^2-40x+16$. **9.** $4a^2-20ab+25b^2$.

10. x^3-x^2-x+10. **11.** $3x^3-8x^2-10x+21$. **12.** a^3+b^3.

13. x^3-3x+2. **14.** $5x^3+7x^2+14x-12$. **15.** $x^2+ax+bx+ab$.

16. $x^2+(a-b)x-ab$. **17.** $x^2-(a+b)x+ab$. **18.** $y^2-y(z+1)+z$.

19. $x+1$. **20.** $3x+1$. **21.** $x-4$. **22.** $2x+9$.

23. $x+5$; $x+2$. **24.** $x+1$; $2x^2+9x+7$. **25.** $x+1$.

26. Qu. $x-2$; rem. 8. No. **27.** $a+6b$. **28.** $a+b$.

29. $2x-11$; $(2x-11)(2x-11)$; $2x-11$; $4x^2-44x+121$.

30. x^2+7x+8. **31.** Qu. x^2-x-3; rem. -14. **32.** $6x^2+4x+1$

33. $x+b$; $(x+a)(x+b)$. **34.** $x+b$. **35.** $x-z$. **36.** $x-z$; $(x-y)(x-$

VIII. c. (p. 126.)

1. (i) 10. (ii) -1. (iii) $\frac{1}{10}$. (iv) -1. **2.** (i) -3. (iv) $-\frac{1}{4}$.

3. (ii) $-a$. (iv) $-\frac{1}{a}$. **4.** (i) $\frac{1}{5}$. (ii) $\frac{1}{6}$.

5. (i) $\frac{a}{b}$. (ii) $-\frac{a}{b}$. (iii) $-\frac{a}{b}$. (iv) $\frac{a}{b}$. (vi) ab. (vii) ab. (ix) a^2. (x) $b-a$

6. -1. **7.** $b-a$. $-\frac{3}{4}$. **9.** $x-y$ *or* $-(x-y)$.

10. $ab(a-b)$ *or* $-ab(a-b)$. **11.** $(a+b)(a-b)$. **12.** $-(x-y)^2$.

13. 1. 14. $a-b$. 15. $\frac{1}{x-1}$. 16. 1. 17. 0.
18. $\frac{2a}{a-b}$. 19. $\frac{a^2+b^2}{ab(a-b)}$. 20. $\frac{1}{ab}$. 21. $-\frac{1}{x}$. 22. $\frac{3}{x-1}$.
23. $\frac{2xy}{(x+y)(x-y)}$. 24. $y-x$.

VIII. d. (p. 129.)

1. $ac+bc+ad+bd$. 2. $ac+bc-ad-bd$. 3. a^2-b^2.
4. $x^2+xy+xz+yz$. 5. x^2-y^2. 6. $ax+2ay-3bx-6by$.
7. $6ax+4ay-3bx-2by$. 8. $x^2-xy-x+y$. 9. $2xy-2ay-2bx+2ab$.
10. $1+x+x^2+x^3$. 11. Yes. 12. Yes.
13. Yes. 14. No. 15. Yes.
16. (i) No. (ii) Yes. (iii) No. 17. Yes.
18. No. 19. $(p+q)(x+y)$. 20. $(p-q)(x+y)$.
21. None. 22. $(x+y)(c+d)$. 23. None.
24. None. 25. $(a-b)(x-y)$. 26. None.
27. $(a+1)(2x+y)$. 28. None. 29. $(a+b)(x+1)$.
30. $(b-c)(a-1)$. 31. $(x+y)(x+z)$. 32. $(x-y)(x+z)$.
33. $(x-y)(x-z)$. 34. None. 35. $(a+b)(a+2)$.
36. $(x+y+z)(a+b)$. 37. $(x^2+a^2)(x^2+b^2)$. 38. $(x^2+1)(x^4+1)$.
39. $(a-b)(a+1)$. 40. None. 41. $(a+b)(a-1)$.
42. $(x+a)(x^2-bc)$. 43. None. 44. $(a+3b)(2+y)$.
45. $(x+c)(x+d)$. 46. $(1+ax)(1+x)$. 47. $2(x+y)(a+b+c)$.
48. $(b+c)(ax+d)$. 49. $(a-b)(x-q)$. 50. $z(z+1)(z^2+1)$.

VIII. e. (p. 131.)

1. x^2+5x+6. 2. $x^2+12x+35$. 3. $x^2+3x-54$. 4. $x^2-11x+24$.
5. $a^2-10a-11$. 6. $2+3x+x^2$. 7. $14-5x-x^2$. 8. $z^2-18z-40$.
9. $24+5x-x^2$. 10. $1+5x+6x^2$. 11. $36-13a+a^2$. 12. $x^2-xy-6y^2$.
13. 39. 14. -30. 15. 0. 16. -16. 17. 14. 18. -3.
19. -150. 20. 48. 21. 15. 22. -28. 23. 1. 24. -4.
25. $(x+3)(x+5)$. 26. $(x-c)(x-d)$. 27. $(x+2)(x+4)$.
28. $(x+4)(x+5)$. 29. $(x-3)(x-5)$. 30. $(x-4)(x-5)$.
31. $(x+1)(x+2)$. 32. $(x+1)(x+6)$. 33. $(x-1)(x-10)$.
34. $(x+5)(x-7)$. 35. $(x+7)(x-10)$. 36. $(x+3)(x+4)$.
37. $(x+6)(x+6)$ *or* $(x+6)^2$. 38. $(x-1)(x-7)$.

39. $(x-2)(x-2)$ *or* $(x-2)^2$. **40.** $(x+5)(x+100)$. **41.** $(x+1)^2$

42. $(x+2)(x+20)$. **43.** $(y+4)(y-11)$. **44.** $(a-2)(a+7)$.

45. $(a+5)(a-5)$. **46.** $(z+3)(z-12)$. **47.** $(l-20)(l-30)$.

48. $(1-2x)(1+5x)$. **49.** $(2+y)(5+y)$. **50.** $(3-x)(8+x)$.

51. $(3-x)(4+x)$. **52.** $(5+y)(7-y)$. **53.** $(1+6x)(1-7x)$.

54. $3(x+1)(x+2)$. **55.** $10(x+4)(x+7)$. **56.** $(xy+9)(xy-1)$.

57. $(4-ab)(7+ab)$. **58.** $2(x+3)(x-6)$. **59.** $5(x-1)^2$.

60. $x(x+1)^2$. **61.** $\frac{x+2}{x}$. **62.** $x+4$. **63.** 9. **64.** -2.

65. 16. **66.** $x+3$. **67.** $6(x+1)(x+2)$. **68.** $\frac{1}{x-1}$. **69.** $2x-4$.

70. $x-30$. **71.** $\frac{x+1}{x+4}$. **72.** $\frac{1}{(x-1)(x-2)}$. **73.** $(x+2)(x+3)$.

74. $\frac{x-4}{x-2}$. **75.** $\frac{3x-4}{2(x+3)(x+4)}$. **76.** $(x+1)^3$. **77.** $\frac{x-3}{x(x-5)(x+3)}$.

78. $b+1$. **79.** $bc+1$. **80.** $2a-2$. **81.** $\frac{x^2+4x+5}{(x+2)(x+3)}$.

82. $\frac{-1}{(x+2)(x+3)}$. **83.** $\frac{x+1}{x}$. **84.** $\frac{12}{(x-3)(x+4)}$.

85. $\frac{3}{x(x+2)}$. **86.** $\frac{1}{x(x-2)}$. **87.** $\frac{10}{(x-2)(x+3)(x-4)}$.

88. $\frac{a+b}{a-b}$. **89.** $\frac{1}{(x+2)(x+3)}$. **90.** $\frac{2xy}{(x-y)(x-3y)}$.

91. 1. **92.** $-\frac{1}{7}$. **93.** $5\frac{2}{5}$. **94.** $-3\frac{3}{4}$. **95.** -4.

96. $-\frac{41}{45}$. **97.** 0. **98.** $-\frac{4}{7}$. **99.** -5. **100.** $-\frac{1}{3}$.

VIII. f. (p. 134.)

1. $2x^2+9x+4$. **2.** $12x^2+23x+10$. **3.** $12x^2-23x+10$.

4. $12x^2-7x-10$. **5.** $6x^2-23x+7$. **6.** $2x^2+5xy+2y^2$.

7. $4x^2-11xy-3y^2$. **8.** $12+23x+10x^2$. **9.** $7-15x+8x^2$.

10. $-3a^2+10ab+8b^2$. **11.** $10x^2-13x-77$. **12.** $21-44x+15x^2$.

13. -15. **14.** 10. **15.** 45. **16.** -95. **17.** 5. **18.** 23.

19. 132. **20.** 17. **21.** -40. **22.** 11. **23.** 36. **24.** -50.

25. $(3x+1)(x+2)$. **26.** $(3x-1)(x-2)$. **27.** $(2x+1)(x+2)$.

28. $(2x-1)(x-2)$. **29.** $(x+1)(5x+3)$. **30.** $(x+2)(2x+7)$.

31. $(x+1)(3x-2)$. **32.** $(x-1)(3x+2)$. **33.** $(x+1)(5x-7)$.

34. $(2x-1)(3x+5)$. **35.** $(3x-1)(5x-2)$. **36.** $(2x+3)(3x-2)$.

37. $(3x-1)(x+5)$. **38.** $(2a-3)(4a-1)$. **39.** $2(a-1)(3a+1)$.

40. $(2x-3y)(x+7y)$. **41.** $(1-x)(3-2x)$. **42.** $(2+5x)(1-3x)$.
43. $(2a+5)(7a-3)$. **44.** $(2-3x)(1+5x)$. **45.** $(x+3y)(2x-7y)$.
46. $(3+4a)(5-2a)$. **47.** $(2a+5)(10a-3)$. **48.** $(x^2-y^2)(4x^2-9y^2)$
49. $2(x^2+2yz)(3x^2-yz)$. **50.** $(x+2y)(2x-y)(5x-y)$.
51. $6a^2+43ab-210b^2$. **52.** $\dfrac{-2x+6}{(2x-1)(2x-3)}$. **53.** $5(x+2)$.
54. $2x^2-x-3$. **55.** $2x-13$. **56.** $-(2x-1)(x-2)(x+1)(2x-7)$.
57. $(2x+5)(x+2)$. **58.** $\frac{2}{11}$. **59.** $\dfrac{5x}{(x+1)(3x-2)(4x-1)}$. **60.** 3

VIII. g. (p. 135.)

1. x^2-y^2. **2.** $x-y$. **3.** $x+4$.
4. $(x+a)(x-a)$; 3, 43. **5.** $(x+2)(x-2)$. **6.** $(a+7)(a-7)$.
7. $(a+1)(a-1)$. **8.** $(10+b)(10-b)$. **9.** $(2a+1)(2a-1)$.
10. $(2a+3)(2a-3)$. **11.** $(3x+y)(3x-y)$. **12.** $(2a+3b)(2a-3b)$.
13. $(4y+5)(4y-5)$. **14.** $(xy+4)(xy-4)$. **15.** $(a+5b)(a-5b)$.
16. $(xy+z)(xy-z)$. **17** $(a^2+3)(a^2-3)$. **18.** $(5x^3+y^3)(5x^3-y^3)$.
19. $3(a+2)(a-2)$. **20.** $5(1+2x)(1-2x)$. **21.** $(x+1)(x+13)$.
22. $(x+8)(x-10)$. **23.** $(x+1)(x-1)(x^2+1)$. **24.** $(x+y+1)(x+y-1$
25. $(2x+7)(2x-1)$. **26.** $(a+b+c)(a+b-c)$. **27.** $(4+a+b)(4-a-b$
28. $(b+a-3x)(b-a+3x)$. **29.** $(2a+6b+3)(2a+6b-3)$.
30. $(10a-5b+7)(10a-5b-7)$. **31.** $(4+3x+3y)(4-3x-3y)$.
32. $9a^2-49$. **33.** $x-1$. **34.** $\dfrac{x+3}{x-3}$.
35. $y(x+1)(x-1)$. **37.** $4x^2(x+1)(x-1)$. **38.** $2x-4$.
39. $83+27$, *i.e.* 110, is a factor. **40.** 0. **41.** No. **42.** $\dfrac{2a^3}{(x+a)(x-a)}$.
43. One factor of $(2n+2)^2-(2n)^2$ is $(2n+2)+2n$, *i.e.* $2(2n+1)$.
44. -4. **45.** -61. **46.** a^2+1. **47.** $(x+\surd 11)(x-\surd 11)$.
48. $16a^4-1$. **49.** 23^2-1^2. **50.** $(x-1)(x+3)(x-3)$. **51.** 1.
52. $\dfrac{x^4}{(x+y)(x-y)}$. **53.** $\dfrac{x+4}{6(x+2)(x-2)}$. **54.** 2.
55. $(x+\surd 3)(x-\surd 3)$. **56.** $(x-1+\surd 2)(x-1-\surd 2)$. **57.** $(x-20)(x-38)$

VIII. h. (p. 138.)

1. x^2+6x+9. **2.** x^2-2x+1. **3.** $x^2+8xy+16y^2$.
4. $9x^2-6xy+y^2$. **5.** $4a^2-12ab+9b^2$. **6.** $25x^4+70x^2y+49y^2$.

7. $4x^2y^2+4xyz+z^2$. **8.** $x^4+2x^2y^2+y^4$. **9.** $a^2+\frac{2a}{x}+\frac{1}{x^2}$.

10. $4b^2+\frac{12b}{c}+\frac{9}{c^2}$. **11.** $25b^2c^2-\frac{20bc}{d}+\frac{4}{d^2}$. **12.** $9x^2-24+\frac{16}{x^2}$.

13. $x-1$. **14.** $x+3y$. **15.** $2x+5y$. **16.** $ab+5c$. **17.** $3a-7b$.

18. $5a-2b$. **19.** a^2+b^2. **20.** $10+3x$. **21.** $x+\frac{1}{2}$. **22.** 9.

23. 25. **24.** $\frac{25}{4}$. **25.** 49. **26.** $\frac{49}{4}$. **27.** $\frac{a^2}{4}$.

28. $10x$. **29.** $4x^2$. **30.** $-6ax$. **31.** $-4ax$. **32.** -6.

33. -7. **34.** $+4b$; $16b^2$. **35.** 2; $-12a$. **36.** Yes. $x+2y$

37. No. **38.** No. **39.** Yes; $x+\frac{1}{2}$. **40.** Yes; $a+\frac{1}{a}$.

41. Yes; a^2+3b^2. **42.** No. **43.** Yes; $1-x$. **44.** Yes; $x+\frac{5}{2}$.

45. Yes; $5xy-3y$. **46.** -42. **47.** 45. **48.** $+14x$, $-14x$.

49. $+x^2y^2$, $-3x^2y^2$. **50.** Yes. **51.** 0. **52.** 0.

53. $36(x-1)^2$. **54.** 49. **55.** $-(2x+1)$.

VIII. k. (p. 139.)

1. $3x(x-1)$. **2.** $(x+8)(x+1)$. **3.** $pq(p-q)$.

4. $a(1-4a)$. **5.** $(a^2+b)(a^2-b)$. **6.** $(a-2)(a-13)$.

7. $5(n^2-5)$. **8.** $(x+10)(x-9)$. **9.** $(b+c)(x+y)$.

10. $(1+n)(1+n^2)$. **11.** $(y+8z)(y-8z)$. **12.** $(t+5)(t+8)$.

13. $ab(a+b)(a-b)$. **14.** $(z+2)(3z-1)$. **15.** $(m-n)(x-z)$.

16. $(x+1)(x-1)^2$. **17.** $a^2(a+2b)(a-2b)$. **18.** $(x+1)(a+b+c)$.

19. $7(x+2z)(x-2z)$. **20.** $x(x+1)(x+2)$. **21.** $(x+y)(4x-5y)$.

22. $(a-b)(a-d)$. **23.** $(x+1)(x-1)(x^2-3)$. **24.** $x(x^2-x+1)$.

25. $(x-y)(x-z)$. **26.** $(ab-7)(ab+5)$. **27.** $(u+2v)^2$.

28. $x^2(x+6)(x-6)$. **29.** $ab(a^2+ab+b^2)$. **30.** $10(a+2x)(a+3x)$.

31. $(a-x)(b-y)$. **32.** $(1-x)(1-4x)$. **33.** $(a-1)(x-1)$.

34. $(2x+5y)^2$. **35.** $\left(1+\frac{3}{x}\right)\left(1-\frac{3}{x}\right)$. **36.** $(x-p)(x+q)$.

37. $3(x-5)(x+8)$. **38.** $(c-d)(x-q)$. **39.** $(8-x)(9-x)$.

40. $-(x-8)(x-9)$, *or* $(x-8)(9-x)$, *or* $(8-x)(x-9)$. **41.** $(a-b)(a^2+b^2)$.

42. $(a-b+10)(a-b-10)$. **43.** $(2y-3z)(x-1)$. **44.** $(x+z)(x+z+5)$.

45. $(a+b+c)(h+k)$. **46.** $(x-3)(x-a)$. **47.** $(2a+b+c)(2a-b-c)$.

48. $(a-1)^2(a+1)(a^2+1)$. **49.** $(p+q)(p-q)(m-l)$. **50.** $5(x-y)^2$.

51. $(z+4x-4y)(z-4x+4y)$. **52.** $(3a+2c)(2a-3b)$.

53. $(2x-5)(3-5x)$. 54. $8(a+b)(5a-2b)$. 55. $(x-1)(y-1)$.

56. $(a+d)(a-d)(b+c)(b-c)$. 57. $(3+x)(5-x)$. 58. $(n-1)(m+n-1)$.

59. $\left(1+\frac{1}{a}\right)^2$. 60. $n(n+1)(n-1)(n^2+1)$.

VIII. l. (p. 140.)

1. (i) $y(x-1)$. (ii) y. 2. $(x-7)(x+6)$. 3. x^2-64.

4. $x^2-14x+13$. 5. H.C.F. $x-5$; L.C.M. $(x+4)(x+5)(x-5)$.

6. $\frac{5}{x(x+5)}$. 7. $\frac{x(x-3)}{x^2-9x+18}$.

8. $x^3+4x^2-15x-18$, $(x+1)(x-3)(x+6)$.

9. Product of 4, $x-1$ and $x-1$; $4(x-1)$.

10. $\frac{2x-4}{(x-1)(x-2)}$; $\frac{3x-3}{(x-1)(x-2)}$. 11. 17, 29. 12. $\frac{x}{2(x+2)}$.

13. (i) $a+b$. (ii) $a(a+b)$. 14. $(x-7)(x+3)$. 15. $2\frac{5}{8}$.

16. $x+1$. 17. $\frac{1}{x^2-1}$. 18. (i) $(x-6)(2x-3)$. 19. $2x^2-11x+14$.

20. $-\frac{19}{11}$. 21. 193. 22. $\frac{3x+3}{x+3}$. 23. 49; $x+7$.

24. $\frac{-3x}{(x+3)(x-3)}$. 25. 3. 26. 1. 29. x. 30. $\frac{1}{x+1}$.

31. 1. 32. $(x-1)(x+2)$. 33. $(x-1)(x+2)(x-3)$. 34. $x-2$.

36. $(x+2)(2x^2+9x+1)$. 37. $(x+1)^2(x+4)$. 38. $(b-3a)(a-2c)$.

39. $(x^2+1)(y^2+1)$. 40. $ac(b+3c)(b-3c)$. 41. 1.

42. $(x+2)(x-3)$. 43. $(a-2b-3)(a-2b-4)$. 44. 0. 45. $2\frac{1}{7}$.

46. $9x^2+42xy+49y^2-640000$. 47. $(x-3)(x-3+ab)(x-3-ab)$.

48. $3x^2-9x+9$. 50. $(2x+3y-4)(2x+3y+3)$. 51. $(b-7)(a+5)$.

52. -2. 53. -3; $x-7$. 54. $9b^2-12bc+4c^2$. 55. -9.

56. 20. 57. $(x-1)(x-z)(x+2y+z)$. 58. $\frac{2}{x-3}$.

59. 69300 ($\frac{2240}{64}\times 66\times 30$). 60. $-2y^3$. 61. $\frac{121a^2}{4}$. 62. $4\frac{1}{2}$.

63. $\left(a+\frac{2}{a}\right)^2$. 64. $(a+b)^2$. 65. $2(1-x)$. 66. $-4\frac{1}{3}$.

68. $(a+b)(a-b-1)$. 69. $2a$. 70. $2(p+y)(p-q+x-y)$.

IX. a. (p. 145.)

1. $4a$. 2. $\frac{7a}{3}$. 3. $b-a$. 4. $\frac{3(p+q)}{2}$. 5. $\frac{a+b}{2}$.

6. $\frac{b}{a}$. 7. $\frac{a+b}{ab}$. 8. b. 9. $\frac{mn}{m-n}$. 10. 1.

11. $\frac{b}{a+1}$. **12.** $\frac{c}{a+b}$. **13.** $\frac{a^2+b^2}{a}$. **14.** $h+k$. **15.** 0.

16. $2a$. **17.** a. **18.** $\frac{ab}{3a-2b}$. **19.** $\frac{2b}{(a+b)(a-b)}$.

20. ab. **21.** $\frac{p}{q}$. **22.** ½. **23.** $\frac{p^2}{q}$. **24.** $3a$.

25. $\frac{pb-qa}{p-q}$. **26.** $\frac{ab}{b-a}$. **27.** $\frac{ab}{a+b}$. **28.** $\frac{a+b}{2}$. **29.** $\frac{1}{ab}$.

30. $a+b$. **31.** $\frac{a+b}{2}$. **32.** $\frac{2pr}{p^2-2pq-q^2}$. **33.** $\frac{ad-bc}{a-b-c+d}$.

34. $\frac{a+b}{2}$. **35.** $\frac{bc+ca+ab}{a+b+c}$. **36.** 0. **37.** $-\frac{a^2+b^2}{2(a+b)}$.

38. $\frac{ac-b^2}{a-2b+c}$. **39.** abc. **40.** $\frac{4abc}{bc+ca+ab}$.

IX. b. (p. 146.)

1. (i) 4½, 13½. (ii) $\frac{l}{4}$, $\frac{3l}{4}$ inches.

2. (i) 36, 72, 72. (ii) $\frac{180}{2n+1}$, $\frac{180n}{2n+1}$, $\frac{180n}{2n+1}$ degrees.

3. (i) 18, 19. (ii) $n+1$, $n+2$. **4.** (i) 1½. (ii) $\frac{m-n}{2}$ lb.

5. $\frac{180-d}{2}$ degrees. **6.** $\frac{10a+3p}{4}$ shillings. **7.** $b-2a$.

8. $\frac{ab}{a+b}$ ft. **9.** $(n+21)$ years. **10.** $\frac{a-b}{2}$ ft.

11. £$\frac{160a}{a-b}$. **12.** $\frac{10c}{7}$ sh. per ton. **13** $5+\frac{a}{2}$, $5-\frac{a}{2}$ sh.

14. $\frac{300a}{a+b}$, $\frac{300b}{a+b}$. **15.** £$\frac{100p}{r}$. **16.** (i) $\frac{4p}{5}$ lb. (ii) $\frac{ap}{a+b}$ lb.

17. (i) 4·2″, 2·8″. (ii) $\frac{ac}{b+c}$, $\frac{ab}{b+c}$.

IX. c. (p. 148.)

1. (i) $x=180-2y$. (ii) $y=\frac{180-x}{2}$. **2.** (i) $b=\frac{2A}{h}$; $h=\frac{2A}{b}$. **3.** $r=\frac{C}{2\pi}$.

4. (i) $P=\frac{100I}{RT}$; (ii) $T=\frac{100I}{PR}$. **5.** $\frac{100(y-x)}{x}$; 41⅔. **6.** $r=\sqrt{\frac{A}{\pi}}$.

7. $h=\frac{3V}{a}$. **8.** $C=\frac{5}{9}(F-32)$; $-13\frac{1}{3}$. **9.** $\frac{pv}{10}-273$; 52.

10. (i) $\frac{v-u}{f}$. (ii) $\frac{v-u}{t}$. **11.** $\sqrt{\frac{A}{6}}$; 5 in. **12.** (i) $\frac{3V}{\pi r^2}$; $\sqrt{\frac{3V}{\pi h}}$.

13. $\frac{A}{2\pi r} - r$. 14. (i) $\frac{2A}{x+y}$. (ii) $\frac{2A}{h} - y$. 15. $\sqrt{\frac{s}{2y}}$.
19. $\sqrt{c^2 - a^2}$. 17. $\frac{fv}{2v-f}$ 18. $\frac{gt^2}{4\pi^2}$. 19. $\frac{2s-2an}{n(n-1)}$. 20. $\frac{2(s-ut)}{t^2}$

IX. d. (p. 150.)

2. One ; ($x = -1$). 3. Equation. 4. Identity.
27. $a=1$, $b=2$. 28. $a=2$, $b=1$.

X. a. (p. 154.)

1. (i) $y=0$. (ii) Nothing. (iii) $x \neq 0$. (iv) $x=0$, unless $a-b=0$. (v) $y=2$ (vi) Nothing.
2. 0. 3. 0, 3. 4. 0, -1. 5. 2, 3. 6. $\frac{1}{3}$, -2.
7. 0, 5. 8. 0, 3. 9. ± 3. 10. $\pm \frac{2}{3}$. 11. 0, $-1\frac{1}{3}$.
12. 1, 2. 13. -1, -2. 14. -4, 5. 15. 0, $1\frac{2}{3}$. 16. ± 2.
17. -7. 18. 2, 6. 19. 3, -5. 20. $2\frac{1}{3}$, 5. 21. 1, $-3\frac{1}{3}$.
22. $\pm 3\frac{1}{3}$. 23. 0, 4. 24. $-\frac{1}{3}$. 25. 3, 5. 26. $\frac{1}{2}$, 12.
27. -3, -8. 28 -5, -6. 29. $-\frac{1}{3}$, 1. 30 8, -9. 31. $\frac{1}{3}$, 2.
32. $\frac{2}{3}$, $-1\frac{1}{2}$. 33. 2, 11. 34. 1, 2. 35. -1, -4. 36. $\frac{1}{2}$, $\frac{2}{3}$.
37. $-\frac{1}{3}$, 3. 38. -3, 7. 39. -2, 8. 40. 7, -8. 41. 1, 3.
42. 2, 4. 43. 1, 9. 44. 10, -12. 45. 8, $1\frac{4}{11}$. 46. 3, $-3\frac{1}{3}$.
47. 12, -6. 48. -1, 3. 49. $x^2 - 3x = 0$. 50. $x^2 + x = 0$.
51. $x^2 - 13x + 22 = 0$. 52. $x^2 - 4 = 0$. 53. $x^2 + 6x - 7 = 0$.
54. $x^2 + 11x + 30 = 0$. 55. $2x^2 - 7x + 3 = 0$. 56. $8x^2 + 2x - 3 = 0$.
57. $12x^2 + 43x + 35 = 0$. 58. -18. 59. -5; $2x$. 60. -2; -6.
61. 0·63. 62. $\pm a$. 63. $\pm \frac{3c}{2}$. 64. $\pm \frac{n}{m}$. 65. 0, a.
66. 0, $-\frac{b}{a}$. 67. 0, $m+n$. 68. $2a$, a. 69. $2p$, $-3p$. 70. $\frac{5b}{3}$, $2b$.

X. b. (p. 157.)

1. $\pm 1·4$. 2. $\pm 0·58$. 3. $\pm 3·5$. 4. $\pm 1·8$. 5. $\pm 5·2$.
6. $\pm 2·5$. 7. 0·45, $-4·4$. 8. $-0·45$, 4·4.
9. 0·41, $-2·4$. 10. $-0·44$, $-4·6$. 11. 0·54, $-5·5$.
12. $-1·6$, 2·6. 13. 0·61, $-4·11$. 14. 0·28, 2·39.
15. 2·30, 8·70. 16. 0·69, $-1·44$. 17. 10·46(7), $-11·46(7)$.
18. 0·37, $-0·27$. 19. 0.54(3), $-6·54(3)$. 20. 0·92, $-5·42$.
21. $-0·45$, 1·11(2). 22. $-0·56$, 3·56. 23. $-0·24$, 8·24. 24. 0·65, $-4·65$.

X. c. (p. 158.)

1. $-1,\ -1\frac{1}{2}\left(\frac{-5\pm1}{4}\right)$. **2.** $\frac{2}{3},\ -2\left(\frac{-4\pm8}{6}\right)$. **3.** $-1,\ 2\left(\frac{1\pm3}{2}\right)$.

4. $-1,\ 1{\cdot}4\left(\frac{2\pm12}{10}\right)$. **5.** $2,\ 1\frac{1}{2}\left(\frac{7\pm1}{4}\right)$.

6. $-0{\cdot}29,\ -1{\cdot}71\left(\frac{-4\pm\sqrt{8}}{4}\right)$. **7.** $0{\cdot}62,\ -1{\cdot}62\left(\frac{-1\pm\sqrt{5}}{2}\right)$.

8. $2{\cdot}68,\ 9{\cdot}32\left(\frac{12\pm\sqrt{44}}{2}\right)$. **9.** $1{\cdot}29,\ 2{\cdot}71\left(\frac{8\pm\sqrt{8}}{4}\right)$.

10. $-0{\cdot}88(9),\ 1{\cdot}88(9)\left(\frac{3\pm\sqrt{69}}{6}\right)$. **11.** $5{\cdot}84(5),\ -6{\cdot}84(5)\left(\frac{-1\pm\sqrt{161}}{2}\right)$.

12. $-0{\cdot}13,\ 0{\cdot}63\left(\frac{3\pm\sqrt{21}}{12}\right)$. **13.** $0{\cdot}09,\ 10{\cdot}91\left(\frac{11\pm\sqrt{117}}{2}\right)$.

14. $-0{\cdot}73,\ 2{\cdot}73(1\pm\sqrt{3})$. **15.** $-0{\cdot}46,\ 6{\cdot}46\left(\frac{6\pm\sqrt{48}}{2}\right)$.

X. d. (p. 160.)

1. 0, 1. **2.** 0, 2. **3.** 0, −1. **4.** −1, 0, 1; 3, 4, 5.

5. 9, 27; −9, −27. **6.** 8·5 in. **7.** 7·07...($\sqrt{50}$) cm.

8. 3·46...($2\sqrt{3}$) in. **9.** 7 ft. approx. **10.** 14, 21.

11. 2·5 cm. **12.** 8 ft. **13.** ±6. **14.** 5 in. **15.** ±1.

16. 6, 2 cm. **17.** 11 cm. **18.** 13. **19.** 3 sec. **20.** 21·2... mi.

21. Ten; 39. **22.** No ($2n^2+2=2n^2$). **23.** 1 yd.

24. 2·28..., 3·28... in. **25.** 18·7... ft. **26.** ½ in. approx.

27. After 5 sec. **28.** 5 hr. **29.** 24, 7 in. **30.** 10 tons.

31. 12 m.p.h. **32.** 18. **33.** 6 sec. **34.** 20 ft., 20 ft

35. Any rate greater than 24·5... ($\sqrt{600}$) m.p.h. **36.** 6·18 in.

37. 2·07..., 5·43... sec. **38.** $AP=9{\cdot}74$ *or* 2·26.

39. 2 m.p.h. **40.** 11 people. **41.** 4·14... cm.

X. e. (p. 164.)

1. 2, 3. **2.** (ii) 0·75. (iii) $x=2{\cdot}5$.

3. (i) 0·59, 3·41. (ii) 0·59, 3·41. (iii) 0·59, 3·41. (iv) C less than 4. (v) C greater than 4. (vi) 4. (vii) 2.

4. (i) −0·59; 2·26. (ii) −0·59, 2·26. (iii) 0·23, 1·43. (iv) between 0 and 1·67.

5. 8·73 m.p.h.

6. (i) 2. (ii) $x=1$. (iii) 0·31, 3·19. (iv) −0·31, −3·19.

7. $OP=OQ=5$ in.

X. f. (p. 166.)

1. $x=5, 2$; $y=2, 5$. 2. $x=y=\pm 1$

3. $x=3, -2$; $y=1, -1\frac{1}{2}$. 4. $x=1, -\frac{1}{4}$; $y=\frac{1}{3}, 1\frac{3}{4}$.

5. $x=-1\frac{1}{3}, 4$; $y=-8, 3$. 6. $x=\pm 2$; $y=\mp 2$.

7. $x=-1, 4$; $y=-2, 3$. 8. $x=\frac{1}{3}, 1$; $y=3, 1$.

9. $x=\frac{4}{5}, 4$; $y=-\frac{2}{3}, 2$. 10. $x=4, 2$; $y=2, 4$.

11. $x=7$; $y=3$. 12. $x=16, -10$; $y=10, -16$.

13. $x=0, 2\frac{1}{2}$; $-5, 0$. 14. $x=2, -1\frac{1}{3}$; $y=3, -4$.

15. $x=12, -2$; $y=2, -12$. 16. $x=3, -2\frac{1}{3}$; $y=1, -\frac{7}{9}$.

17. $x=10, -100$; $y=10, -1$. 18. $x=\frac{1}{3}, \frac{2}{5}$; $y=\frac{2}{5}, \frac{1}{3}$.

19. 3, 7 in. 20. 8, 6 cm. 21. 8 m.p.h.

22. 6, 7 in. 23. 15 ft. by 8 ft. ; 12 ft. by 10 ft. 24. 4 ft., $2\frac{1}{2}$ ft.

25. $BD=3\frac{3}{4}''$, $DC=2\frac{1}{4}''$. 26. 21, 20 in. 27. £320 *or* £360.

XI. a. (p. 170.)

1. $a^3+2a^2b+2ab^2+b^3$. 2. $2a^2+2b^2-9c^2+3bc+3ca-5ab$.

3. x^2-x+1. 4. $2x^2+x-4$. 5. x^2+x+1.

6. $x^4+x^2y^2+y^4$. 7. $3x^2-x-1$. 8. $a-2$. 9. $2x+2$; rem. $2x-2$.

10. $2x^4-3ax^3+6a^2x^2-3a^3x+2a^4$. 11. $a^4+2a^2b^2+4b^4$. 12. $2+a-a^2$.

13. $3-2x+x^2+2x^3-x^4$. 14. $a-b+c$.

15. $x^2+y^2+z^2+2yz+2zx+2xy$. 16. $x^2+y^2+z^2-2yz+2zx-2xy$.

17. $a^2-2ab+2b^2$. 18. $m^3+m^2n+mn^2+n^3$. 19. $a+b+c$.

20. $2a-3b-4c$. 21. $2x^6-4x^5-x^4+11x^3-6x^2-3x+9$.

22. x^2-xy+y^2. 23. $-3x^2-6x-19$; rem. 40.

24. $x^2-2+\frac{1}{x^2}$. 25. (i) x^2-x+3 ; rem. -1. (ii) 2 ; rem. x^3.

26. 17. 27. $x+1+\frac{2}{x+1}$. 28. $x+2+\frac{4}{x-2}$.

29. $x^4+5x^3-x^2-5x$. 30. -8. 31. 1. 32. a^2.

33. -5. 34. $b+c+bc$. 37. $\frac{3x^2-2x^3}{(1-x)^2}$.

38. $2+5-11-9+16-3$. 39. $3+5-9-3+20$. 40. $6+15-16-10+8$.

41. $2-1-5$. 42. $3+4+2$. 43. $6+4+3$.

XI. b. (p. 173.)

1. $a^2+b^2+c^2+2bc+2ca+2ab$. 2. $a^2+b^2+c^2-2bc-2ca+2ab$.

3. $a^2+b^2+c^2+2bc-2ca-2ab$. 4. $x^4+2x^3+5x^2+4x+4$.

5. $4a^2+9b^2+c^2-6bc-4ca+12ab$. 6. $x^4+4x^3-6x^2-20x+25$.
7. $y^2z^2+z^2x^2+x^2y^2+2x^2yz+2xy^2z+2xyz^2$. 8. $x^4-4x^3y+2x^2y^2+4xy^3+y^4$
9. $a^4-2a^3b+4a^3c+a^2b^2+4a^2c^2-4a^2bc$.
10. $25x^2+16y^2+z^2-8yz-10zx+40xy$. 11. $2x^2+3x+1$.
12. $x^2-xy+5y^2$. 13. $3a^2-a-4$. 14. x^3-2x+7.
15. $3x+y+z$. 16. $2a-4b-3c$. 17. $a^2-ab-bc$.
18. $a+2b+2c$. 19. $a-2b-2c$. 20. $x^2+y^2+z^2$.
21. $bc-ca+ab$. 22. x^2+3x+1. 23. x^2-3x+1.
24. $5a^2-a+4$. 25. $10x^2-2x-1$. 26. $x^3+6x^2y+12xy^2+8y^3$
27. $x^3-3x^2y+3xy^2-y^3$. 28. x^3+3x^2+3x+1.
29. x^3-3x^2+3x-1. 30. $8a^3+36a^2b+54ab^2+27b^3$.
31. $x^6-3x^4y^2+3x^2y^4-y^6$. 32. $a+2b$. 33. $a-2b$.
34. x^2+y^2. 35. $2x+3y$. 36. $x-4y$.

XI. c. (p. 175.)

1. a^2+ab+b^2. 2. $x^2-2xy+4y^2$. 3. $x^4-21x^2y^2+4y^4$.
4. $(x+1)(x^2-x+1)$. 5. $(x-1)(x^2+x+1)$. 6. $(2-y)(4+2y+y^2)$.
7. $(x+2y)(x^2-2xy+4y^2)$. 8. $(10-a)(100+10a+a^2)$.
9. $x(x+1)(x^2-x+1)$. 10. $ab(a+b)(a-b)$.
11. $(a+b+c)(a+b-c)$. 12. $(2a+2b+3)(2a+2b-3)$.
13. $(a^2+b^2)(a+b)(a-b)$. 14. $4(2a-1)(4a^2+2a+1)$.
15. $(8+5)(8^2-8\,.\,5+5^2)$, *i.e.* $13\times7\times7$. 16. $(4a+7b)(-2a+3b)$.
17. $(ab+1)(a^2b^2-ab+1)$. 18. $(x+a+b)(x-a-b)$.
19. $(x+y)^2(x^2-xy+y^2)^2$. 20. $(x+y)(x+y+1)$.
21. $(a+b+x)(a+b-x)$. 22. $(x-y+1)(x-y-1)$.
23. $(x-y-2)(x-y-3)$. 24. $(x+y+z)(x-y-z)$.
25. $(a-1+b)(a-1-b)$. 26. $(x+y)(x-y+1)$. 27. $(a+b-c)(a-b+c$
28. $(a+b)^3$. 29. $(x-y)(x^2+xy+yz)$. 30. $(x-y)(2x-y-z)$.
31. $(n-1)(n^2-n-1)$. 32. $4(4x-y-z)(x-2y+3z)$.
33. $(x+y+a-b)(x+y-a+b)$.
34. H.C.F. $x+y$; L.C.M. $(x+y)^2(x^2-xy+y^2)$. 35. $(x-2)^2(x^2+2x+4)$
36. $x(x+y)(x-3y)$. 37. m^3-27n^3. 38. $x+y+z$.
39. $2(y^3-z^3)$. 40. $-(2n^2+4n+1)$. 41. $(x^2-2)^3$.
42. One factor is $12+1$. 43. $(x^2+x+1)(x^2-x+1)$.
44. $(a+b+c)(a^2+2ab+b^2-ac-bc+c^2)$. 45. $(a+b)^2(a-b)$.
46. $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$.

47. $(x-y-z)(x^2+xy+xz+y^2+2yz+3^2)$. 48. $(b+c)(b^2-bc+c^2+1)$.

49. $(x^2+3xy+y^2)(x^2-3xy+y^2)$. 50. $\left(a-\frac{1}{a}\right)\left(a^2+1+\frac{1}{a^2}\right)$.

51. $(x^2+4x+2)(x^2-4x+2)$. 52. $(x^2+1)(x^4-x^2+1)$.

53. Expn. $=(x+y)^3-(2z)^3$. 54. One factor is $(2a-b)-(a-2b)$.

56. x^2+xy+y^2. 57. $a^4+9a^2b^2+81b^4$.

58. $(x+y)^3(x+z)^2$. 59. $(x+y+z)^2(x+y-z)$.

60. One factor is $(8m-3n)-(3m-8n)$.

XI. d. (p. 177.)

1. $\frac{a^2+b^2}{a}$. 2. 1. 3. $\frac{x^3}{x^3+1}$. 4. $\frac{x^2-xy+y^2}{x+y}$.

5. $\frac{x+y+2z}{x-y+2z}$. 6. $\frac{2x+y-1}{3}$. 7. $\frac{b^3-a^3}{ab}$. 8. $\frac{x(x+y)}{y(x-y)}$.

9. $y(x+y)$. 10. $u+v+1$. 11. $\frac{x+y}{x-y}$. 12. $\frac{(-x+c)}{(x-a)(x-b)}$

13. $-\frac{2ab(a-b)}{a^3+b^3}$. 14. $\frac{2a}{a^2-b^2}$. 15. $-\frac{x-y}{(x+y)^2}$. 16. 0.

17. $\frac{2x^3}{x^4-1}$. 18. $a-b$. 19. 0. 20. $\frac{x+y-z}{x+y+z}$.

21. $-\frac{3x^2+1}{x(x^4-1)}$. 22. $\frac{4ax^3}{x^4-a^4}$. 23. $\frac{a(a+b)^2(a^2+b^2)}{(a-b)^3}$.

24. 0. 25. $\frac{-48b^3}{(a^2-b^2)(a^2-9b^2)}$. 26. $\frac{1}{abc}$.

27. $\frac{2a(a-b)}{b(a+b)}$. 28. 1. 29. $\frac{a^2b^2-3ab+1}{b(ab-2)}$.

30. $\frac{x^3-2x^2+1}{x^2-x+3}$. 31. $x=-7\frac{1}{3}$. 32. $x=2\frac{4}{7}$.

33. $x=-1\frac{4}{5}$. 34. $x=-\frac{1}{4}$. 35. $x=5$.

36. $x=-3\frac{1}{3},\ -2\frac{4}{5}$. 37. $x=-2$ ($x=\pm 1$ introduces division by 0).

38. $x=\frac{-8\pm\sqrt{48}}{2}$ $(-0{\cdot}54\ldots,\ -7{\cdot}46\ldots)$. 39. $x=\frac{6\pm\sqrt{28}}{4}$ $(2{\cdot}82\ldots,\ 0{\cdot}18\ldots)$.

40. $x=a+b,\ \frac{a+b}{2}$.

XI. e. (p. 181.)

1. $x=1,\ y=1,\ z=2$. 2. $x=2,\ y=1,\ z=2$. 3. $x=1,\ y=1,\ z=1$.

4. $x=3,\ y=2,\ z=2$. 5. $x=4,\ y=2,\ z=3$. 6. $x=\frac{1}{2},\ y=\frac{1}{3}$.

7. $x=2,\ y=1$. 8. $u=\frac{1}{25},\ v=\frac{1}{25}$. 9. $x=3\frac{4}{7},\ y=\frac{7}{15}$.

10. $x=\pm\sqrt{3},\ y=\pm 1$ (any pair of these values).

11. $x=7,\ y=3;\ x=3,\ y=7$. 12. $x=11,\ y=1;\ x=-1,\ y=-11$.

13. $x = \pm 5,\ y = \mp 3;\ x = \pm 3,\ y = \mp 5.$ 14. $x = 4,\ y = 1;\ x = 1,\ y = 4.$

15. $x = 4,\ y = 3.$ 16. $x = 5,\ y = 3,\ x = -3,\ y = -5.$ 17. $x = 6,\ y = 1$

18. $x = \pm 4,\ y = \pm 2.$ 19. $x = 2,\ y = 1;\ x = -3,\ y = -4.$

20. $x = 3,\ y = 3;\ x = -1\tfrac{1}{2},\ y = -6.$ 21. $x = 2,\ y = -1;\ x = -\tfrac{5}{8},\ y = \tfrac{9}{4}$

22. $x = 3,\ y = 6;\ x = \tfrac{3}{4},\ y = -3.$ 23. $x = 0,\ y = -2;\ x = 1\tfrac{1}{2},\ y = 1.$

24. $x = \pm 2\sqrt{6},\ y = \pm\sqrt{6}.$ 25. $x = \pm 1,\ y = \pm 1;\ x = \pm\sqrt{\tfrac{1}{5}},\ y = \pm 2\sqrt{\tfrac{1}{5}}$

26. $x = \pm 2,\ y = \pm 1;\ x = \pm\sqrt{3},\ y = \pm\sqrt{3}.$

27. $x = \pm 3,\ y = \pm 1,\ x = \pm 7\sqrt{\tfrac{1}{8}},\ y = \mp\sqrt{\tfrac{1}{8}}.$

28. $x = \pm 2,\ y = 0$ (other roots impossible).

29. $x = \pm 5,\ y = \pm 2$ (any pair of values). 30. $x = \pm 1.$

31. $x = \dfrac{2 \pm \sqrt{24}}{2}$ (3·45..., −1·45...). 32. $x = 12,\ \tfrac{19}{11}.$

XI. f. (p. 183.)

1. $x = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}.$ 2. $x = -p \pm \sqrt{p^2 - q}.$ 3. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

4. $x = \pm\sqrt{ab}.$ 5. $x = \pm a,\ \pm b.$ 6. $x = \dfrac{a^2 + ab + b^2}{a + b}.$

7. $\dfrac{x}{y} = -\dfrac{a}{b}.$ 8. $\dfrac{x}{y} = \pm 1.$ 9. $x = \dfrac{r - q}{p - q},\ y = \dfrac{r - p}{q - p}.$

10. $x = y = a - b.$ 11. $x = b,\ y = a.$

12. $x = \dfrac{b}{ab' + a'b},\ y = \dfrac{a}{ab' + a'b}.$ 13. $x = \dfrac{aa'(b - b')}{a'b - ab'},\ y = \dfrac{bb'(a - a')}{ab' - a'b}.$

14. $x = \dfrac{cb' - c'b}{ab' - a'b},\ y = \dfrac{ca' - c'a}{ba' - b'a};\ x = 1{\cdot}6,\ y = 0{\cdot}8.$ 15. $x = \pm 1,\ y = \mp 1.$

16. $x = 0,\ y = 0;\ x = \dfrac{4a}{m^2},\ y = \dfrac{4a}{m}$ 17. $x = \dfrac{a}{m^2},\ y = \dfrac{2a}{m}.$

18. $x = \tfrac{1}{2}(c \pm \sqrt{2r^2 - c^2}),\ y = \tfrac{1}{2}(c \mp \sqrt{2r^2 - c^2}).$

19. $x = \dfrac{pr}{p^2 - q^2},\ y = \dfrac{qr}{q^2 - p^2}.$ 20. $x = ct_1,\ y = \dfrac{c}{t_1};\ x = ct_2,\ y = \dfrac{c}{t_2}.$

21. $TB = \dfrac{p^2 r}{q^2 - p^2},\ TC = \dfrac{pqr}{q^2 - p^2}.$ 22. $x = \dfrac{a - b}{2}.$

XI. g. (p. 185.)

26. $A = 1,\ B = 1.$ 27. $A = 1,\ B = -1.$ 28. $A = 2,\ B = -5.$

29. $A = 1,\ B = 2.$ 30. $A = \tfrac{1}{7},\ B = -\tfrac{1}{7}.$ 31. $A = 1,\ B = -2,\ C = 1.$

32. $A = 1,\ B = 1,\ C =\ .$ 33. $A = 1,\ B = -3,\ C = 2.$

A. 1. (p. 187.)

1. $4n^2$ is a perfect square.
3. (i) 15. (ii) 5. (iii) 4.
4. $10+x$; $10+x+y$.
5. (i) $16a^6$. (ii) $2x^2$. (iii) $\frac{2x}{y}$.
6. (i) x. (ii) b^4. (iii) $\frac{q-p}{q}$.

A. 2. (p. 188.)

1. (i) $\frac{1}{8}$. (ii) 18. (iii) 6.
2. (i) $8-(u+v)$. (ii) $8-\frac{1}{2}(u+v)$ miles.
3. H.C.F. $3x^2$; L.C.M. $6x^2y$. (i) $\frac{1}{2y}$. (ii) $\frac{2y+z}{6x^2y}$.
4. $2z(x+y)$ sq. ft.
5. a^2-b^2; $2c-2b$.
6. $\frac{2x}{3}$; $42-14$, *i.e.* 28.

A. 3. (p. 188.)

1. $6a$.
2. 1s. 8d.; 6.
3. $\frac{a\times 2}{b\times 2}=\frac{a}{b}$.
4. (i) $2\frac{1}{6}$. (ii) $2\frac{4}{15}$. (iii) $\frac{a^2+b^2}{ab}$.
5. H.C.F. $2abc$; L.C.M. $24a^2b^2c$.
6. $ac-ab$.

A. 4. (p. 189.)

1. $20n$.
2. (i) 6. (ii) $\frac{1}{2}$. (iii) 4.
3. The result of taking $y+z$ from x; $x-y+z$; subtract $x-(y+z)$ from $x-y+z$ and the result is $2z$.
4. $2x^3-2x+3$.
5. $a+7b$.
6. $\frac{4x}{3}$ shillings; $\frac{x}{3}$ shillings.

A. 5. (p. 189.)

1. 18s.
2. (i) 4. (ii) 10. (iii) $x+1$. (iv) 7. (v) 10 min. past 4 (vi) 10 min. past $x-1$. (vii) y min. past $x-1$ o'clock.
3. (i) $4(a+b)$ in. (ii) $(a+b)^2$ sq. in.
4. $6a$.
5. (i) $4a^2$. (ii) a^2+2a. (iii) $\frac{a}{2}$.
6. $\frac{3x}{8}$.

A. 6. (p. 190.)

1. For 'x' put 'x shillings' or 'x pence'; similarly for $2x$.
2. (i) $2a$. (ii) ac. (iii) $2x$, xy.
3. (i) $3x$. (ii) $6x^2-4x+6$.
4. $3n+2$ or $3n-1$, where n is a positive integer.
5. x^3-2.
6. $a+2b+3c-4x$.

A. 7. (p. 190.)

1. $7\times 8=6^2+3\cdot 6+2=56$.
2. $10n+3$, where n is a positive integer
4. $2y$; $a-b$.
5. 240 cu. in.
6. $x-y$.

A. 8. (p. 191.)

1. $n+n+n=3n$. **3.** (i) a^{20}. (ii) a^{44}.

4. (i) $\frac{a-b}{2}$. (ii) $\frac{1}{2c}$. (iii) $x-2$. **5.** $20-y$ shillings. **6.** $\frac{7x}{3}$ hours.

A. 9. (p. 191.)

1. What must be added to x to obtain 20? $20-x$.

3. (i) $9ab$. (ii) $\frac{7x}{4}$. (iii) 1. **4.** $3x^2-x$.

5. (i) 6. (ii) 9. (iii) 3. (iv) 5. (v) 1. **6.** $\frac{14x}{5}$; 14.

A. 10. (p. 192.)

1. $x+2$; $3x+3$. **2.** (i) $\frac{3}{22}$. (ii) $\frac{a}{bc}$. (iii) $\frac{7}{5}$. (iv) $\frac{y}{x}$.

3. (i) Multiply $y+z$ by 3 and subtract the result from x. (ii) Subtract 3 from x, add z to y, multiply the results.

4. $2240x \div (a+56)$. **5.** 3, $6b^2$, $2b$, $3bc$; 3, $2b$, $3bc$. **6.** £$(180+19a)$

A. 11. (p. 192.)

1. $4+25<7^2$. **2.** $360-(x+y+z)$ degrees.

3. (i) 128. (ii) 1. (iii) 24. (iv) $1\frac{1}{2}$. **4.** $a^2bc+ab^2c+abc^2$.

5. $330-60x$ ft.; 5 min. **6.** $500x+19500a$.

A. 12. (p. 193.)

2. (i) $9a^2b^4$. (ii) $5ab$. (iii) $3x^4-6x^3$.

3. $7n$; $7n+1$, where n is a positive integer.

4. 62; $2y-x$ degrees. **5.** (i) 1. (ii) 4. (iii) $\frac{5a}{6}$.

6. $a-\{30(b+c)+d\}$ acres.

A. 13. (p. 193.)

1. $12\frac{5}{8}$. **2.** $5ab$; $\frac{3}{7y}$. **3.** (i) $81a^6x^4$. (ii) $\frac{9x^4}{25y^2z^2}$. (iii) $10m^3n$.

4. $90-x$ degrees. **5.** $x+y$ in.

6. (i) 20; (ii) $20+\frac{1}{12}(x-500)$ pounds.

A. 14. (p. 194.)

1. $n(n+1)$. **2.** $7n+3$ or $7n-4$. **3.** $\frac{1}{2}(x+y+z)$; x^4+x^3+1

4. $\frac{a}{2}$. (i) $\frac{a^3}{b}$. (ii) $\frac{a-b}{2ab}$. (iii) $\frac{a^3}{b^3}$.

5. $5(2x+y)+3(x+2y)$ pence; $240-\{5(2x+y)+3(x+2y)\}$. **6.** $\frac{2240w}{abc}$.

A. 15. (p. 194.)

1. $\frac{n}{11}$ is the same as $\frac{1}{11}$ of n or $n \times \frac{1}{11}$. 2. $4c$. 3. (i) 15. (ii) 65 yd
4. $2\frac{1}{4}$; $4a^2$. 5. $\frac{5x}{100}$; $5x$. 6. $\frac{5x}{4}$.

A. 16. (p. 195.)

1. $(100-1)23$; 2277 ; $(100-1)x$ or $100x - x$.
2. (i) $9a^4$. (ii) $7abc$. (iii) $6a$. (iv) $2x^2y^2$. 3. $180 - x$. 4. $x + 2y$.
5. £$2r$. 6. Twice the sum of the first n numbers is $n^2 + n$

A. 17. (p. 195.)

1. 12 sq. in. 2. $\frac{a+3b}{4}$. 4. $a^3 + 3a^2b + 3ab^2 + b^3$.
5. $100 - n$; $\frac{(100-n)a}{100}$. 6. $\frac{x-100}{2}$ (if $x > 100$).

A. 18. (p. 196.)

1. (i) $x - 17$. (ii) $x + 6$. 2. (i) 4. (ii) 16. (iii) 10. (iv) [illegible]
4. (i) $\frac{1}{3}$. (ii) $\frac{ab+1}{b}$. (iii) 15. (iv) ab. 5. $20x$.

A. 19. (p. 196.)

1. 700. 3. $a^2 + b^2 + c^2$. 4. $\frac{x+y}{2}$ in. 5. 5. 6. $\frac{39n}{112}$.

A. 20. (p. 197.)

1. The nth leap year is $1900 + 4n$. 3. $\frac{10x}{33}$ lb.
4. 2, $2c$, ac, c, $2a$; $8a^2c^2$, $4ac^2$, $12a^2c^2x^2$. 5. (i) 1. (ii) $p^2 + pq + q^2$
6. $10000A$ sq. m. ; $\frac{x}{100}$ c. per sq. m.

A. 21. (p. 197.)

1. $10x + y$ s. per cwt. is the same price as $\left(\frac{x + y + \frac{y}{2}}{14} + x\right)$ d. per lb.
2. $2x - 4y + 6$. 3. x^{15}. 4. $\frac{15x}{22}$. 5. $\left(2l + \frac{2A}{l}\right)$ in. 6. $\frac{2}{3x}$ in.

A. 22. (p. 198.)

1. $x - 4\frac{1}{2}$ sh. 2. $2xy$. 3. $11a^3b$. 4. $30 - a$
5. $\frac{xyh}{ab}$ ft. 6. £$\frac{x+qy}{k}$.

A. 23. (p. 199.)

1. (i) £$10(x-2)$. (ii) $y+2$ pence. 2. $\frac{a}{2}$. 3. (i) a^2. (ii) $2x^2$. (iii) $2a$.

4. $x-2y$ in. 5. $\frac{x^3}{1728}$ cu. in. 6. $\frac{16x-y}{12}$ oz.

A. 24. (p. 199.)

1. $\frac{x+7}{4}$ pence. 2. $\frac{3x}{10}$. 3. $36a^3b^2$. (i) a. (ii) $6b$. 5. x in. 6. $\frac{x}{y}$.

A. 25. (p. 200.)

1.

Figure.	Compartments.	Corners.	Edges.
	x	y	z
(i)	3	4	6
(ii)	4	6	9
(iii)	5	7	12

$x+y=z+1$.

2. (i) $3a+4b-2c$. (ii) $2x^2$. 3. (i) $\frac{1}{10x}$. (ii) $\frac{99x}{100}$.

4. $3+\frac{x}{2}$ pence. 5. $4x+4y+4z+1$ feet. 6. $\frac{10}{a}-\frac{10}{b}$ cm.

B. 1. (p. 201.)

1. $b-a$; -2; 7. 2. $24a+6$ mi.; $3a$ sh. and 9d.

3. (i) 6. (ii) a^2. (iii) $\frac{r}{3}$. (iv) $\frac{x}{2}$. 4. $6y-2x$; $a^3-a^2b+ab^2-b^3$.

5. $x^2-(x-1)^2=39$ *or* $(x+1)^2-x^2=39$. 6. $x=4\frac{1}{2}$.

B. 2. (p. 201.)

1. $1\cdot7x^2$ sq. in. 2. (i) (-2). (ii) (-2). (iii) 3. (iv) -8.

3. (i) $\frac{x}{40}$ hr. $+\frac{150-x}{25}$ hr. $=5$ hr., *or* $40x$ mi. $+25(5-x)$ mi. $=150$ mi.

4. $x=2$, $y=2$. 5. Between $1\frac{1}{2}$ and 3; any number greater than 3

6. Q is 20 yd. from B.

B. 3. (p. 202.)

1. 25. 2. (i) $\frac{1}{b}$. (ii) b. (iii) $\frac{8}{13}$. (iv) $\frac{ac}{bc+1}$. 3. $212\frac{1}{4}$ cu. in.

5. Inc. from May to Nov.: dec. Nov. to May (roughly). 6. 36 min.

B. 4. (p. 202.)

1. $\frac{x}{12}$ sh.
2. (i) $\frac{1}{a}$. (ii) x. (iii) $\frac{b}{a}$. (iv) $\frac{q}{p+q}$.
3. (i) (-4). (ii) $3a$. (iii) $(-3a)$. (iv) $3a$.
4. $54-\{5(a+b)+c\}$.
5. (i) $x=6$. (ii) $x=1$, $y=2$.
6. Lavant. Henfield.

B. 5. (p. 203.)

1. $(45-x)$ degrees.
2. H. -4, S. $+3{\cdot}6$, G. $-1{\cdot}6$, W. $-$ 9.
3. $-3x^2+6x-2$.
4. 3.
5. $2x+\frac{1}{2}(x+2)=24$.
6. $75x + 25y$ sh

B. 6. (p. 204.)

1. $\frac{a}{2}+\frac{b}{2}$; $a+\frac{b}{2}$.
2. (i) $4x$ oz. (ii) $\frac{120+n}{30}$.
3. $a^2+b^2+c^2$; $-a-b-c$
4. (i) $x=1\frac{3}{4}$. (ii) $x=5$, $y=2$.
5. $2\frac{1}{3}$d.

B. 7. (p. 204.)

1. 3 p.m., 4 a.m.
2. 2, 1, 0, -1, -2; 4, 1, 0, 1, 4.
3. (i) $x=2$. (ii) $x=2\frac{1}{3}$.
4. 4 ft.; 3 ft.
5. 60.
6. $262\frac{1}{2}$.

B. 8. (p. 205)

1. (i) 33. (ii) 52. (iii) 19 ft.
2. Increases continuously from -10 to 2.
3. $x-2$; $x-2$. Yes.
4. $7+3x-x^2$.
5. (i) $x=3{\cdot}5$. (ii) $x=22$, $y=11$.
6. $11\frac{1}{2}$ doz.

B. 9. (p. 205.)

1. $\frac{b-a}{b}$; b not less than a.
2. (i) $x=3\frac{3}{4}$. (ii) $x=3\frac{1}{3}$. (iii) $x=-\frac{1}{4}$.
3. $x-y$; -2 mi. per hr. Losing ground at 2 mi. per hr.
4. $a+(n-2)d$; $2a+(n-1)d$.
5. $1801+10n$, where n is a positive integer between 0 and 11.

B. 10. (p. 206.)

1. $+3$ ft. above sea level.
2. $33\frac{1}{4}$.
3. Less; x between 1 and 0.
4. $2y$; x^2+4x-5.
6. $4x$ ft.

B. 11. (p. 206.)

1. $b-a$ in.; -2 in. (*i.e.* 2 west of A); 5 in.
2. (i) 1. (ii) -4. (iii) $1\frac{1}{3}$. $\frac{x}{6}-\frac{y}{2}$.
3. $17{\cdot}6$; $3{\cdot}42$.

5. (i) $x=75$. (ii) The two equations are the same; the equations are contradictory.

6. 8s.

B. 12. (p. 207.)

2. 5·1; −0·7; 2·1; 3·3; 1·5; 4·3; −1·0; −6·9; 0·5; −1·7; −2·0.

3. $8abc$; $\dfrac{3a+2c}{48a^2bc^2}$; $\dfrac{2c}{3a}$. 4. 8.56 p.m.; 14.56 p.m. (really 2.56 a.m.).

6. (i) $x=-3\frac{13}{18}$. (ii) $x=6$, $y=2$.

B. 13. (p. 208.)

1. $n+2$ pence. 2. $-2x^3+2xy^2-2y^3$.

3. (i) $\dfrac{xy}{xy}, \dfrac{x^2}{xy}, \dfrac{y^2}{xy}$. (ii) $\dfrac{ab+b^2}{ab}, \dfrac{a^2+ab}{ab}$.

4. $A+aA$, $A-aA$ gm. $\frac{121}{100}A$, $\frac{99}{100}A$.

5. (i) $x=2$. (ii) $x=-12$, $y=-19$. 6. 3·1 in.

B. 14. (p. 208.)

2. $a=0·6$, $b=-1·3$. 3. $\dfrac{3x}{2}$; $\dfrac{15x}{8}$ pence. 4. $F=28\frac{4}{7}$ (28·57...).

5. (i) $-4x^2+3x$. (ii) $\dfrac{1}{a^2+ab+b^2}$. 6. 220.

B. 15. (p. 209.)

1. (i) 6. (ii) 8. (iii) 24. (iv) 216.

2. 11, −5, mi. per hr., $(32-5x)$ mi −3 mi. (3 mi. W. of O).

3. 18 lb. 4. (i) $x=1·2$. (ii) $x=-23$, $y=-5$. 5. $\frac{3}{16}$.

6. $a=·55\ldots$, $b=31·3\ldots$.

B. 16. (p. 210.)

1. 0·013 in.; $0·001\times n$ in. 2. Thursday.

4. $2(x-2)+1$. 5. (i) 1. (ii) −2. 6. 3 lb

B. 17. (p. 210.)

1. The number of which the ten digit is x and the unit digit y is $10x+y$.

2. $\dfrac{16x-15}{20x^2}$. 3. $1\frac{1}{4}$, $1\frac{1}{4}$. 4. (i) 0. (ii) −96 ft. 5. 160 ft. per sec

B. 18. (p. 211.)

1. 262 tons. 2. $6ab$; 24. 3. (i) 9. (ii) 9. 0.

4. (i) $x=6$. (ii) $x=2\frac{1}{3}$, $y=1\frac{1}{3}$. 5. $x+1$, $x-y$ o'clock.

6. 5 lbs (ht. 67 in.).

B. 19. (p. 212.)

2. $\frac{495}{2px}$ min.

3. Av. -3° C. S., -7; M., -3; T., 6; W., 3; T., -1; F., 8; S., -4 degrees. S., 1; M., 5; T., -4; W., -1; T., 3; F., -6; S., 6 degrees.

5. $-2a$; $2a^2-2$; $-4a^3+4a$. 6. £250.

B. 20. (p. 212.)

1. $6400x$ tons. 2. $x=$ [illegible].

4. $6x^3y^2-8x^2y^3$; $18ab(2a^2-ab+2b^2)$. 6. $\frac{3}{4}$ mi.

B. 21. (p. 213.)

1. 42·6 m.p.h. 2. (i) 4. (ii) -9. (iii) 2. 3. $x=3$.

4. $a=2$, $b=-7$; $3\frac{1}{2}$. 5. 486 cwt. 6. $AY=3$; $BX=4$.

B. 22. (p. 214.)

1. (i) $2a-2$. (ii) $-72ab^4$. 2. (i) 29. (ii) $2a-b$. 3. $12(x-y+z)$ ft.

4. (i) $x=14\frac{1}{3}$, (ii) $x=\frac{1}{2}$, $y=2$. 5. 80 : 57. 6. $nd-n+1$ in.

B. 23. (p. 214.)

1. 12·3 lbs. per sq. ft. 2. $x=5$. 3. (i) $-3x^2-4x^3$. (ii) $\frac{x^2-3y^2}{3xy}$.

4. (i) $2x^3$. (ii) $6x^2y$. 5. $\frac{3}{5}B+\frac{188T}{B^2}$, *or* $\frac{3B^3+940T}{5B^2}$.

6. (i) $\frac{2y+2z-3x}{4}$ in. *or* $\frac{2y+2z-x}{4}$ in. (ii) $\frac{y+z-x}{2}$ in.

B. 24. (p. 215.)

1. (i) $2-2a$. (ii) $4x^3-x^2+3x$. 2. (i) 0. (ii) $\frac{4-2x}{9}$.

3. $x=7$, $y=-5$. 4. (i) $b=0$. (ii) 11. 5. 2°.

6. $8\frac{3}{4}$ cu. in. ($3\frac{1}{2}\times 2\frac{1}{2}\times 1$).

B. 25. (p. 216.)

1. $\frac{3y}{2}$. 2. (i) $\frac{x}{2}$. (ii) $\frac{5a}{4}$. (iii) $\frac{27x}{4}$. 3. $a=-2$. 4. $\frac{10x}{3}$.

5. 120. 6. 11 : 40 : 81.

C. 1. (p. 217.)

1. (i) $4a(a+b)$. (ii) $6b(a+b)$; L.C.M. $12ab(a+b)$. (i) $\frac{2a}{3b}$. (ii) $\frac{2a+3b}{12ab(a+b)}$.

3. $x=3$, $y=7$. 4. $2x$ shillings, $2y$ pence; $2x+1$ shillings, $2y-12$ pence.

5. 12, -12. 6. $\frac{S}{\pi r}-r$, or $\frac{S-\pi r^2}{\pi r}$.

C. 2. (p. 217.)

1. $11500-2500x$ ft.; -1000 ft.; if he could descend till 5 o'clock, he would be 1000 ft. below A.

2. $(n-1)\times(n+1)<n^2$ by 1.

3. (i) $10(x-1)(x+1)$. (ii) $2(x-1)^2$. (i) $2(x-1)$. (ii) $x-1$.

4. 42, 70 lb. 5. $\frac{p^2}{q}$. 6. 11, -17. Either 18, 10 or -10, -18.

C. 3. (p. 218.)

1. £$\frac{19a}{20}$. 2. (i) If $x<\frac{2}{3}$. (ii) If $x>\frac{2}{3}$. (iii) $x=\frac{2}{3}$.

3. (i) $x=9$, $y=1\frac{1}{3}$. (ii) $x=-1$.

4. $(4x-1)(3x+2)$. $x=\frac{1}{4}$, $-\frac{2}{3}$. $x=0$, $-\frac{5}{17}$.

5. (i) $a^8(a-1)$. (ii) $a^3(a-1)$. L.C.M. $a^8(a-1)$. $\frac{1}{a^5}$.

6. $(n+1)^2-n^2\equiv 2n+1$.

C. 4. (p. 218.)

1. $\frac{Ir}{E-IR}$. 2. $76x-4x^2$ sq. in.; 72 sq. in. 3. $3+\frac{5x}{4}$ pence.

4. (i) $\frac{x+1}{x}$. (ii) $-\frac{1}{x^2(x^2-1)}$. 5. $x-2$; x^2+x+9. 6. 12.

C. 5. (p. 219.)

1. $25a^2-30ax+9x^2$; $49y^2$; $(2x-7y)^2-(7y)^2$.

2. $-\frac{25}{(x-3)(x-4)}$; $x=-2a$. 3. $a=\frac{c^2}{4\pi}$. 4. $-3y^3$.

5. It is isosceles. 6. $6\frac{3}{4}$ or 0.

C. 6. (p. 219.)

1. $s=\frac{v^2-u^2}{2a}$. 2. $x+y+z$; no factors. 3. 21243.

4. $\frac{2\pm\sqrt{10}}{3}$ $(1{\cdot}72, -0{\cdot}39)$; $4(x=3, y=-1)$. 5. $\frac{4y^2}{(x^2-y^2)^2}$. 6. $16\frac{2}{3}$

C. 7. (p. 220.)

2. $\frac{2}{3}$; all values; $\frac{3x-22}{12(x-1)}$. **3.** (i) $x=2{\cdot}79$, $-1{\cdot}79$. (ii) 0. 1.

5. 5 ft. 3 in.; 3 ft. 9 in.; $\frac{a+x}{2}$, $\frac{a-x}{2}$ ft.

6. (i) $x=\frac{1}{a}$. (ii) $x=1{\cdot}7$, $-1{\cdot}2\left(\frac{1\pm\sqrt{33}}{4}\right)$.

C. 8. (p. 220.)

1. (i) 8. (ii) -4. (iii) x^3. (iv) $6x^3$. (v) $\frac{2}{3}$.

2. (i) $(x+1)(2x+3)$. (ii) $(x+4)(x-3)$; 9984.

3. (i) $x=\frac{4}{5}$. (ii) $x=1\frac{1}{3}$, $y=1\frac{1}{3}$. **4.** 4. **5.** 0.

C. 9. (p. 221.)

1. $(x+7)(x+4)$; $\frac{(x+7)(x+4)}{x}$. **2.** 6; -6.

3. $x_1=\frac{rx_2}{2x_2-r}$; interchange x_1 and x_2. **4.** 1, 0·33; the same; $-21{\cdot}1$.

5. (i) $x=100$. (ii) $x=abc$. **6.** $\frac{ax+by}{a+b}$; $1\frac{1}{5}$ lb.

C. 10. (p. 221.)

1. $130+0{\cdot}02w$ in. **2.** $\frac{x^2-1}{x^2+1}$. **3.** 88 yd.

4. (i) $x=-2$, $y=1$. (ii) $x=ab$. **5.** 1·02. **6.** 10s. per week.

C. 11. (p. 222.)

1. x mi. per hr. $=\frac{22x}{15}$ f.s.; $2\frac{2}{11}$.

2. x^3+y^3. (i) $x+y$. (ii) $(x+y)(x^2-xy+y^2)$; $\frac{3(x^3+y^3)}{x^3+y^3}$. **3.** 3·41, 0·59.

4. $t=\frac{pv-2730}{10}$. **5.** 600 of first, 400 of second.

C. 12. (p. 223.)

1. (i) 5. (iii) -5. **2.** 4d., 5d. per lb.

3. (i) $x=0$. (ii) $x=2$, $y=12$; $x=3$, $y=8$.

4. $\frac{5}{x+4}-\frac{4}{x+3}$. **5.** (i) $a-b$. (ii) 0.

C. 13. (p. 223.)

1. (i) $\frac{wv^2}{Pg}$. (ii) $\sqrt{\frac{Pgr}{w}}$. **2.** x and $x+2a$; 0.

3. (i) $x=16$, $y=3$; $x=-3$, $y=-16$. (ii) 0·80, $-0·63$. **5.** $\frac{77}{60}ab$ sq. yd

C. 14. (p. 224.)

1. $(x-y)(x+y+z)$; $\frac{(x-y)(x+y+2z)}{(x+z)(y+z)}$. **2.** $\frac{ku}{k-u}$.

3. $2\frac{1}{4}$ hrs. after soldiers started ; walking.

4. 1 dollar = 50d., 1 franc = 10d. ; only two equations are necessary ; no.

5. (i) $x=-2\frac{2}{3}$, $y=1\frac{1}{3}$. (ii) $x=a-b$.

C. 15. (p. 224.)

1. $n-m+1$. **2.** (i) $x=4$, -1. (ii) $x=7$, $y=1$; $x=-6$, $y=-12$.

3. $a=bq+r$; 92. **4.** $x=1$, 0·62, $-1·62$.

5. £$\left(45+\frac{9x}{20}\right)$; $45+\frac{9x}{20}=40\frac{1}{2}$; $x=-10$; a loss of 10 %.

6. 3 in. from either end.

C. 16. (p. 225.)

1. $x^3+ax^2+(ab+ac+bc)x-abc$.

2. $y=3·1$; $x=4·5$ from each equation.

3. (i) $(4a+d)(3a-2c)$; (ii) $(x-3y)(x+3y+1)$.

4. (i) $x=a+b$. (ii) $x=-pq$. **5.** $\frac{x(x^2+4x+7)}{(x-1)(x+1)^2}$. **6.** 48 mi. per hr.

C. 17. (p. 225.)

1. $x=\frac{5}{9}$; $x=1\frac{4}{5}$. **2.** $6\frac{17}{25}$ tons. **3.** 26 min. **4.** $\frac{(a-c)x}{(x+a)(x+c)}$.

5. $x=1·45$ (6), $y=0·15$ (6) ; $x=-2·45$ (6), $y=-3·75$ (6).

6. (1) Freezing pt., 20° ; boiling pt., 120° ; (2) $-4°$ on A, $2\frac{2}{5}°$ on B ; (3) 5°.

C. 18. (p. 226.)

2. $x=2\frac{3}{4}$. **3.** $\frac{a-b}{a+b}$.

4. $t_1+t_2-t_3$ deg. ; t_2-t_3 deg. ; $t_1-(t_2-t_3)=t_1-t_2+t_3$.

5. $a=6$, $b=0·009$, $R=17$ appr.

C. 19. (p. 226.)

1. 100x tons.
2. $\frac{2(a^4+1)}{a^4-1}$.
3. $\frac{y(1-2y)}{1-y^2}$.
4. £9600.
5. $\frac{-7x^2-12x}{(x+1)(x+2)(x+6)}$; $x=0, -1{\cdot}71$.

C. 20. (p. 227.)

1. 40.
2. $x^3(x-1)$.
3. $x=0, y=0$; $x=\frac{b}{a}, y=\frac{b}{a}$.
4. $a \bumpeq 1{\cdot}43$; $x \bumpeq 1{\cdot}8$.
5. $x \bumpeq 3{\cdot}73, 0{\cdot}27$.
6. $\frac{20(2x+y)}{1869}$ lb.

C. 21. (p. 227.)

1. $P=\frac{0{\cdot}00000025l^3}{dt^2}$; time is multiplied by $\sqrt{8}$, *i.e.* 2·82....
2. $3x^2+2x$.
3. $x=3{\cdot}6, 0{\cdot}4$.
4. (i) $(x^2+2y)(x^2-9y)$. (ii) $-(a-b)^2$. (iii) $(x+1)^2(x-1)^2$.
5. 0·055 lb.
6. $8x^3$ cu. in.

C. 22. (p. 228.)

1. (i) $P=\frac{V^2S}{620}$; multiplied by 8.
3. (i) $x=\frac{a^2}{a+1}$. (ii) $x=\frac{1}{3}, -8$.
4. $\frac{r}{2}(S-2\pi r^2)$.
5. (i) $9\frac{1}{4}$. (ii) $(x-3)^2-4^2$.
6. 100.

C. 23. (p. 229.)

2. $-\frac{(1-x)^2}{x}$.
3. $a=8$.
4. $36\pi V^2=S^3$.
5. $330x+3y$ pence; $11x+\frac{y}{10}$ hf. crowns; $y=770$.
6. $\frac{s(pt-k)}{t-s}$ mi.

C. 24. (p. 229.)

2. $x+y$ hr.; $x(x+y)$ mi.
3. $x=1, -7$.
4. $A=\frac{C^2}{4\pi}$.
5. $\frac{3x+1}{x(x-1)^2}$.
6. (i) $T=t+\frac{h}{16t}$; (ii) $a=4T^2$; (iii) 144 ft., $1\frac{1}{4}$, $4\frac{3}{4}$ sec.

C. 25. (p. 230.)

1. (i) $s=\frac{4d^2+6l^2}{3l}$; (iii) $\frac{1}{133}$ ft. (0·09 in).
2. 7, 24, 25 in.
3. $x=2, 2\frac{2}{3}$.
5. $BC=10$ in.
6. $\frac{(c-b)n}{a-b}$ men.

D. 1. (p. 231.)

1. $5 - x$. 2. (i) $a^3(2a+b)(4a^2-2ab+b^2)$. (ii) $(a+x)(b+x)(a-b)$.

3. $x = 10a,\ -9a$. (ii) $x = \dfrac{b^2+c^2-a^2}{2a},\ y = \dfrac{a^2+c^2-b^2}{2b}$. 4. $-\dfrac{a}{b}$.

5. $\dfrac{1}{1-x}$. 6. $\dfrac{100}{an}$; $100\dfrac{ad_1{}^2 - \left(a+\dfrac{1}{n}\right)d_2{}^2}{ad_1{}^2}$.

D. 2. (p. 231.)

1. $x^5 + y^5$. 2. (i) $x = \dfrac{ac}{a+b},\ y = \dfrac{bc}{a+b}$. (ii) $x = 17,\ y = 15,\ z = 10$.

3. $x^2 - 2xy - 2xz$; no. 4. $\dfrac{4y^2}{(x^2-y^2)^2}$.

5. (i) 2; from 0 to 4; any value of x less than 2. 6. $\dfrac{10}{n+1},\ \dfrac{10n}{n+1}$ in.

D. 3. (p. 232.)

1. $-(c-a)(a-b)$. 2. $x = \dfrac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1},\ y = \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1},\ x = \dfrac{164}{53},\ y = \dfrac{71}{53}$.

3. $\pm\left(a - 1 + \dfrac{1}{a}\right)$. 4. $\dfrac{(x^2+ax+a^2)(x^2+bx+b^2)}{x^2-ax-bx-ab}$.

5. $x = 12,\ y = 8;\ x = 3,\ y = -8$. 6. $r = 3\tfrac{1}{3}$.

D. 4. (p. 232.)

1. (i) $(x+1)(x^2-x+1)(x^3+4)$. (ii) $(1+a)(1+b)(2+a+b)$.

2. $x = 6,\ y = \tfrac{1}{3},\ \dfrac{x}{y} = 7,\ -1\tfrac{1}{2}$. 3. $-\dfrac{1}{(x+a)(x-b)}$. 4. $6a$.

5. $(a-6)x + b - 8$; $a = 6,\ b = 8$. 6. $r^2 = a^2 + p^2$; $\sqrt{11}$ (3·32) in.

D. 5. (p. 233.)

1. Not true for the first 12. 2. $a = 5,\ 1,\ -1,\ -5$.

3. (i) $\dfrac{2a^2+2a+1}{2a^2-1}$. (ii) $\dfrac{x-y+z}{x+y+z}$. 4. $x = 6,\ y = 4,\ z = 3$. 6. 4·55 ft

D. 6. (p. 234.)

1. $\dfrac{9ab}{4}$ sq. ft.; 125. 4. $\dfrac{(m+1)(2m+1)(4m+3)}{3}$.

5. $v \simeq 2{\cdot}8$ (2·7). 6. 4, 3, 1 in.

D. 7. (p. 234.)

1. 0·343.
2. (i) $(x+y)(x-y-1)$. (ii) $(x+y-1)(x-y-1)$
3. $(31-a)x$; $a=31$.
4. $(3x-2)(x-3)(2x-1)$; $2x^2-7x+3$.
6. $BD=\frac{ac}{c-b}$; $DC=\frac{ab}{c-b}$.

D. 8. (p. 235.)

1. $x+3a-2b$; the other is $19x^2+(24a+14b)x+9a^2+6ab+4b^2$.
2. $\frac{x+y-z}{x-y+z}$.
3. 7, 4.
4. $\sqrt{3}\,.\,a^2$.
5. $x=\frac{ab}{a+b-c}$.
6. 12 o'clock; P 125 mi. from A.

D. 9. (p. 235.)

1. $x=2$, $y=-1$.
2. 1.
4. H.C.F. a^2-x^2; L.C.M. $(a+x)(a-x)^2(a^2+x^2)$.
6. $21\frac{9}{11}$, $54\frac{6}{11}$ min. past 7.

D. 10. (p. 235.)

1. (i) $x=a+1$, $y=a-1$. (ii) $x=5$, $-\frac{1}{2}$.
2. $\frac{a^4-10a^3b-6ab^3-b^4}{a^4+10a^3b+6ab^3-b^4}$.
3. $a=1$; $x=6{\cdot}16$, $-0{\cdot}16$.
6. 25.

D. 11. (p. 236.)

1. (i) $x=7\frac{1}{2}$. (ii) $x=7$, $y=2$; $x=2$, $y=7$.
2. $x=5$.
3. (i) $\frac{3x+3}{4}$. (ii) $\frac{(a^2+b^2)(2a^2-ab+b^2)}{a(a-b)}$.
4. $(ac+b^2)(bc+a^2)$
5. £41. 13s. 4d.
6. After $2\frac{4}{13}$ min.; the roots are imaginary; he is never in range.

D. 12. (p. 237.)

1. $4y^4-8y^3-3y^2+7y-2$; $(y+1)(y-2)(2y-1)^2$.
2. $x=\frac{1}{2}$, $-3\frac{1}{2}$.
3. (i) 69, -70. (ii) $x=\frac{ac}{a^2+b^2}$; $y=\frac{bc}{a^2+b^2}$.
4. $\frac{1}{abc}$.
6. Tea, 5s.; sugar, 4d. per lb.

D. 13. (p. 237.)

1. $x=\frac{c(b-c)}{a(b-a)}$, $y=\frac{c(a-c)}{b(a-b)}$.
2. $x^3+(a+b+c)x^2+(bc+ca+ab)x+abc$; 23.
4. 1·015.
5. Values between 5 and 2.
6. $\frac{c-b}{a-b}$; $\frac{13}{19}$.

D. 14. (p. 237.)

1. $-a-b$.

2. (a) 80, (b) 80·75, (c) 80·99, (d) $81-x^2$ sq. in. $x=0$. Each side $=9$ in

3. $x=2,\ y=-\frac{1}{2}$; $x=-2,\ y=\frac{1}{2}$. 6. $21\frac{9}{11}$ min. after 4.

D. 15. (p. 238.)

1. 0. 2. $2\frac{1}{4}$.

3. (i) $x=-2,\ y=0,\ z=2$. (ii) $x=\pm(a+b),\ y=\pm(a-b)$; $x=\pm(a-b)$,
$y=\pm(a+b)$.

4. $(x+3y+3)(x-2y+2)$. 5. 3·0719(6); $3+7x+19x^2+55x^3+163x^4$

6. $a=25,\ b=15,\ c=11\frac{1}{4}$.

D. 16. (p. 239.)

1. (i) $x=48,\ y=7$. (ii) $x=7,\ y=9$; $x=-10,\ y=-8$.

2. $\dfrac{2x^3}{x^4-1}$. 4. $(a+b-c-d)^2$.

D. 17. (p. 239.)

1. $x^3+y^3+x^3-3xyz$. 2. x^2+6x-7. 3. $x=2$.

4. $\dfrac{1}{x(2x-1)}$. 5. $\frac{1}{2}(b-a)$; $\dfrac{c}{a}(b-a)$ in. 6. $5\frac{3}{8}$ in.

D. 18. (p. 240.)

1. (i) $x=1,\ y=-4$; $x=3,\ y=-2$. (ii) $x=2a,\ \dfrac{a}{2}$. 2. $a=-6$.

5. £$\dfrac{10000k}{(100+p)(100+q)}$. 6. $b=\sqrt{\dfrac{1000g}{\sqrt{2}\,l}-cd}$. (ii) vol. is multiplied by 4

D. 19. (p. 240.)

1. $x^2+2xy+2y^2$. 3. $x=\pm2,\ y=\pm3$. 5. £1 $=27\frac{1}{2}$.

6. 3 in. from pt. of suspension.

D. 20. (p. 241.)

1. $d=r\sqrt{\sqrt[3]{\left(\dfrac{m}{m-r^2h}\right)^2}-1}$. 2. $x=6,\ y=1$.

3. (i) $3(x+4)(x-4)$. (ii) $2(x-1)(1-2x)$ 4. (ii) $x=3,\ -4$.

5. *A* was absent $6\frac{1}{2}$ min.

D. 21. (p. 241.)

1. $a=\frac{1}{2}$, $b=\frac{4}{5}$; $x=6{\cdot}53$, $-1{\cdot}53$. 2. $2r^2+2pr+pq-q^2=0$; $r=2\frac{1}{2}$, -6

3. -7. 4. $A=-2$, $B=-3$. 6. (i) 0·36. (iii) 4 ft.

D. 22. (p. 242.)

3. (i) 4, 3, -1, -2. (ii) $x=5$, $y=-\frac{3}{4}$. 5. $y=\frac{48-7x}{10-x}$.

6. 8·94 ($\sqrt{80}$) cm.

D. 23. (p. 243.)

2. $\frac{120y}{x}+\frac{6z}{5}$ sec. 4. $x=2$; $(x-2)(x^2+x+1)$. 5. 16″, 18″.

6. $a=0{\cdot}3$; $b=0{\cdot}8$; 2·125 lb.

D. 24. (p. 243.)

1. 9. 2. $-\frac{a(a+1)^2}{a-1}$

3. (i) $x=\frac{1}{2}$, $y=-1\frac{1}{2}$, $x=-\frac{3}{5}$, $y=-\frac{4}{5}$; (ii) $x=-3a$.

4. -2. 5. y^3. 6. $a=0{\cdot}60$, $b=0{\cdot}39$.

D. 25. (p. 244.)

2. $51\frac{1}{24}$. 3. $p=\frac{1}{c(c+1)(c+2)}$, $q=-\frac{c+3}{c(c+1)(c+2)}$, $r=\frac{1}{c}$.

4. (i) $x=\pm\frac{a}{bc}$, $y=\pm\frac{b}{ca}$, $z=\pm\frac{c}{ab}$. (ii) $x=a$, $\frac{1}{a}$.

6. $\sqrt{(a^2+b^2)x^2-2bkx+k^2}$ mi.

D. 26. (p. 245.)

2. $\frac{x^3-1}{1-y^3}$. 3. $y=2am$ or $-2am-\frac{4a}{m}$.

4. (i) $x^2-1{\cdot}1x+0{\cdot}1=0$. (ii) $x^2-(a-1)^2=0$. (iii) $x^3-x^2-2x=0$.

5. $\frac{p(p+q)}{q}$ hr. 6. $3(z-x)(x-y)$.

D. 27. (p. 245.)

1. (i) $(3x^2-y)(x-3y^2)$. (ii) $(a-b)(2a-b+c)$.

2. (i) 205 and 235. (ii) -5 and -35. (iii) $\frac{19}{26}$ and $\frac{21}{22}$ ($\simeq 0{\cdot}73$ and $0{\cdot}95$)

3. $x=\frac{a}{2}\pm\frac{a}{\sqrt{b}}$; $x=2{\cdot}89$. 4. $\frac{(n+1)(n+2)}{2}$.

5. (i) $x=-1$, $y=-1$; $x=3$, $y=3$. (ii) $x=4$, $y=2$, $z=-3$.

6. $OB=5\frac{1}{4}$, $BT=6\frac{3}{8}$ in.

D. 28. (p. 246.)

2. $(b_1 - b_2)x + (a_2 - a_1)y + (a_1 b_2 - a_2 b_1) = 0$.
3. (i) $x = 4$. (ii) $x = 9, 3, -5, -15$; $y = 5, 15, -9, -3$.
5. $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$.

D. 29. (p. 247.)

1. (i) L.C.M. $xy(x+y)^2(x^2 - xy + y^2)$. (ii) $(c - a^2)(c - b^2)$.
2. (i) $x = 3, y = 45$; $x = \pm 7, y = 5$. (ii) $x = a + b, -(b + c)$.
3. $a^2(b - c) + b^2(c - a) + c^2(a - b)$; $-(b - c)(c - a)(a - b)$.
4. $1 - 4x + 10x^2 - 20x^3$; $x^{12} - 4x^{11} + 10x^{10} - 20x^9$.
6. $(x + a)(y + b) = \frac{1}{2}(a + b)^2$; $y = 2\frac{1}{8}$.

D. 30. (p. 248.)

1. $a + b + c$. 2. (i) 1. (ii) $\dfrac{2}{x^3}$.
3. $2x - (m + n)y + 2amn = 0$; $x = am^2, an^2$; $y = 2am, 2an$.
4. $(a^3 + b^3)c^2 - a^2b^2d$. 5. 36.
6.

	Solid.	Cube.	Pyramid.	Prism.	Barn.	*H.*	*L.*	Block.
Corners	*C*	8	5	6	10	24	12	10
Edges	*E*	12	8	9	17	36	18	15
Faces	*F*	6	5	5	9	14	8	7

$C + F = E + 2$.

D. 31. (p. 248.)

1. $\lambda = -\dfrac{c}{c'}$. 2. $\dfrac{n^2}{m+n}$ c.c. 3. $4n - 2$ months.
4. $\sqrt{2}\left(am + \dfrac{a}{m}\right)$. 5. $\dfrac{ab}{a+b}$ min.; 6 min. 6. $x = 6{\cdot}81, 0{\cdot}59$; $m = \pm 2$.

D. 32. (p. 249.)

1. 205·4° F. 3. $x = a$, $y = -b$.
4. (i) 50·02. (ii) 10000·0001. $\frac{1}{0}$ has no meaning. No limit to its value as x approaches 0.
5. $10\frac{1}{2}$ sea miles. 6. $x = 2$.

D. 33. (p. 250.)

1. $\dfrac{x^2-ax-bx-ab}{x-a}$; $\dfrac{y^2+ay+by-ab}{y+b}$.

2. $\dfrac{x}{240}(m-y)$£, $\dfrac{x}{m}(m-y)$ pence per lb.; $x=21$, $y=1000$; £2333, £2100, £1909.

3. $x=4a$. 5. $(x^2+2x-1)^2$. 6. 5 % aluminium, 95 % copper approx.

D. 34. (p. 250.)

1. $\dfrac{a^2+b^2}{a^2-b^2}$. 3. $x=y=\dfrac{1}{l+m}$. 5. $3x-2-\dfrac{1}{x}$. 6. $x=59$, $y=24$

D. 35. (p. 251.)

1. $\pi hl(2r-h)$. 2. $a_2b_3-a_3b_2+a_3b_1-a_1b_3+a_1b_2-a_2b_1=0$.

4. $x=y=1, -2\frac{1}{3}$. 5. $\dfrac{v^2-x^2}{v}$ mi. per hr. 6. A, 576; B, 700.

ANSWERS.

E.P. 1.

1. $\frac{x}{x}=1.$

2. $x+x+x=3\times x.$

3. $\frac{x}{2\times x}=\frac{1}{2}.$

4. $\frac{a}{b}\times b=a.$

5. $N\times 0=0.$

6. $x\times x\times x=x^3.$

7. $\frac{a}{b}\times\frac{b}{a}=1.$

8. $1+\frac{a}{b}=\frac{b+a}{b}.$

9. $0\div x=0.$

10. $1\div a=\frac{1}{a}.$

11. $1\div\frac{a}{b}=\frac{b}{a}.$

12. $x^3\div x=x^2.$

13. 7 lb. at xd. per lb. cost $x\times 7$d.

14. N balls cost $2\times N$s.

15. In £P there are $P\times 20$s.

16. When N people are seated, $20-N$ chairs are empty.

17. In x years, age $27+x$.

18. x in. $=\frac{x}{12}$ ft.

19. For x pence, $\frac{x}{1\frac{1}{2}}$ stamps.

20. If x are used, $100-x$ remain.

21. After x miles, $45-x$ miles more.

22. x bricks weigh $2\times x$ lb.

23. x cwt. $=\frac{x}{20}$ ton.

24. When n are done, $50-n$ remain.

25. In t min. I walk $t\times 100$ yd.

26. At v miles an hour, he goes $v\times t$ miles in t hours.

27. Empty w lb., full W lb., coal weighs $W-w$ lb.

28. From x a.m. to y p.m. is $(12-x)+y$ hours.

29. Saves £P a year; in n years saves £$(P\times n)$.

30. Scores R runs in n innings, average $\frac{R}{n}$ runs.

31. Cost price £C, selling price £S, gain £$(S-C)$.

32. Stride x inches; in S yards $\frac{S\times 36}{x}$ paces.

33. Of n beds, k beds are occupied ; $(n-k)$ beds are empty.

34. Loses t sec. an hour, loses $t \times 24 \times n$ sec. in n days.

35. When man is x years old, son is y years old. When man is $x+z$ years old, son is $y+z$ years old.

36. 1 gallon every s miles, $\frac{x}{s}$ gallons for x miles.

37. After x chapters, y chapters remain ; $x+y$ chapters in all.

38. Fare n pence, x whole tickets and y halves cost $\left(n \times x + n \times \frac{y}{2}\right)$ pence

39. Earn £P a year, spend £M a month, save £$(P-12M)$ a year.

40. Burn x tons a week. N tons last $\frac{N}{x}$ weeks.

E.P. 2.

1. 15, 8, 1, 45, 18, $1\frac{1}{5}$, 2, 16. 2. 3, 3, 0, 10, 3, 3, 0, 4.
3. 9, 6, 0, 2, $\frac{1}{6}$, 54, 18, 0. 4. 2, 2, 1, $1\frac{1}{2}$, 8, $\frac{1}{8}$, $\frac{1}{2}$, 8.
5. 2, 3, 6, 0, 81, 6, 2, 0. 6. 1, 6, 0, 4, 1, 8, 1, 6.
7. 60, 50, 0, 15, 2, 0, 27, 0. 8. 4.
9. 24. 10. 1. 11. 1. 12. 2, $\frac{32}{9}$.
13. $\frac{1}{4}$, 1, $\frac{1}{3}$, $\frac{1}{4}$, 2, 4, $\frac{1}{6}$, $\frac{1}{2}$, $2\frac{1}{4}$, $\frac{3}{4}$, $\frac{4}{5}$, $1\frac{1}{2}$, $\frac{1}{3}$, $2\frac{1}{4}$, $1\frac{1}{3}$.
14. 6, 2, 2, $\frac{3}{10}$, 15, $\frac{9}{25}$, 4, 6, $6\frac{1}{3}$, 9, $2\frac{1}{2}$, 0, 1, 4, $\frac{4}{5}$.
15. 0, 6, 1, $\frac{2}{3}$, 18, $\frac{2}{3}$, 9, $\frac{1}{3}$.

E.P. 3.

26. 5 is a factor of 15, 7 of 35. 27. $3 \times \frac{1}{2} = \frac{1}{2} + 1$; $6 \times \frac{1}{5} = \frac{1}{5} + 1$.
28. $\sqrt{36} = 6$, $\sqrt{900} = 30$.
29. At 3 m.p.h., 24 mi. in 8 hr. ; at 4 m.p.h., 24 mi. in 6 hr.
30. $\frac{1}{2}$d. per inch, £132 per mi. ; $1\frac{1}{4}$d. per inch, £330 per mi.
31. Falls 64 ft. in 2 sec., 4 ft. in $\frac{1}{2}$ sec.
32. $8^2 - 4^2 = 8 \times 6$; $10^2 - 6^2 = 8 \times 8$.
33. $7^2 - 3^2 = 8 \times 5$; $11^2 - 7^2 = 8 \times 9$.
34. 20 ft. long, 12 ft. wide ; 16 ft. long, 15 ft. wide.
35. 50° F. = 10° C. ; 95° F. = 35° C.
36. 3 sides, 60° ; 4 sides, 90° ; 5 sides, 108°. 37. 9 mi., 12 mi.
38. 3 lines, 3 points ; 4 lines, 6 points ; 5 lines, 10 points.
39. $h=6$, $l=60$, $6 \times 60 = 12 \times 30$; $h=3$, $l=105$, $3 \times 105 = 15 \times 21$.
40. 2·4 min., $2\frac{1}{4}$ min.

E.P. 4.

1. $3x$. **2.** $2a$. **3.** $6b$. **4.** $7c$. **5.** $4y$. **6.** $4z$.
7. $7t$. **8.** 0. **8.** $30v$. **10.** $2xy$. **11.** $8a^2$. **12.** $3ab$.
13. t^2. **14.** $2ab$. **15.** $10t^2$. **16.** $7cd$. **17.** $5c^2$. **18.** $8pq$.
19. $6x^2$. **20.** $4p$. **21.** $3R$. **22.** $3xy$. **23.** $5xy$. **24.** $8cd$.
25. $3u+v$. **26.** $4a+2$. **27.** $2x-y$. **28.** $a+2b$. **29.** None. **30.** $b-a$
31. $8x-5y$. **32.** $x+5y$. **33.** $2x+4y$. **34.** $7x$. **35.** None.
36. $6ab-7$. **37.** None. **38.** $6p-q$. **39.** None. **40.** $6z$.
41. 0. **42.** $3a^2b$. **43.** None. **44.** $4st$. **45.** None.
46. 0. **47.** $9v^3$. **48.** None. **49.** $x+4y$. **50.** $4a+2f+2$.
51. $2A+2B+1$. **52.** $2A^2-8AB$. **53.** $2x^2y+xy^2$. **54.** $3u+2v+3$.
55. $5ab+a+b$. **56.** 0. **57.** $3t^2-5t+3$. **58.** $5yz-3xz-xy$.
59. x^3-x^2+x+1. **60.** $2q^2$. **61.** x^2-x-2. **62.** c^4+2c^2+3.
63. a^2b+ab^2. **64.** $3a^2b^2$. **65.** $6-9t$. **66.** None.
67. x^3-x^2+2x. **68.** $4x^2-5x+4$. **69.** a^3+2a^2+1.
70. $4t^3-4$. **71.** $2x^4$. **72.** $3-y-4y^2$.
73. $2x^3-x^2+3x+1$, $1+3x-x^2+2x^3$.
74. $5x^4-x^3-2x^2+x+3$, $3+x-2x^2-x^3+5x^4$.
75. $3x^2-6x+3$, $3-6x+3x^2$. **76.** $2x^3+6x+4$, $4+6x+2x^3$.
77. 3, 1. **78.** 5, 3. **79.** $2x^3$. **80.** 1. **81.** 0.
82. 2, 3. **83.** 1. **84.** 0. **85.** a^4-a^2. **86.** x^3+1.
87. $2x^4+2x^2$. **88.** $4a^2$. **89.** $2x^2y$. **90.** x^3-x.

E.P. 5.

1. $16t^2$. **2.** $25a$. **3.** $2a^2b$. **4.** $6c$. **5.** c^2. **6.** $2d$.
7. $20x^3$. **8.** $20xy^2$. **9.** $9x^3$. **10.** a^2bc. **11.** $4abc$. **12.** ab.
13. $6t^4$. **14.** $2st$. **15.** $2pq^2r$. **16.** x^6. **17.** x^4. **18.** $9xy$.
19. p^5. **20.** $2t^3$. **21.** $9a^2b^2$. **22.** $3y^4$. **23.** $3ab$. **24.** $6a^2b$.
25. $8t^6$. **26.** $6c^3$. **27.** $4r$. **28.** $6x^3y^2$. **29.** 10. **30.** $8p^3$.
31. $9a^3b^3$. **32.** $4x^4y^4$. **33.** $6p^2q^2$. **34.** c^3. **35.** 1. **36.** $24x^3y^3$.
37. $16a^2b^4$. **38.** $2a^4$. **39.** $27x^6$. **40.** $9x^6$. **41.** $4xy^2$. **42.** $8y^9z^6$.
43. $2a^2$. **44.** $25p^2q^4r^6$. **45.** $13a^2$. **46.** $16b$. **47.** $27c^4$. **48.** x^6.
49 $24n^3q^4$. **50.** $2x^5y^4$. **51.** $19t^3$. **52.** $18v^3$. **53.** $4z^3$. **54** $3ab^5$.
55. $108x^6$. **56.** $4a^4b^2$. **57.** $8t^2$. **58.** 2. **59.** $x^2y^2z^3$.

E.P. 6.

1. $3x$. **2.** ab. **3.** $6p^2q$. **4.** $6y^2z$. **5.** $5ab$. **6.** $3c^2x^3$.
7. $2xyz$. **8.** $28abxyz$. **9.** $x^2, 6x^3$. **10.** $5b, 10a^2b^2$.
11. $2y^2, 24y^3z$. **12.** $2, 12x^3y^3$. **13.** $1, 20pqr^2$. **14.** $2xy, 40xy$.
15. $2, 60t$. **16.** y^3z, y^6z^2. **17.** $3f, 18ef^2g^2$. **18.** xy, x^2y^2.
19. $1, 36b^3c^3$. **20.** $4b, 24ab^2xy$. **21.** $2x, 60x^3y$.
22. $3a^2b, 36a^4b^3c^2$. **23.** $z^2, 12z^4$. **24.** $1, 60r^2s^2$.
25. $2, 12x^2y^3$. **26.** $xyz, 12x^2y^2z^3$. **27.** $1, 120p^3q^2$.
28. $3y, 36x^3y^3z^2$. **29.** $5x, 60abcx^3y$. **30.** $4x^2, 36a^2b^2c^2x^2$.
31. a. **32.** b. **33.** c. **34.** d. **35.** 1. **36.** $\frac{p^2}{3}$.
37. r^4. **38.** $\frac{1}{t}$. **39.** $\frac{1}{u}$. **40.** $\frac{1}{v^3}$. **41.** $\frac{y}{z}$. **42.** $\frac{2y}{x}$.
43. 1. **44.** $\frac{1}{b}$. **45.** $\frac{c}{2d}$. **46.** 1. **47.** None. **48.** $\frac{1}{k^2l^2}$.
49. $4m^3n^3$. **50.** None. **51.** $\frac{2l^2}{3v^2}$. **52.** $\frac{1}{x^2}$. **53.** u. **54.** a^4b^4.
55. $\frac{c^4}{3}$. **56.** $\frac{9}{4}$. **57.** $8a^6$. **58.** $\frac{x}{4}$. **59.** $\frac{1}{q^2r^2}$. **60.** None.
61. $\frac{3}{2a}$. **62.** $\frac{2}{3b}$. **63.** 0. **64.** $\frac{e^2-1}{e}$. **65.** $\frac{f+1}{f}$.
66. $\frac{gh^2+g}{h}$. **67.** $\frac{4-l^2}{2l}$. **68.** $\frac{n+6m}{2mn^2}$. **69.** $\frac{pq-p^2}{q^2}$. **70.** $\frac{rt-s}{t}$.
71. 0. **72.** $\frac{1-x}{6x^3}$. **73.** $\frac{2z+3y}{12yz^2}$. **74.** $\frac{3a^2-2b^2}{30abc}$.
75. $\frac{9ay^2+4bx^2}{6x^2y^2}$. **76.** c. **77.** $\frac{d}{b}$. **78.** e^2f. **79.** $\frac{p}{q}$.
80. $\frac{r}{s}$. **81.** $\frac{x^3}{y^3}$. **82.** $\frac{1}{8}x^2yz$. **83.** $\frac{1}{ac}$. **84.** b^8.
85. $\frac{4c}{d^3}$. **86.** $\frac{27e^2}{f^3}$. **87.** $24p^6$. **88.** $\frac{1}{2y}$. **89.** p^6. **90.** 2.

E.P. 7.

1. $z-(x+2y)$. **2.** $z^2-(x+2y)^2$. **3.** $3c(a-b)$.
4. $2(r-q)+p$. **5.** $\frac{1}{3}(x+y+z)$. **6.** $2l-(m-n)$.
7. $v-2(d-c)$. **8.** $(N+1)(n-1)$. **9.** $(R+r)^2-(R-r)$.
10. $(A-B)\div(a-b)$. **11.** $N(N-1)(N-2)$. **12.** $2l(2l+2)(2l+4)$.

13. $12(2a-b)$ in. 14. $N-(x-y)$ gall. 15. $8N-t(a+b)$ pints.
16. $40-(x-7)$ mi. 17. $(x+4)(x-4)$ sq. ft. 18. $n(W-w)$ lb.
19. $72+2(n-72)$ pints. 20. $72+5(N-12)$ pence.
21. $20t+30(3\frac{1}{2}-t)$ mi. 22. $20x+y$ sh.; $12(20x+y)+z$ pence.
23. $6a+b+\frac{1}{4}(6a+b)$ sh. 24. $(l+8)(b+8)-lb$ sq. ft.

E.P. 8.

1. $a+b-3c$. 2. b. 3. $f-g$. 4. $2l$. 5. $2p-q$.
6. $r-3s$. 7. $4t-2s$. 8. $7y-x$. 9. 0. 10. y^2.
11. $3a-3z$. 12. $2b$. 13. $c-3d$. 14. $5f+h$. 15. $3+2k$.
16. $4p-3r$. 17. $5a+12q$. 18. $u-7v$. 19. $2xy$. 20. x^2-y^2.
21. z^2. 22. $x-5$. 23. $2a-3b$. 24. 1. 25. $4e^2-11ef+6f^2$
26. $a^6+a^3b^3+2b^6$. 27. $6y-12z$. 28. 0. 29. $3x-4y$.
30. $x-y$. 31. $1+a$. 32. $5p-15$. 33. $2x-2y$. 34. $x-5y$.
35. $9b-10c$. 36. a^3+ab^2. 37. $a-2(b+c)$. 38. $p-(q-r)$.
39. $x(x-y)$. 40. $m-(n-l)$. 41. $2(a-c)-3(x+z)$.
42. $a(a+b)-p(p-q)=a(a+b)+p(q-p)$.
43. $x(x+y-z)=xy-x(z-x)$.
44. $4l-(l-m+n)=3(l+m)-(2m+n)$.
45. $1+x+x^2(1+x)=1+x^2+x(1+x^2)$. 46. $2x-4z$.
47. 0. 48. xy. 49. $2q+2r$. 50. $r-2s+2t$.

E.P. 9.

1. $s+4$ mi. 2. $3+W+w$ lb. 3. $10-m$ gall. 4. $d-n$ pence.
5. $2x+20+2y$ books. 6. $n+2$ apples. 7. $3t-3$ ban.
8. $W+n-c$ lb. 9. 15, $t-15$, $45-n$ min.
10. 12, 10, $4b$, $20b$ mi.
11. $5l$ pence, $2x$ pence, $12m$ sh., xp sh., $2nR$ pence, NP sh.
12. 48, 57, $12l$, $12l+9$, $12l+m$, $12p+q$, $36v+z$ in.
13. 35, $20N-5$, $20g-f$ sh. 14. $2b+7a$ in., $\frac{2b+7a}{5}$ oz.; 30 in., 6 oz.
15. $30-3z$ pence. 16. $b+6n$ sh.
17. 60, 18, $24t$ mi.; $\frac{1}{24}$, $\frac{5}{12}$, $\frac{s}{24}$ gall. 18. $\pounds(P-40)$.

19. $z-10$ sh. **20.** $\frac{v}{5}$, $\frac{v}{5+x}$ apples. **21.** $\frac{k}{2}$, $4n$ jugs.

22. $\frac{100}{g}$, $\frac{n}{g}$, $\frac{8z}{g}$. **23.** $\frac{16}{3}$, $\frac{2}{3}$, $\frac{l}{3}$ yd. **24.** 32, 24, $16W$ oz.

25. $3n$, $21t$ loaves ; $\frac{x}{3}$ days.

E.P. 10.

1. $24s$, $9t$ in. **2.** $(nl+c)$ ft. **3.** $\frac{p}{n}$ hours. **4.** $4W$ weeks.

5. $18l$ sq. ft., $2l$ sq. yd. **6.** $\frac{p^2}{4}$ sq. yd. **7.** $\frac{1}{2}x^2$ sq. ft.

8. $60x$ sq. ft. **9.** $6t$ cu. in. **10.** $6tW$ oz. **11.** $72V$ tins.

12. $54t^3$ cu. ft., $2t^3$ cu. yd. **13.** $\frac{20}{p}$ pencils. **14.** $2s$ mi.

15. $\frac{50}{v}$ hours, $\frac{50}{v-10}$ hours. **16.** $\frac{N}{6}$. **17.** $19y-19x$ s.

18. $5p^2$ sq. in. **19.** $100n$ seats. **20.** $\frac{20}{n}$ trains. **21.** $(12t+4h)$ in.

22. $(12k+48)$ pence, $(k+4)$ sh. **23.** $\frac{c}{6}$ ft., $2c$ in. **24.** $44p^2$ sq. ft.

25. $\frac{11N^2}{5}$. **26.** $\frac{1}{2}t$ days. **27.** $\frac{3xy}{2}$ sq. in. **28.** $6b^2$ tiles.

29. $3x$ mi. ; 4. **30.** £$(S+P)$, £$(S+p-P)$. **31.** $n+b+c$ marks.

32. £$(125P)$. **33.** Pay $(2y-6x)$ pence.

34. $\frac{1}{2}(p+q+r)$ lb. **35.** 3 mi.

E.P. 11.

1. $2a$, $-2b$, $2y$, $8x$, $-7p$, $4z$. **2.** 0, $-5b$, $4c$, $3x$, $-2y$, $-5z$.

3. a, $-4b$, $12x$, $-y^2$, z^3, $-10c$. **4.** $4a$, b, $-3y$, $-3x$, x, $-2a$.

5. 2. **6.** −3. **7.** 0. **8.** 4. **9.** 0. **10.** −6.

11. −1. **12.** 8. **13.** 0. **14.** −4. **15.** 2. **16.** −3.

17. −4. **18.** −1. **19.** −1. **20.** 0. **21.** 0. **22.** 2.

23. −2. **24.** $\frac{1}{2}$. **25.** 4. **26.** −1. **27.** 16. **28.** 1.

29. 0. **30.** 1. **31.** 1. **32.** 1. **33.** −5. **34.** 7.

35. 0. **36.** −2. **37.** −4. **38.** −2. **39.** −1. **40.** 2.

41. −7. **42.** −7. **43.** −9. **44.** 2. **45.** 5. **46.** −6.

47. 0. **48.** 7. **49.** −1. **50.** 12. **51.** 0. **52.** 0.

53. -8. **54.** 5. **55.** -2. **56.** -12. **57.** 9. **58.** $-3p^2$.
59. $2r^2$ **60.** $-6st$. **61.** $4t$. **62.** $-2k$. **63.** $-4l$. **64.** $4a$.
65. $3b$. **66.** $-2c$. **67.** $-6x$. **68.** 0. **69.** 0. **70.** $-3p$.
71. $-a$. **72.** $-8t$. **73.** $-2x$. **74.** $-2y$. **75.** $3z$. **76.** $2a$.
77. $-2c$. **78.** 0. **79.** $2c-2b$. **80.** $3r+3s$. **81.** $-2k-l$.
82. $-\frac{1}{b}$. **83.** $-\frac{1}{c}$. **84.** $\frac{1}{2}$. **85.** $\frac{1}{q}$. **86.** $-x$.
87. $-\frac{x}{y}$. **88.** 1. **89.** $\frac{a}{2}$. **90.** $-ab$. **91.** $-b$.
92. $-c+2d$. **93.** $-x+2y$. **94.** $-p+q$. **95.** $-r+2s$. **96.** $-3x+y$.
97. $y-z$. **98.** $2a-2b$. **99.** $-2x$. **100.** $2x^2$. **101.** $2a+b-c$
102. $-x+3y+1$. **103.** $-\frac{3x}{y}$. **104.** $(-1)(-x+y)$.
105. $-abc$, x^6, $-y^4$, $-6pq$. **106.** $-3x^2-5x-9$.
107. $5x+2$. **108.** $-x^3y+xy^3$. **109.** $a-a^2$. **110.** $2b^2-b-1$.
111. $-\frac{1}{2}$. **112.** 3. **113.** 1. **114.** $2a-2b$.
115. $\frac{1}{2}$. **116.** $11, 3$. **117.** $b+a-1$.
118. $-x^2y+xy^2-xyz$. **119.** $-(4s-3r)$. **120.** -2.

E.P. 12.

1. $a=8\frac{1}{2}$, $b=4\frac{1}{2}$. **2.** $u=6$, $v=-5$. **3.** $x=4\frac{1}{2}$, $y=-4\frac{1}{2}$.
4. $p=2\frac{1}{2}$, $q=5\frac{1}{2}$. **5.** $l=4$, $m=4$. **6.** $u=5$, $v=3$.
7. $x=4$, $y=3$. **8.** $x=2$, $y=12$. **9.** $x=-1$, $y=2$.
10. $x=2$, $y=5$. **11.** $x=\frac{7}{3}$, $y=0$. **12.** $x=-2$, $y=3$.
13. $l=\frac{7}{12}$, $m=1\frac{3}{4}$. **14.** $a=2\frac{3}{4}$, $b=1\frac{1}{4}$. **15.** $x=-1\frac{1}{2}$, $y=-3$.
16. $t=1\frac{1}{3}$, $z=5\frac{1}{3}$. **17.** $c=7$, $d=-2$. **18.** $p=-\frac{5}{23}$, $q=\frac{21}{23}$.
19. $x=7$, $y=0$. **20.** $x=-3$, $y=4$. **21.** $x=2$, $y=-7$.
22. $r=-3$, $s=-4$. **23.** $p=-2$, $z=5$. **24.** $x=-5$, $y=2$.
25. $c=12$, $d=3$. **26.** $p=\frac{1}{2}$, $q=-3$. **27.** $x=2$, $y=-1$.
28. $x=3$, $y=-4$. **29.** $x=4$, $y=-1$. **30.** $x=-2$, $y=5$.
31. $p=4$, $q=-3$. **32.** $x=7$, $y=-3$. **33.** $x=\frac{1}{5}$, $y=\frac{1}{3}$.
34. $y=\frac{1}{2}$, $z=\frac{1}{5}$. **35.** $l=5\frac{1}{4}$, $m=-\frac{3}{4}$. **36.** $r=-\frac{7}{8}$, $s=-\frac{1}{4}$.
37. $x=2$, $y=3$. **38.** $x=5$, $y=2$. **39.** $p=9$, $q=7$.
40. $y=6$, $z=9$. **41.** $x=2\frac{4}{5}$, $y=\frac{2}{5}$. **42.** $x=7$, $y=2$.

43. 3s. 3d., 2s. 9d. **44.** $x=73$, $n=7$. **45.** 27, 63.
46. 1s. 4d., 2s. **47.** £14, £2 10s. **48.** 11, 15.
49. 2s. 6d., 5s. ; 2s. **50.** 44, 12 years old. **51.** 16s., 10s.
52. 15 children, 13s. 6d. **53.** *A*, £3 10s. ; *B*, £4.
54. 17, 26. **55.** 10d. ; 2s. 6d. **56.** 12 half-crowns, 10 florins.
57. $\frac{n-1}{3n}$ where n has any value. **58.** *A*, $3\frac{1}{2}$ pt. ; *N*, $2\frac{1}{2}$ pt.

E.P. 13.

1. $\frac{x-4}{2(x-2)}$. 2. 6. 3. $\frac{7}{6x}$. **4.** $10\frac{1}{2}$.
5. $\frac{2x-3}{15}$. 6. 9. 7. $\frac{x}{x+1}$. 8. -1.
9. $\frac{1}{3x(x+1)}$. 10. $\frac{1}{3}$. 11. $-\frac{9}{x-3}$. 12. $-\frac{2}{3}$.
13. $\frac{1}{x(x-1)}$. 14. $\frac{x}{6(x-2)}$. 15. -12. 16. $\frac{1}{2}$.
17. $\frac{2}{(x+1)(x-1)}$. 18. $\frac{x-2}{x(x-1)}$. 19. 3. 20. $2\frac{3}{5}$.

E.P. 14.

1. $(a-c)(b-d)$. 2. $(t-3)(t-4)$. 3. $(p-4)(p+1)$.
4. $(a-2b)(x+y)$. 5. $(x+7)(x-3)$. 6. $(y+4)(y+6)$.
7. $(b-2c)(a+2d)$. 8. $(3a-1)(a+2)$. 9. $(1-3x)(1-8x)$.
10. $(b-c)(b+d)$. 11. $(z-7)(z+2)$. 12. $(2b-3)(b+2)$.
13. $(y-z)(x-y)$. 14. $(1+3t)(1-5t)$. 15. $(3a+2)(4a-3)$.
16. $(1+a)(1+a^2)$. 17. $(5c+1)(2c-1)$. 18. $(5l-2m)(2l-3m)$.
19. $(x-y)(x-z)$. 20. $3(t+5)(t-2)$. 21. $(3+4p)(2-3p)$.
22. $(1-x)(1-y)$. 23. $(a-4)(a-12)$. 24. $(4b-5)(5b-6)$.
25. $(a-1)(a+1)$. 26. $4(t+6)(t-3)$. 27. $(3+4p)(5-2p)$.
28. $(1-p)(1+q)$. 29. $(4y-1)(3y-1)$. 30. $(2x-3)(6x+1)$.
31. $(2a-3b)(2x-3y)$. 32. $(3x-2y)(x+y)$. 33. $(3x-y)(2x-a)$.
34. $(6r+s)(2r+5s)$. 35. $(x+a)(x+b)$. 36. $(4a+3b)(7a-6b)$.
37. $(3x-5y)(x^2+2y^2)$. 38. $(7c+2d)(3c+4d)$. 39. $(x-p)(x+q)$.
40. $(4y+9z)(5y-6z)$. 41. $(1+xy)(1+x^2)$. 42. $(2a^2-bc)(a^2+2bc)$.

43. $(a-2)(x-2)$.
44. $(2x^3+5)(x^2+3)$.
45. $(c+d)(2a+b+1)$.
46. $(3+4t^2)(1-3t^2)$.
47. $(x-y)(1+x-y)$.
48. $(12a-7b)(3a+4b)$.
49. $(c-d)(2c-d)$.
50. $(15+4p)(7-3p)$.
51. $(x+1)(4-x)$.
52. $(x-2y)(x-3y)$.
53. $(4x-9y)(3x+8y)$.
54. $-(x-y)(x+y)$.
55. $(a-x)(y-a)$.
56. $2(x+3)(x-2)$.
57. $(ab-c)(b-ac)$.
58. $(xy+2z)(xy-9z)$.
59. $(2a-3b)(7a+4b)$.
60. $(3x-2y)(2x-5z)$.

E.P. 15.

1. $\frac{x-y}{x+y}$.
2. $\frac{2}{5}$.
3. $\frac{a}{x}$.
4. $\frac{c}{3}$.
5. $\frac{p-2}{p+2}$.
6. $\frac{x}{x+y}$.
7. $\frac{a+b}{c}$.
8. $\frac{2}{x-y}$.
9. $-\frac{2}{3}$.
10. $\frac{a+b}{a-b}$.
11. $\frac{3(x-y)}{2y}$.
12. $\frac{3}{x-1}$.
13. $\frac{b}{c}$.
14. $\frac{1}{4}$.
15. $\frac{2}{3xy}$.
16. $\frac{1}{x+2}$.
17. $\frac{a-2}{2}$.
18. $\frac{b}{a+2b}$.
19. $\frac{x+4}{x-2}$.
20. $\frac{x+2y}{x-y}$.
21. $\frac{x-3}{x+7}$.
22. $\frac{y-1}{y+4}$.
23. $\frac{x}{4(x-1)}$.
24. $\frac{x+2}{x+1}$.
25. $\frac{x(x-1)}{x+1}$.
26. $\frac{a-1}{a-2}$.
27. $\frac{y}{z}$.
28. $2x$.
29. $\frac{x^2}{6y}$.
30. $\frac{5(3a-b)}{2(a-3b)}$.
31. $\frac{x(a-1)}{a^2}$.
32. $\frac{x^2}{y^2}$.
33. $\frac{x-y}{x}$.
34. $\frac{b(b+c)}{c}$.
35. $\frac{b+c}{(b+2c)(b-c)}$.
36. $\frac{a-2b}{a}$
37. $\frac{x^2+4}{x^2-4}$.
38. $\frac{x}{6(x-3)}$.
39. $\frac{1}{2(a-b)}$.
40. $\frac{a^2+b^2}{a^2-b^2}$.
41. $\frac{4x}{x^2-9}$.
42. 1.
43. $\frac{2x+3}{(x+1)(x+2)}$.
44. $\frac{b^2}{a^2-b^2}$.
45. $\frac{2}{(x^2-4)(x+4)}$.
46. 0.
47. $\frac{3x}{x^2-4}$.
48. $\frac{a^2+2b^2}{ab(a+b)}$.
49. $-\frac{2xy}{x^2-y^2}$.
50. $\frac{x^2+2}{(x-1)(x-3)(x+2)}$.
51. $\frac{1}{a+b}$.
52. $-\frac{c}{b+c}$.

E.P. 16.

1. 3, no, 3.　**2.** -2, -2, any value.　**3.** Not zero.

4. $x=3$ if $y \neq 5$, x has any value if $y=5$.　**5.** $x^2-9x+18=0$.

6. $y^2-7y+10=0$.　**7.** $t^2=4$.　**8.** $z^2=4z$.

9. $(x-1)(y-1)=0$.　**10.** $x^2-x=6$.　**11.** $p^2+11p+28=0$.

12. $(y-3)(z+2)=0$.　**13.** $2x^2-7x+3=0$.　**14.** $21t^2-26t+8=0$.

15. $4y^2+3y=0$.　**16.** $6z^2+23z+20=0$.　**17.** $x^2=25$.

18. $9s^2=4$.　**19.** $y^3-7y^2+10y=0$.　**20.** $z^3+2z^2-z=2$.

21. $8t^3+2t^2-23t+10=0$.　**22.** $16v^3=9v$.　**23.** $10x^2-21x=10$.

24. $6y^3+5y^2-8y=3$.　**25.** 0 or 3.　**26.** 2 or -3.

27. 4.　**28.** 0 or -2.　**29.** $1\frac{1}{2}$ or 5.　**30.** -3 or $-\frac{1}{3}$.

31. 0 or $\frac{2}{3}$.　**32.** $-1\frac{1}{2}$.　**33.** 0 or -1 or 2.　**34.** $3\frac{1}{2}$ or $-\frac{2}{7}$.

35. ± 4.　**36.** $1\frac{1}{2}$ or -5.　**37.** $\frac{1}{4}$ or -4.　**38.** 1 or 2 or -3.

39. 3 or 5.　**40.** 0 or 8.　**41.** -4 or -7.　**42.** 0 or -11.

43. 1 or -6.　**44.** 10 or -7.　**45.** 3.　**46.** -6.

47. 3 or -12.　**48.** 15 or -4.　**49.** 9 or -8.　**50.** 9 or -11.

51. 5 or $\frac{1}{2}$.　**52.** $\frac{2}{3}$ or $-\frac{1}{2}$.　**53.** ± 5.　**54.** $\pm 1\frac{1}{2}$.

55. $\frac{2}{3}$ or $-\frac{4}{5}$.　**56.** $1\frac{1}{2}$ or $-\frac{2}{7}$.　**57.** 1 or -4.　**58.** $\frac{3}{4}$ or $-\frac{2}{3}$.

59. 0 or $1\frac{3}{4}$.　**60.** $-2\frac{1}{2}$ or $-\frac{4}{5}$.　**61.** 5 or -3.　**62.** 1 or $-\frac{5}{6}$.

63. 1 or 3.　**64.** 3 or -2.　**65.** 9 or -2.　**66.** 5.

67. 3 or $\frac{1}{6}$.　**68.** -1 or -7.　**69.** 4 or -5.　**70.** 4 or $-\frac{1}{3}$.

E.P. 17

1. 4·6 or $-0{\cdot}6$.　**2.** 1·1 or $-7{\cdot}1$.　**3.** 6·1 or $-1{\cdot}1$.

4. 1·2 or $-4{\cdot}2$.　**5.** $-2{\cdot}6$ or $-5{\cdot}4$.　**6.** 8·6 or 1·4.

7. 9 or -2.　**8.** 1·1 or $-6{\cdot}1$.　**9.** 1·3 or $-2{\cdot}3$.

10. 7·1 or $-1{\cdot}1$.　**11.** 0·9 or $-2{\cdot}4$.　**12.** 1 or $-\frac{1}{3}$.

13. 1·2 or $-0{\cdot}5$.　**14.** 1·3 or 0·3.　**15.** 1·3 or $-0{\cdot}6$.

16. 2·6 or 0·4.　**17.** 0·4 or $-5{\cdot}4$.　**18.** 3·4 or 0·1.

19. 2·9 or $-0{\cdot}4$.　**20.** $\frac{2}{3}$ or -1.　**21.** 5 or -2.

22. 8 or $-0{\cdot}4$.　**23.** 2·2 or $-4{\cdot}2$.　**24.** 0·5 or $-2{\cdot}5$.

25. 4 or $-1\frac{1}{2}$.　**26.** 3·6 or $-0{\cdot}1$.　**27.** 2·6 or 1·4.

28. 0 or -11.　**29.** $-4{\cdot}6$ or $-0{\cdot}4$.　**30.** 5 or $\frac{1}{3}$.

E.P. 18.

1. (i) $-1{\cdot}2$, $-1{\cdot}2$, $6{\cdot}7$, $8{\cdot}0$, $3{\cdot}0$ feet; minus values denote above B; (ii) $2\frac{1}{4}$ ft. above B when $t=1\frac{1}{2}$; (iii) $t=-1$ or 4, $0{\cdot}4$ or $2{\cdot}6$, $-1{\cdot}7$ or $4{\cdot}7$, $0{\cdot}8$ or $2{\cdot}2$; (iv) $t=3$; (v) $-1{\cdot}4$ or $4{\cdot}4$, $0{\cdot}4$ or $2{\cdot}6$, $-0{\cdot}8$ or $3{\cdot}8$, $0{\cdot}6$ or $2{\cdot}4$.

2. $0{\cdot}9$ or $3{\cdot}1$; $0{\cdot}2$ or $3{\cdot}8$; $-0{\cdot}6$ or $4{\cdot}6$; $-1{\cdot}5$ or $5{\cdot}5$; $0{\cdot}5$ or $3{\cdot}5$; no values; 20.

3. $0{\cdot}2$ or $-4{\cdot}2$; $-3{\cdot}4$ or $-0{\cdot}6$; -2; no values.

4. $2{\cdot}2$ or $-0{\cdot}7$; $0{\cdot}5$ or 1; no values; $1{\cdot}3$ or $0{\cdot}2$.

5. $-0{\cdot}3$ or 2; $1{\cdot}4$ or $0{\cdot}2$; $0{\cdot}7$ or 1; no values; $1{\cdot}6$ or $0{\cdot}1$; $1{\cdot}9$ or $-0{\cdot}2$.

E.P. 19.

1. $x=3$ or $-2{\cdot}6$, $y=-5$ or $6{\cdot}2$. 2. $x=2$ or -1, $y=\frac{1}{2}$ or -1.
3. $x=5$ or 3, $y=-3$ or -5. 4. $x=1$ or $\frac{1}{3}$, $y=1$ or $\frac{5}{3}$.
5. $x=0{\cdot}5$ or $0{\cdot}06$, $y=1{\cdot}5$ or $-1{\cdot}58$.
6. $x=-3$ or $-0{\cdot}6$, $y=-3$ or $1{\cdot}8$.
7. $x=1$, $y=1$. 8. $x=y=1$ or 0. 9. $x=2$, $y=-1$.
10. $x=\frac{2}{3}$ or $\frac{1}{7}$, $y=\frac{1}{4}$ or $-\frac{1}{7}$. 11. $x=3$ or $\frac{3}{8}$, $y=4$ or $-3\frac{7}{8}$.
12. $x=3$ or $-\frac{1}{3}$, $y=2$ or $\frac{1}{2}$. 13. $x=4$ or $-\frac{14}{11}$, $y=-1$ or $\frac{76}{11}$.
14. $x=3$ or -2, $y=-1$ or $6\frac{1}{2}$. 15. $x=2$ or $-\frac{4}{13}$, $y=1$ or $-\frac{7}{13}$.
16. $x=-2$ or $\frac{22}{41}$, $y=3$ or $-\frac{7}{41}$. 17. $x=\frac{2}{3}$ or $\frac{1}{3}$, $y=0$ or $-\frac{1}{2}$.
18. $x=2$ or -2, $y=3$ or 6. 19. $x=4$ or 7, $y=-3$ or -5.
20. $x=1$ or $2\frac{1}{3}$, $y=3$ or $5\frac{2}{9}$. 21. $x=3$ or 20, $y=4$ or -30.
22. $x=\pm 3$, $y=\mp 2$. 23. $x=\pm 9$, $y=\pm 1$. 24. $x=\pm 4$, $y=\pm 5$.
25. $x=0$ or ± 3, $y=\pm 2$ or ± 1. 26. $x=\pm\frac{3}{4}$, $y=\pm\frac{2}{3}$.
27. $x=2\pm\sqrt{2}$, $y=2\mp\sqrt{2}$.
28. $x=\pm 3$ or $\pm\sqrt{2}$, $y=\pm 2$ or $\mp 4\sqrt{2}$.
29. $x=\pm 3$ or $\pm\frac{1}{3}\sqrt{3}$, $y=\mp 1$ or $\pm\frac{4}{3}\sqrt{3}$. 30. $x=\pm 1$, $y=\pm 2$.

E.P. 20.

1. $1+a$ or $1-a$. 2. $\dfrac{a}{b}$ or $\dfrac{b}{a}$. 3. $-b$ or $-\dfrac{1}{a}$.
4. a or $\dfrac{c^2}{a}$. 5. $p+q$ or $p-q$. 6. a or a^3.
7. $\dfrac{b^2}{a}$ or $-\dfrac{1}{a}$. 8. -1 or $-\dfrac{b+c}{a+b}$. 9. a or b.

10. a^2 or $2ab - a^2 - 2c$. 11. $a + b$ or $2b - a$. 12. 1 or $\frac{c(a-b)}{a(b-c)}$.

13. $\frac{ac}{b}$ or $\frac{bc}{a}$. 14. 0 or $a + b$ or $\frac{a^2+b^2}{a+b}$. 15. $x = y = 0$ or $a + b$.

16. $x = a$ or $-1 - \frac{1}{a}$, $y = a$ or $-\frac{1}{a+1}$. 17. $x = a + b$, $y = a - b$.

18. $x = b$ or $a - b$, $y = \frac{a^2}{b}$ or $2a$. 19. $x = a$ or $\frac{a^2 - b^2}{2a}$, $y = b$ or $\frac{b^2 - a^2}{2b}$.

20. $x = a$ or $\frac{a^3 - 3ab^2}{a^2 + b^2}$ $y = -b$ or $\frac{3a^2b - b^3}{a^2 + b^2}$.

E.P. 21.

1. $2x^3y^3 - 20xy^5 + 6y^6$. 2. 2. 3. $p^3 + 4$.

5. $\frac{1}{(q-r)(r-p)}$. 6. $(2x+3)(x+1)$, $-(x-y)^2$. 7. 1.

8. $\frac{(x-b)^2}{(x-a)^2}$. 9. 4. 10. $x = 3$, $y = 12$. 11. 0·6.

12. $(2-3x)(1+4x)$, $(a+p)(a-q)$. 13. $\frac{1}{3x-2y}$. 14. $\frac{a-c}{2}$.

15. 8s., 11 half-crowns. 16. $\frac{10}{15}$. 17. $a = \frac{47}{15}$, $b = -\frac{11}{15}$.

18. $\frac{4}{3}$. 20. 2·14 or $-0·94$. 21. a.

23. $(1-3x)(1-6x)$. 24. $2\frac{1}{3}$ or $-\frac{2}{3}$. 25. $a = 3$, $b = -2$.

26. $\frac{a+b}{b}$. 27. c or $2a - c$. 28. $(x-y)(5x-4y)$.

29. A 2s., B 2s. 6d. 30. $\frac{8x^2 - 4x^3}{(1-x)^2}$. 31. $\frac{2fv}{f+v}$.

32. $\frac{x-y}{2x+y}$. 33. $R = \frac{t}{2} + \frac{V}{2\pi lt}$. 34. 1 or 2.

35. $\frac{x}{(1-x)^3}$. 36. $x = 2$ or $\frac{1}{5}$, $y = -3$ or 15.

ANSWERS.

PART II.

Note.—Where only one kind of unit occurs in the question, the unit is not specified in the Answer.

EXERCISE XII. a. (p. 253.)

2. $a^5, a^8, a^6, a^8, a^{16}, a^{12}, a^{11}, x^{15}$. **3.** $a^m \times a^n = a^{m+n}$.
4. $a^m \div a^n = a^{m-n}$. **6.** $a^{20}, a^{100}, a^{45}, 27a^{27}$.
7. $(a^m)^n = a^{mn}$. **8.** x^{14}. **9.** 3. **10.** x^{40}.
11. x^8. **12.** Yes. **13.** Yes. **14.** x^{20}.
15. x^3. **16.** 1. **17.** a^{27}. **18.** a^{34}. **19.** $64x^{13}$. **20.** 2.
21. 30,020,000,000 cm. per sec. ; 4,500,000 ; £12,200,000.
26,000,000,000,000 miles ; 2000 × 4,500,000 = 9,000,000,000.
22. $6{\cdot}28 \times 10^7, (6{\cdot}71)^3 \times 10^9$.

XII. b. (p. 256.)

1. Indices 2, 1, 0, − 1, − 2 ; values 100, 10, 1, 0·1, 0·01.
2. Indices 3, 2, 1, 0, − 1, − 2 ; values 8, 4, 2, 1, ½, ¼.
3. Indices 5, 4, 3, 2, 1, 0, − 1, − 2, − 3 ; values 16807, 2401, 343, 49, 7, 1, $\frac{1}{7}, \frac{1}{49}, \frac{1}{343}$.
4. 7, − 5, 17, 0. **5.** 11, 3, − 2, 10, 0. **6.** x^4, x^{-3}, x^0.
7. x^{-5}, x^{-5}, x^0, x^0. **8.** $\frac{1}{x^{12}}$. **9.** $a^7, a^7, a^5, a^8, 0, 1, a$.
10. $\frac{1}{a^2}, \frac{1}{a}, a^3, \frac{1}{a^2}$. **11.** $\frac{1}{9}, \frac{1}{8}, \frac{1}{4}$, 1, 16, $\frac{81}{16}$, 100, 1, 1, $\frac{3}{2}$.
12. $\frac{5}{8}, \frac{1}{3}, \frac{2}{3}$, − 1. **13.** 8, 7, 7½. **14.** 12, 14, 13.
15. ½, 0. **16.** $x^4, x^{\frac{7}{2}}, x^{\frac{1}{3}}$. **17.** $x^2, x, \sqrt[3]{x^2}, \sqrt[3]{x}$.
18. $\sqrt[4]{x}, \sqrt[4]{x^3}$. **19.** 15, 6, 1. **20.** 9, $\frac{2}{5}, \frac{3}{4}$.
21. 4, 27, 2, 9. **22.** 256, 32, 27, 128, 64. **23.** 7, 5, $\frac{7}{12}, \frac{1}{8}$.
24. ½, $\frac{1}{8}, \frac{1}{8}$, 0·001, 1. **25.** $x^{\frac{3}{2}}, x^{1\frac{1}{4}}, x^{\frac{1}{2}}$. **26.** $6x^{\frac{5}{6}}, x^{\frac{1}{2}}$.

28. $1/\sqrt[4]{x^3}$, $1/\sqrt{x^5}$. **29.** 1·414, 1·189, 1·090, 1·044, 1·022, $2^0=1$.

30. 3·162, 1·778, 1·334, 1·154, $10^0=1$.

31. Number of zeros after decimal point is 9, 12, 23, 7, 9, 7, 4.

32. $\frac{l}{t^2}$, $\frac{m}{t^3}$, $\frac{ml^3}{t^2}$; $dt^3=k$, $\mu R^2=a^3$, $Nd^3=Hk^3$; $\frac{lt^2}{LT^2}$, $\sqrt{\frac{l_2}{l_1}}$, $p\sqrt[4]{v}$
$\sqrt{a}+\sqrt{b}=\sqrt{c}$, $\sqrt[3]{x^2}+\sqrt[3]{y^2}=\sqrt[3]{a^2}$, $gt^2=4\pi^2 l$; $ye^4=1$.

33. 1·005, 1·01, 0·99, 0·99, 0·995, 1·01, 0·998, 0·098, 10·01, $10^{-10}\times 9{\cdot}91$.

34. $3x^2$, $\frac{3}{2}\sqrt{x}$, $-\frac{2}{x^3}$, $\frac{1}{2\sqrt{x}}$, $-\frac{5}{x^6}$, $-\frac{1}{2}x^{-\frac{3}{2}}$, 0, $-\frac{5}{3}x^{-\frac{8}{3}}$

35. $\frac{1}{3}x^3$, $\frac{2}{3}x^{\frac{3}{2}}$, $-\frac{1}{x}$, $\frac{3}{4}x^{\frac{4}{3}}$, $-\frac{1}{2x^2}$, $\frac{2}{5}x^{\frac{5}{2}}$, $\frac{1}{2}x^2$, x, $\frac{3}{2}x^{\frac{2}{3}}$.

XII. c. (p. 260.)

1. $x^{\frac{1}{3}}$, $a^{\frac{2}{3}}$, $a^{\frac{3}{5}}$, $a^{\frac{1}{2}}b^{\frac{1}{3}}$, $2a^{\frac{1}{2}}b$, $3a^{\frac{1}{3}}b^{\frac{3}{2}}$, $a^{\frac{1}{3}}b$. **2.** $3x^{-2}$, x^{-1}, $x^{-1}y^{-1}$, $5x^{-1}y^{-\frac{1}{2}}$.

3. 4, 2. **4.** 3, $\frac{1}{3}$, -2, $-\frac{3}{2}$, $-\frac{1}{2}$.

5. $2a^{\frac{3}{2}}$, $a^{\frac{5}{12}}$, $a^{\frac{3}{2}}b^{\frac{3}{2}}$. **6.** $a^{\frac{1}{2}}$, $2a^{\frac{3}{2}}$, $a^{\frac{2}{3}}$. **7.** $a^{\frac{3}{2}}$, 1. **8.** x^2+1.

9. $x^{\frac{1}{6}}$. **10.** $\frac{x+1}{x}$. **11.** 16. **12.** x^3.

13. 0. **14.** 10^{3n}, $\frac{3y^3}{10x^2}$, x^{-4}, x^2, $(x+y)^2$.

15. 7, 13, -1, 2^n+2^{5n}. **16.** $\frac{1}{2}$.

XII. d. (p. 262.)

1. 7, 5, 12, 12, 2, 2. **2.** 12, 24, 0·3.

3. $3\sqrt{2}$, $6\sqrt{2}$, $5\sqrt{2}$, $20\sqrt{5}$, $b\sqrt{a}$, $\frac{1}{x}\sqrt{a}$. **4.** 17·3, 5·20, 0·866.

5. 0·577, 1·44. **6.** $9\sqrt{2}$, $\frac{5}{2}\sqrt{2}$, $10\sqrt{10}$, $\sqrt{3}$, $b\sqrt{b}$

7. 7·07, 22·4, 70·7, 0·894. **8.** 1, 2·41.

9. 38, $\frac{3}{19}(5\sqrt{2}-2\sqrt{3})$. **10.** 0·268, 3·15, 1·24.

11. 4·31. **12.** 1. **13.** $2\sqrt{2}$, $\frac{1}{2}\sqrt{3}$.

14. 2·90. **15.** 24. **16.** 1·465, 2·07. **17.** 3·72.

21. $a+b+2\sqrt{(ab)}$; $a+b$; x or y must be zero. **22.** xy, xy, yes.

23. $\frac{2\sqrt{x}}{x-y}$, $\frac{x+y}{\sqrt{(xy)}}$.

XII. e. (p. 265.)

1. 1·15, 0·707, 4·59. **2.** 2·32. **3.** -7, -10.

5. 2, $\frac{1}{4}$, 1, 10, 0·01, 2.

9. (i) See p. 267. (iii) 1·58, 2·51, 5·01, 7·94. (iv) 0·48, 0·70, 0·91. (v) 1·58, 4·64, 3·98.

10. 0·903, 1·43, 2·301, 1·699.

12. Indices are 0·4771, $-1 + 0{\cdot}9542$, 0·4313.

XIII. a. (p. 268.)

Note.—Answers are given correct to three figures but, if according to four-figure tables the fourth figure is 5, this is shown.

1. 0·845. **2.** 0·544. **3.** Indices are 1·301, 2·301, 3·301, 6·301.
4. 2, 6. **5.** 1·6377, 2·6377, 4·6377, 6·6377.
6. 72·86, 728·6, 72860, 7,286,000. **7.** 1, 2, 4, 5.
8. 1, 3, 0, 5, 5, 3, 4, 5, 7, 8. **9.** Less, greater, less, greater.
10. Greater, less, less. **11.** 224, 51300, 14100000.
12. (i) Indices are 1·46, 3·785, 6·92. (ii) 1·52, 7·73, 3·87.

XIII. b. (p. 272.)

1. 35·0. **2.** 2·78. **3.** 76·2. **4.** 41900. **5.** 5·88.
6. 2040. **7.** 2170. **8.** 341000. **9.** 11·1. **10.** 9·09.
11. 128. **12.** 2·01. **13.** 3·02. **14.** 20·0. **15.** 13,200.
16. 845500. **17.** 24·7. **18.** 17·9. **19.** 5·73. **20.** 7·39.
21. 342. **22.** 1·51. **23.** 416. **24.** 1·24. **25.** 38·0.
26. 1·96. **27.** 18·7. **28.** 91·3. **29.** 5·91. **30.** 35·6.
31. 2·06. **32.** 6·52. **33.** 1·15. **34.** 2·48. **35.** 5·34.
36. 6·44. **37.** 2·08, 4·48, 9·65, 20·8. **38.** 17·6. **39.** 4·28.
40. 1·04. **41.** 50100. **42.** 3·23. **43.** 1·06. **44.** 1·195.
45. 6·08.

XIII. c. (p. 273.)

1. 60·0 sq. cm., 6·91 cm. **2.** 5·32 cu. cm., 1·06 cm.
3. 42·8 ft., 2·27 sec. **4.** 7·83 sec., 3·26 ft.
5. 168. **6.** 11·7, 4290. **7.** 3·90. **8.** 4·34.
9. 1300. **10.** 12·5. **11.** 2·64. **12.** 4·52 tons.
13. 12·2. **14.** 430. **15.** 21·5. **16.** 1·58.
17. 11550. **18.** $1{\cdot}1 \times 10^{9}$, 5×10^{47}.

XIII. d. (p. 276.)

1. 1·8. **2.** 2·1. **3.** 0·9. **4.** $\bar{5}\cdot6$. **5** $\bar{5}\cdot2$.

6. $\bar{1}\cdot3$. **7.** 0·1. **8.** 3·1. **9.** $\bar{5}\cdot3$. **10.** 3·8.

11. $\bar{5}\cdot7$. **12.** $\bar{1}\cdot8$. **13.** $\bar{1}\cdot3$. **14.** $\bar{2}\cdot7$. **15.** 0·8.

16. $\bar{2}\cdot8$. **17.** $\bar{1}\cdot2$. **18.** $\bar{4}\cdot1$. **19.** $\overline{13}\cdot82$. **20.** $\bar{5}\cdot2$.

21. 3·9. **22.** $\bar{1}\cdot3$. **23.** $\bar{1}\cdot8$. **24.** $\bar{3}\cdot85$. **25.** $\bar{1}\cdot3$.

26. $\bar{2}\cdot8$. **27.** $\bar{1}\cdot76$. **28.** $\bar{1}\cdot9$. **29.** $\bar{2}\cdot9$. **30.** 1·45.

XIII. e. (p. 276.)

1. 3·25. **2.** 0·246. **3.** 0·004205. **4.** 0·365. **5.** 0·533.

6. 0·04935. **7.** 12·2. **8.** 4·43. **9.** 0·0243. **10.** $10^{-6}\times2\cdot57$

11. 0·885. **12.** 0·280. **13.** 0·428, 0·199, 0·09215. **14.** 0·126.

15. 0·0267. **16.** 0·120. **17.** 0·00508. **18.** 21·7. **19.** 0·103.

20. 1·035. **21.** 0·6585. **22.** 0·598. **23.** 7·16, 0 452.

24. 0·734. **25.** 4·86. **26.** 0·667. **27.** 0·807.

28. $10^{-4}\times9\cdot685$. **29.** 0·794. **30.** $10^{-14}\times1\cdot05$, $10^{-5}\times2\cdot6$.

XIII. f. (p. 277.)

1. $10^{5}\times1\cdot60$. **2.** 0·779. **3.** 0·0994. **4.** 11·5. **5.** 335.

6. 338. **7.** 112 sq. in. **8.** 631. **9.** 15·6. **10.** 0·694, 0·492

11. 1·44. **12.** 3·27. **13.** 15·6. **14.** 0 0931. **15.** 951 tons.

16. $10^{-5}\times4\cdot81$. **17.** 33·8, $10^{-10}\times9\cdot93$. **18.** 2330. **19.** 0·708.

20. 3·63. **21.** 0·516. **22.** 2·69, 2·72.

23. £2080; 20s. 9d.; £$10^{12}\times8$. **24.** £294. **25.** 0·909.

26. £2940; annuity is worth £5830; £368.

XIII. g. (p. 279.)

1. 7·39. **2.** 7·71. **3.** 82·5. **4.** 82·5. **5.** 15·1.

6. 21·2. **7.** 37·1. **8.** 0·0111. **9.** 3·73. **10.** 37·9.

11. 1·54. **12.** 2·605. **13.** 0·236. **14.** 9·90. **15.** 4·27.

16. 1·69. **17.** 2·23. **18.** 2·51.

XIII. h. (p. 281.)

1. 2 log 2, 3, 0, $\frac{1}{2}$. **2.** 2, 2, 2. **3.** $\log(1+3)$, $\log 2+\log 4$.

4. $\log 2-\log 1$, equal, $\log(6-3)$. **5.** $\sqrt{2}$.

6. 2, $\frac{1}{2}$, 2 log y. 7. 0, log x. 8. $\frac{3}{4}$.

9. $xy=5$, $xy^3=100$, $3^x=1000$, $x^3=10y^2$, $5^x=6^y$, $5^x=100\times 6^y$.

10. 13, 2, -1. 11. 1·06. 12. 1·87. 13. 1·585.

14. 4·8, 0·053. 15. 2, 3, 2. 16. 0·312. 17. 23·5, 14·2.

18. 4·1. 19. 4780 ft. 20. 1·13.

XIV. a. (p. 285.)

1. 8 : 3, 2 : 3, 1 : b^2, 1 : x, $5x : 4y$, 1 : 4, 1000 : 1, $(a+b) : a$, $3a : b$, $5x : 2y$.

2. 9 : 2. 3. -4.

5. $\frac{a}{c}=\frac{d}{b}$, $\frac{p}{q}=\frac{r}{p}$, $\frac{a+b}{p+q}=\frac{e+f}{c+d}$, $\frac{x}{p}\cdot\frac{y}{q}\cdot\frac{z}{r}=1$, $\frac{a}{d}\cdot\frac{b}{e}\cdot\frac{c}{1}=1$.

6. $\frac{bx}{a}$. 7. $\frac{10}{11}$. 8. $\frac{1}{3}$.

9. $\frac{x+y}{x}$. 10. $\frac{1}{3}$. 11 $\frac{4a}{5b}$.

12. £$\frac{ap}{a+b}$, $\frac{bp}{a+b}$. 13. $aq : bp$. 14. 28, 20 ; $\frac{pc}{p-q}$, $\frac{qc}{p-q}$.

15. $\frac{5x}{3}$ 16. 8 : 3 17. 6 : 1.

18. $\frac{bx}{a}$, $\frac{100(b-a)}{a}$. 19. Greater. 20. Less.

21. $\frac{1}{2}$. 22. 8 : 6 : 9. 23. $\frac{l(a-b)}{2(a+b)}$ ft.

24. $\frac{px}{qy}$, $\frac{px}{q(x+y)}$. 25. $\frac{6}{5}$ or $-\frac{2}{11}$. 26. 3 : 2.

27. 5 : 4. 28. $\frac{ab}{a+b}$. 29. $\frac{61}{56}$.

30. 1. 31. 3 : 2. 32. 1 : 2.

33. 27 : 8, 75 : 4. 34. Increased in ratio 3 : 2.

35. $\frac{a+x}{b+x}<\frac{a}{b}<\frac{a-x}{b-x}$. 36. $\frac{b-a}{a}$.

37. $\frac{(a+b)(qc-pd)}{qa-pb}$. 38. 2000. 39. $\frac{x}{y}$. 40. 8 : 5.

XIV. b. (p. 288.)

1. 6, $\frac{c^2d}{a}$, x. 2. $4\frac{1}{2}$, $\frac{bc^2}{a}$, xy^2. 3. 6, a^2b^3, $\sqrt{(ab)}$.

4. 11, 5, 4. 5. 1, 14 : 21 : 6. 6. 29 : 9 : 6.

7. 5 : 7 : 4. 10. $b+d+f$, $a-c+e$, $10b-7d+2f$, $\frac{b}{2}+\frac{f}{3}$.

12. $\frac{11}{24}$. **13.** 5 or $-\frac{1}{3}$. **14.** -2, $\frac{1}{3}$. **15.** $\frac{3}{7}$ or 0.

20. 5, 20 (19), 8, 20 ; ($z=x^2y+2x+y$). **21.** 0·104, 0·170.

XIV. c. (p. 293.)

1. Yes, doubles y, doubles x, $y=88x$.

2. No, halves y, doubles x, $y=\frac{480}{x}$.

3. Trebles y, halves x, line through origin.

4. Halves y, divides x by 10, curve, area is constant.

5. $y \propto \frac{1}{x}$; $x=6, 3, 2, 1{\cdot}5, 1{\cdot}2$; $\frac{1}{x}=\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}$.

6. Halves x, divides y by 16, curve.

8. $C \propto x^2$.

9. $y \propto \frac{1}{x^2}$.

10. 3 : 1, 10 : 3, 18 : 5, no.

11. $y=4x$, $y=2x^2$, $xy=10$, $y=\sqrt{x}$.

12. $A \propto h$; 9, 12, 15, 24, 30

13. 0, 3, 12, 27, 75, 300.

14. 8, 2, 0·8, 0·4.

15. 2·6, 20·8, 166·4.

16. 0, 2, 2·83, 4, 10, 20.

17. $A \propto r^2$, $A \propto x^2$, $t \propto \frac{1}{x}$, $y \infty x$, $h \propto \frac{1}{d^2}$, $y \propto x$.

18. $y=\frac{3}{4}x^2$.

19. 7·5.

20. Quadruples y.

21. Increases y in ratio 3 : 2.

22. Divides y by 4.

23. $x^2=5y$.

24. $xy=8$.

26. Increases in ratio 512 : 1, $V^2 \propto S^3$.

27. Yes, $P=0{\cdot}00492v^2$

28. 7·78 mi., $N=1{\cdot}23\sqrt{h}$.

29. 6¾ gallons.

30. £4. 13s. 9d.

31. 1·41 : 1, 1·26 : 1.

32. 1s. 9d. **33.** 1·755x lbs.

34. 1·59 : 1.

35. 1 sec., $t=0{\cdot}2\sqrt{l}$. **36.** 20·6 ft.

37. Multiplies sag by 27.

38. 95·2, 64, 8·16 ; 98·75, 97·5, 75 lbs.

39. $C=80{\cdot}2d^{\frac{3}{2}}$, 147 amp., 13·4 amp.

40. 0·414 units, 66·2 ft.

XIV. d. (p. 301.)

1. $V \propto h$, $V \propto r^2$, $V \propto r^2h$, $V=\frac{1}{3}\pi r^2h$.

2. $W \propto b$, $W \propto d^2$, $W \propto \frac{1}{l}$, $W \propto \frac{bd^2}{l}$, $W=\frac{3bd^2}{16l}$.

3. $T \propto \frac{1}{l}$, $T \propto v^2$, $T \propto \frac{v^2}{l}$, $T=\frac{3v^2}{32l}$, 28·3 ft. per sec., $v \propto \sqrt{l}$

4. $T \propto ln^2W$, $T = \frac{\pi^2 ln^2 W}{8}$, $n \propto \frac{1}{\sqrt{l}}$, $l \propto \frac{T}{n^2}$, doubles T.

5. Top row 270, 315, 360 ; second, 216, 252, 288 ; third, 180, 210, 240.

6. Top row 3, 6, 9 ; second 12, 24, 36 ; third 27, 54, 81.

7. $\frac{24}{xy^2}$.

8. $S \propto lr$, $p \propto \frac{T}{V}$, $t \propto \frac{d}{v}$, $p \propto \frac{lx^2}{d^5}$, $t \propto \sqrt{\left(\frac{l^3}{dp}\right)}$.

9. $d = 0 \cdot 8 \frac{Wv^2}{R}$.

10. $d = \lambda . \sqrt[4]{\left(\frac{i^2H}{P}\right)}$, reduced 27 per cent.

11. $H = 0 \cdot 00504 d^{\frac{2}{3}} v^3$.

12. $C \propto Tv^3$, $C = \frac{d^3}{250T^2}$.

13. $P = \frac{900W}{D} + \frac{22500W}{D^3}$, 50 lbs.

14. $x^2z = 36y^3$.

15. $A \propto \frac{y^4}{x}$.

16. $v \propto \sqrt{d}$.

17. $v \propto \sqrt{\frac{H}{W}}$, $v \propto \sqrt[3]{\frac{H}{A}}$, 15·9 lbs., 79 lbs.

18. 27·6 H.P., $H = 10^{-6} \times 2 \cdot 87 Av^3$.

19. 1·52 lbs. per sq. in.

20. Decreases in ratio 1 : 6.

XV. a. (p. 307.)

1. 1, 3 ; $x > 3$ and $x < 1$.

2. No.

3. 2, −3 ; $2 > x > -3$.

4. 9 ; 4, −2 ; $-\infty$, $-\infty$.

7. $x > 10$ or $x < -10$; $0 \cdot 1 > x > -0 \cdot 1$.

8. Small, positive ; small, negative ; large, positive ; large, negative.

10. $\sqrt{(25 - x^2)}$; circle, centre O ; $\pm\sqrt{(25 - x^2)}$.

14. Yes, 2·01, 1·999, 1 ; $2 > x > 1$; no, 1002, −998.

15. 1, 3, 4 ; $x > 4$ and $3 > x > 1$; 378, −12, −2002 ; yes, *e.g.* 104.

16. Yes, *e.g.* 2·999 and 3·001 ; no ; 1, 5 ; 0·3, 0·3 per cent.

18. 98, −102, −6, −201, 199 ; small, 0·000001 ; −0·000001 ; $3 > x > 2$ and $1 > x$.

23. $1 > x > -2$; 1, −2.

XV. b. (p. 311.)

1. $y = \frac{1}{2}(x + 4)$.

2. $y = \sqrt{(8x - x^2)}$.

3. $y = \sqrt{(8x - 16)}$.

4. $y = \sqrt{(6 - x^2)}$.

5. $z = \sqrt{(4x - x^2)}$; $y = \sqrt{\frac{x^3}{4 - x}}$.

6. $y = \frac{4}{3}\sqrt{(9 - x^2)}$.

7. $z(1 - x)(3x - 1)$.

8. $y = \frac{abx}{x^2 + b^2}$.

9. $y = \frac{d}{r}\sqrt{(d^2 - x^2)}$.

10. $x = \frac{c(by + az)}{(a + b)(b + c)}$.

11. $V = 0 \cdot 0293 A^{1 \cdot 5}$.

12. $\frac{1}{2}(x^3 \sim 1)$. 13. $\frac{\pi x^2}{192}\sqrt{(16-x^2)}$. 14. 2^{-x}, 2^{x-2}.

15. $z=\frac{bx-ay}{x-y}$. 16. $\frac{y}{\sqrt{2}}$, $\frac{y}{\sqrt{2}}$; $y=\frac{ax\sqrt{2}}{x+a}$.

17. $z=\sqrt{(xy)}$; $z=\sqrt{\frac{x}{y}}$. 18. $z=\sqrt[5]{(x^3y^2)}$; $z=2\sqrt[5]{(x^3y^2)}$.

19. $(x-2)(x-3)$. 20. $\frac{x-1}{x-2}$. 21. $(3-x)(x-1)$.

22. $-(x^2-1)^2$. 23. $\sqrt{(4-x^2)}$. 24. $(x-1)^2+2$.

XV. c. (p. 315.)

1. 2·5. 2. 8. 3. 9·2. 4. 7·5, $\frac{1}{2}x\sqrt{(10x-x^2)}$.

5. 2·94. 6. $A=\frac{10x(x-10)}{x-20}$, 5·86.

7. $A=x^2+\frac{800}{x}$, 3·68, 10·1 or 1·54. 8. $V=4x(12-x)^2$, 1024 cu. in.

9. $y=\frac{x^2+25}{2x}$, 5, 2·10 or 11·90. 10. $\frac{3}{2}x(8-x)$, 4.

1. $V=\frac{1}{2}x^2(15-4x)$, 15·6. 12. $V=2\pi r^2(6-r)$, 201.

13. -4, 1, 4, 5, 4, 1, -4; (i) 3·24 or $-1·24$; (ii) $3·24 > x > -1·24$; (iii) 5, $x=1$; (iv) 2·87 or $-0·87$; (v) 2, 2·73 or $-0·73$; (vi) 3·45 or $-1·45$.

14. 4·2, 1·8, 0·2, $-0·6$, $-0·6$, 0·2, 1·8, 4·2, 7·4; (i) 0·82 or $-1·82$; (ii) $0·82 > x > -1·82$; (iii) $-0·7$, $x=-0·5$; (iv) 1·56 or $-2·56$; (v) 4, 0·8, 1·44 or $-2·44$; (vi) 0·62 or $-1·62$.

15. $-9·6$, $-3·6$, $-1·2$, $-1·2$, $-2·4$, $-3·6$, $-3·6$, $-1·2$, 4·8; (i) 4·27; (ii) $x>4·27$; (iii) $-3·83$, $x=2·53$; (iv) $-0·974$, $x=-0·528$; (v) 4·52; (vi) (a) 3·78, 0·71, $-1·49$; (b) $-2·31$; (c) 4·62; (vii) 3, 2, -2.

16. 1, 1·62, $-0·62$; 2·20; 0·66. 17. 2·11, $-0·25$, $-1·86$.

18. No real solutions.

XV. d. (p. 320.)

1. 3, 2, 3, $4a^2+2$, b^6+2. 2. 10, 100, 1, 0·1, 10^{2x}.

3. 3, 0·301, 1·301, 0, 3 log x. 4. 3, $(x+h)^2-3(x+h)+5$, $\frac{1}{x^2}-\frac{3}{x}+5$.

5. $2x+h+3$; $2x+3$. 6. 1, $-\frac{1}{1+h}$, -1. 7. $9x^2+5$, 2.

8. x^6, $2h^3$. 11. 1, 2. 12. $\frac{1}{2}$. 13. $\frac{x-1}{x}$.

14. $(x-3)(x-4)$. 15. $8(x-3)$. 16. $\frac{3(x-3)}{x-1}$.

17. $3x^2$ or x^3+2. **18.** $\log x$. 19. 10^x. 20. x^3-x. 21. $(-1)^x$.

XVI. a. (p. 325.)

1. Yes, *e.g.* 3000 ; no ; 2. **2.** 11, no, 2.

3. 0·490, 0·4975 ; 0·5 ; no. 4. Approx. 0·09, 0·01, 0·001, 0·0001 ; 0.

5. 1, 7. 6. 1·5, 1·67, 1·91, 1·99 ; 2 ; no ; $0{\cdot}9991 < x < 1{\cdot}001$.

7. $\frac{1}{2}(h^2+3h+3)$, 1·5. 8. $16(h+1)$, 16. 9. 5·25, $5-2a$, 2·5.

10. $-\frac{1}{4}$, $-\frac{1}{a^2}$, ± 2. 11. $\frac{n}{2}$, $\frac{1}{2}$. 12. $\frac{50(n-1)}{n}$, 50, 50.

13. $\frac{1}{3}$, $\frac{(n+1)(2n+1)}{6n^2}$, $\frac{1}{3}$. 14. $\frac{1}{3}$; $\frac{(n-1)(2n-1)}{6n^2}$, $\frac{1}{3}$; $\frac{1}{3}$.

XVI. b. (p. 330.)

1. 70 ; 250 ; 1890 to 1900, 390.

2. $\frac{x}{a}$, $\frac{y}{b}$, $\frac{y-x}{b-a}$, $y-x$ miles per min.

3. 8·4, 24·9, 27·3, 29, 37·2 m.p.h.

4. 1·5, 2·6, 2·25, 3·2, 1·3, 2·4 ft. per sec.

5. 0·3 ; 1·8 ; 3 ; 2·4, 1·2 tons per in.

6. For $t=0$, 5, 10, 15, 20, 25, rate is 1·76, 0·88, 0·43, 0·21, 0·11, 0·05 cu. ft. per sec.

XVI. c. (p. 334.)

1. 0·47, −2·9, 0. 2. 0·021, −0·020, 0·021, 0·009, −0·0077.

4. 0·93, 0·38 ins. 5. 2, 2, 2. 6. −5. 7. 0·6.

8. 3 ft., 3 ft. per sec. ; 12 ft., 6 ft. per sec., 9 ft. per sec. ; 13·23 ft., 12·3 ft. per sec. ; $3(2+h)^2$ ft., $12+3h$ ft. per sec. ; 12·3, 12·03, 12·000003, 12 ft. per sec.

9. 144 ft., 48 ft. per sec. ; 64 ft., 80 ft. per sec. ; 134·56 ft., 94·4 ft. per sec. ; $16(3-h)^2$ ft., $96-16h$ ft. per sec. ; 94·4, 95·84, 95·999984, 96 ft. per sec.

10. 32 ft. per sec. 11. $32a$ ft. per sec.

12. 23 ; $3h^2+17h+23$; $3h+17$ ft. per sec. ; 17 ft. per sec.

13. 3, 2, 0, −2. 14. 0·15, 0·18 cm. per sec.

15. 4, $6-h$, $12-2a-h$, $12-2a$, 6, $y=12x-x^2$. 16. 5·1, 5, −1·5.

17. $-\frac{1}{4}$, $-\frac{1}{a^3}$. 18. $6c^2$. 19. $\frac{1}{4}$, $\frac{1}{2}$. 20. m.

21. $4a+b$. 22. $2ax_1+b$, $-\frac{b}{2a}$. 24. $2x$. 25. 0.

XVII. a. (p. 341.)

1. $5 . \delta x$, 5. **2.** $2x . \delta x + (\delta x)^2$, $2x$.

3. $10x . \delta x - 3\delta x + 5(\delta x)^2$, $10x - 3$. **4.** $(1 + x + \delta x)^2$, $2x + 2$.

5. $1 + 2x$. **6.** $2x$, $6x$, $14x$, $200x$.

7. $3x^2$, $12x^2$, $15x^2$, $87x^2$. **8.** $12x^3$, $40x^3$.

9. $35x^4$, $5x^4$. **10.** $4\frac{1}{2}x^5$, $-6x^5$, $1 - 12x^5$.

11. $\underset{\delta x \to 0}{Lt} \dfrac{(x + \delta x)^n - x^n}{\delta x} = nx^{n-1}$. **12.** $3x^2 + 4x$, $6x + 3x^2$, $2x^2 + x$.

13. $3u + 5v$. **14.** 3, 5, $30x$, 8, no, yes.

15. Yes, yes, no, no. **16.** $\dfrac{d}{dx} x^n = nx^{n-1}$.

17. $\dfrac{d}{dx}(cx^n) = c\dfrac{d}{dx}(x^n)$. **18.** $\dfrac{d}{dx}(Ax^m + Bx^n) = \dfrac{d}{dx}(Ax^m) + \dfrac{d}{dx}(Bx^n)$.

20. The differentials of x^8, x^{50}, x^{-1}, x^{-3}, x, x^0, $x^{\frac{3}{2}}$, $x^{\frac{1}{2}}$, $x^{-\frac{1}{2}}$, x^{-m} are $8x^7$, $50x^{49}$, $-x^{-2}$, $-3x^{-4}$, 1, 0, $\frac{3}{2}x^{\frac{1}{2}}$, $\frac{1}{2}x^{-\frac{1}{2}}$, $-\frac{1}{2}x^{-\frac{3}{2}}$, $-mx^{-m-1}$.

21. $0{\cdot}41$. **22.** $-\frac{1}{30}$. **23.** $0{\cdot}331$, $0{\cdot}030301$. **24.** $0{\cdot}008$. **25.** -20

XVII. b. (p. 345.)

1. $7x^6$. **2.** $30x^2$. **3.** $1 - \dfrac{1}{x^2}$. **4.** $-\dfrac{5}{x^6}$.

5. $-\dfrac{3}{x^2}$. **6.** $-\dfrac{4}{x^3}$. **7.** $6x - 2$. **8.** x^2.

9. $5x^4$. **10.** $375x^2$. **11.** $2x - 2$. **12.** $-\dfrac{n}{x^{n+1}}$.

13. a. **14.** ax^{a-1}. **15.** $-\dfrac{3c}{x^4}$. **16.** $\dfrac{b}{2\sqrt{x}}$.

17. $\frac{3}{2}\sqrt{x}$. **18.** $-\dfrac{1}{2x^{\frac{3}{2}}}$. **19.** 0. **20.** $\frac{1}{3}x^{-\frac{2}{3}}$.

21. $2x^2 - \frac{1}{5}$. **22.** $\frac{1}{10} - \dfrac{10}{x^2}$. **23.** $21x^2 - 2$. **24.** $4\pi x^2$.

25. $12x^3 - \frac{1}{2} + \dfrac{6}{x^2}$. **26.** $20x^3 + 6x - \dfrac{1}{x^2} + \dfrac{2}{x^3}$.

27. $3x^2 + 4x + 1$. **28.** $2x - \dfrac{2}{x^3}$. **29.** $2nx^{2n-1}$. **30.** $-kx^{-k-1}$

31. 6. **32.** 6. **33.** 1, 2. **34.** $42x$.

35. $\mp\frac{1}{4}x^{-\frac{3}{2}}$ **39.** 68, 100; -32, -32. **40.** $392x^7$, $392x^7$.

41. $12 + 20x$, 5, $2(3 + 5x)^2$, $60 + 100x$.

42. $4 + 6x^2 + 3x^4$, $2x$, $2 + 4x^2 + 3x^4 + x^6$, $8x + 12x^3 + 6x^5$.

43. p. **44.** $\frac{124}{55}$.

XVII. c. (p. 348.)

1. Decr., incr., −, +, yes. **2.** Incr., incr., +, +, no.
3. Incr., incr., incr., decr., decr., +, +, +, −, −, yes.
4. Max. *C*; min. *A*, *E*. **6.** Max.; decr., decr.; −, −.
7. Min.; incr., incr.; +, +. **8.** No; decr., incr.; −, +.
9. No; incr., decr.; +, −. **10.** +, −; −, +; −, −; +, +.
11. Yes.
15. *AB* −, − +; *BC* +, + −; *CD* −, − +; *DE* +, + −; *EF* −, − +. **18.** 1. **19.** 3.

XVII. d. (p. 354.)

1. $x=1$ min.; $x=-2$ min.; $x=0{\cdot}4$ max.; $x=1$ min., $x=-1$ max.; none; $x=1$ min.; $x=\frac{3}{4}$ min., not $x=0$; $x=1$ min., $x=-\frac{1}{3}$ max.; $x=3$ min., $x=-1$ max.; none.
2. 40,000 sq. ft. **3.** 2 cu. ft. **4.** 4 ft. **5.** 8 in.
6. 80,000 sq. ft. **7.** $13\frac{1}{2}$ sq. ft. **8.** 12 knots. **9.** $10\frac{1}{3}$ sq. ft.
10. $1\frac{3}{4}$ hrs. **11.** 18 cu. ins. **12.** 0·927. **13.** 8 ft.
14. 2·55 cu. ft. **15.** 4 ins. from top. **16.** 3 ins.
17. 0·184. **18.** 8, 4 ins. **19.** $\frac{1}{4}$. **20.** 1·5 ft.
22. 12 cu. in. **23.** 1·68 in. **24.** $31\frac{1}{4}$. **25.** $\frac{1}{4}$.
26. $\frac{3\sqrt{3}}{16}$. **27.** $\sqrt{2}$. **28.** 8·16, 7·07 in.
29. On *BA* produced, $PA=3$ in. **30.** $y=\frac{\sqrt{(4x^4+a^2b^2)}}{2x}$, 500, 707 yds.
31. $\frac{1}{2}x(a-2x)(b-2x)$, 1·27 ins.

XVII. e. (p. 358.)

1. $5(\delta x)^2$, $3x(\delta x)^2+(\delta x)^3$. **2.** $x\,.\,\delta x+\frac{1}{2}(\delta x)^2$, $A=\frac{1}{2}x^2$, $\frac{1}{2}(\delta x)^2$.
3. $2\pi r\,.\,\delta r$, 0·63 sq. in. **4.** 0·126 cu. in.
5. $\delta t(10-0{\cdot}6t^2)$, 0·76 ft. **6.** $-\frac{c}{p^2}\,.\,\delta p$.
7. $\delta y=\frac{8x}{3l}\,.\,\delta x$; length of wire increases 0·05 in. **8.** 540 ft.
9. 210 sq. yd. **10.** $\delta M=\frac{\delta k}{10}\left(1-\frac{1}{k^2}\right)$, $-0{\cdot}12$ tons.
11. $\delta A=x^2\,.\,\delta x$. **12.** $\delta V=\frac{1}{4}\pi x^2\,.\,\delta x$.

13. $\delta V=\pi y\,.\,\delta y.$

14. $\delta V=\pi(2ax-x^2)\,.\,\delta x.$

15. $3\,.\,\delta x$; $15(1+3x)^4\,.\,\delta x$; $15(1+3x)^4.$

16. $\frac{1}{2\sqrt{z}}\,.\,\delta z$; $2x\,.\,\delta x$; $\delta y \overset{\frown}{=} \frac{x\,.\,\delta x}{\sqrt{(1+x^2)}}$; $\frac{x}{\sqrt{1+x^2)}}.$

17. $\delta y \overset{\frown}{=} 3(x^3-x+7)^2(3x^2-1)\,.\,\delta x$; $3(x^3-x+7)^2(3x^2-1).$

18. $\delta y \overset{\frown}{=} \frac{-6\,.\,\delta x}{(3x-5)^3}$; $-\frac{6}{(3x-5)^3}.$

19. $\delta u=2y\,.\,\delta y$; $\frac{du}{dx}=2y\frac{dy}{dx}.$

20. $\delta V \overset{\frown}{=} 4\pi r^2\,.\,\delta r$; $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}.$

XVII. f. (p. 361.)

1. 0·12 cu. in. per hr.

2. 1·5 in. per sec.

3. 0·637, 0·326 in. per sec.

4. $\frac{11}{8}$ ft. per sec.

5. 0·057 lbs. per sq. in. per min.

6. 0·31 in. per sec.

7. $10^{-10}\times 3V^5$, 3.

8. 0·96, 0, − 0·96 sq. in. per min

9. 0·00133 in. per min.

10. 0·242 in. per sec.

11. 0·286 in. per sec.

12. 1·39 sq. cm. per sec.

13. − 0·665 mm. per sec.

14. 1·00564.

15. $\frac{dQ}{dt}=k(30-Q)$, where k is a constant.

XVIII. a. (p. 365.)

1. $y=x^4+c.$

2. $y=\frac{1}{5}x^5+\frac{1}{3}x^3+c.$

3. $\frac{dy}{dx}=2x$, parallel curves.

4. $y=\frac{1}{12}x^4+ax+b.$

5. $y=\frac{1}{2}x^2+c.$

6. $y=x^3-x^2+c.$

7. $y=4x+c.$

8. $y=ax+b.$

9. $5x^4$, $y=\frac{1}{5}x^5+c$, $y=\frac{3}{5}x^5+c.$

10. $8x^7$, $y=\frac{1}{8}x^8+c$, $y=\frac{5}{8}x^8-2x+c.$

11. $-\frac{1}{x^2}$, $y=-\frac{1}{x}+c$, $y=-\frac{4}{x}+c.$

12. $-\frac{4}{x^5}$, $y=-\frac{1}{4x^4}+c$, $y=x+\frac{3}{4x^4}+c.$

13. $\frac{1}{2\sqrt{x}}$, $y=2\sqrt{x}+c$, $y=\frac{1}{2}x^2+6\sqrt{x}+c.$

14. $y=\frac{5}{3}x^3-\frac{3}{2}x^2+9x+c.$

15. $y=\frac{1}{6}x^6-\frac{1}{3}x^3+3x+c.$

16. $y=\frac{1}{4}x^4-\frac{1}{2x^2}+c.$

17. $y=\frac{10}{3}x^{\frac{3}{2}}+c.$

18. $y=-\frac{3}{x}-\frac{5}{2x^2}+c.$

19. $y=\frac{1}{3}x^3-\frac{5}{2}x^2+6x+c.$
20. $y=\frac{1}{12}x^4-\frac{1}{6}x^3+ax+b.$

21. $y=\frac{3}{101}x^{101}-\frac{5}{99x^{99}}+c.$
22. $c=-1.$

23. $s=12+50t-16t^2.$
24. $s=\frac{2}{3}t^3+5t+8.$

25. 165 ft.
26. $y=x^3-x+3.$

27. 0 ; 0 ; $y=\frac{0{\cdot}0001}{12}x(20-x)(400+20x-x^2)$, 0 ; $\frac{1}{15}$, $-\frac{1}{15}$; 5 in.

28. 10,000 ft. lb.

29. $x=12t$, $y=16t^2$; $y=\frac{1}{9}x^2$; $2\frac{1}{2}$ sec. ; 30 ft. ; $\frac{20}{3}$; $x=10$, $y=\frac{100}{9}$.

30. $\frac{3}{4}$ in. per sec., 8 in.
31. $\frac{dT}{dx}=a(K-T).$
32. $\frac{dp}{dx}=-ap.$

XVIII. b. (p. 371.)

1. $\frac{1}{3}x^3$, $2\frac{1}{3}$, $\int_1^2 x^2dx.$
2. $\frac{2}{3}x^{\frac{3}{2}}.$
3. 4.
4. 1, 4 ; $4\frac{1}{2}$.
5. $\frac{1}{2}$, 0·83.
7. $7\frac{1}{3}$, $\frac{3}{32}$, $-\frac{2}{3}$, $\frac{11}{24}$
8. πy^2, $\pi(y+\delta y)^2$, $\frac{1}{5}\pi x^5$, 0·148.
9. $\frac{1}{27}\pi x^3.$
10. 1·57.
11. $\sqrt{(100-x^2)}$, $\pi(100-x^2)$, $\pi(100-x^2).\delta x$, $\pi(100x-\frac{1}{3}x^3)$,
$\frac{\pi}{3}(2000-300x+x^3)$, 2090 cu. in.
12. 47·4 cu. in.
13. 14·2 cu. ft.
14. $2\frac{2}{3}$.
15. $\frac{23}{48}$.
16. 0.
17. Each $=8\frac{5}{6}$.
18. 2, $\frac{5}{8}$ in.
19. 1·56 in.
20. $\frac{1}{2}Wr^2.$
21. $\frac{2}{5}Wr^2.$
22. $2\frac{1}{7}$.
23. $1\frac{7}{8}$.
24. 210 ft. lb.
25. $\frac{21}{65}$.
26. $\frac{9}{16}$.
27. $4\frac{1}{7}$.
28. $\pi\rho a^3.$
29. 90 lb.
31. 2520 cu. ft.
32. 3050 cu. in.

XVIII. c. (p. 377.)

1. 0·335 ; $\frac{1}{3}$, 0·6 per cent. ; 0·33.
2. 120·2, 122·5, 118·7.
3. 39·495, 38·687, 39·270 ; 0·5, 1·4 per cent.
4. 41·63 ; 41·59 ; 0·07, 0·02 per cent.
5. 3·32, 3·28 mi.
6. $10^5\times 3{\cdot}040$ cu. ft., each method.
7. Dufton 1090, Simpson 1030 ft.
8. 2088 ; 2094, 2094 cu. cm.
9. 0·1676, 0·1678.
10. 128·1, 127·5.

XIX. a. (p. 381.)

1. $12,\ 2n$; $36,\ n^2$; $\frac{1}{13},\ \frac{1}{2n+1}$; $6\times10^6,\ n\times10^n$; $96,\ 3\times2^{n-1}$; $\frac{11}{64},\ \frac{2n-1}{2^n}$; $40,\ 7n-2$; $-5,\ 13-3n$; $6\times10^{-6},\ n\times10^{-n}$; $1,\ (-1)^n$; $343,\ (n+1)^3$; $10^6+24,\ 10^n+4n$.

2. $1,\ 3,\ 5$; $\frac{1}{6},\ \frac{1}{2},\ \frac{3}{2}$; $\frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}$; $2,\ 6,\ 12$; $8,\ 40,\ 315$; $0,\ 7,\ 26$.

3. $1,\ 3,\ 2,\ 10,\ n$.

4. $1,\ 16,\ n^2$.

5. 29, no, $14\times29=406$.

6. $29,\ 435$; $35,\ 385$; $-15,\ -165$; $9\frac{1}{4},\ 101\frac{3}{4}$

7. 27, 22, 21, 19.

8. $104,\ 7n-1$; $128,\ 2^{n-1}$; $102,\ 11n-8$.

9. $15,\ 3\frac{1}{14},\ 11,\ 7$.

10. 2866, 23.

XIX. b. (p. 383.)

1. $19\frac{1}{3}$.

2. $7\frac{3}{5}$.

3. 87.

4. $56\frac{1}{2}$.

5. $19\frac{1}{4},\ 26\frac{1}{2},\ 33\frac{3}{4}$.

6. 5029.

7. -130.

8. 702·5.

9. $-79\frac{1}{2}$.

10. $2n^2+n$.

11. $12a+66d$.

12. $\frac{1}{2}n[2a+d(n-1)]$.

13. $258\frac{1}{2}$.

14. 540.

15. $\frac{1}{2}n(a+l)$.

16. 29 ; $\frac{1}{2}(n^2-n+2)$.

17. 4215.

18. $2x-a$; $x(n-1)+a(2-n)$.

19. $91,\ n^2-n+1$.

20. $5,\ \frac{n}{2}(5n-17)$.

21. Yes, no.

22. 999000.

23. $l-(n-1)d$.

24. $\frac{1}{2}(b+a)(b-a+1)$.

XIX. c. (p. 386.)

1. $7\times5^{n-1},\ (-2)^{n-1},\ 2^{4-n},\ -10(-0\cdot6)^{n-1},\ \frac{7^{n-1}}{6^{n-2}},\ \frac{3^{2n-3}}{7^{n-2}},\ (-1)^n x^{2n-1},\ \frac{b^{n-1}}{a^{n-2}},\ \frac{1}{a^{n-2}}$.

2. $-\frac{1}{16},\ 2$.

3. $(\frac{3}{4})^{25}$.

4. 1533.

5. $80\frac{242}{243}$.

6. ar^7-a ; $\frac{a(r^7-1)}{r-1}$.

7. $2(3^{10}-1),\ \frac{25}{3}[1-(0\cdot4)^{12}],\ \frac{9}{4}-(\frac{2}{3})^6,\ \frac{7}{3}(2^{11}+1),\ \frac{243}{13}[1-(\frac{4}{9})^{20}]$.

8. $1+r+r^2+r^3+r^4+r^5$; $1+r+r^2+\ \dots\ +r^9$; $\frac{1-r^{21}}{1-r},\ \frac{a(1-r^{21})}{1-r}$.

9. $ar^{n-1},\ ar^{n-2},\ ar+ar^2+ar^3+\ \dots\ +ar^{n-2}+ar^{n-1}+ar^n,\ ar^n-a,\ \frac{a(r^n-1)}{r-1}$.

10. 1·111 ; 0·000001 per cent., $\frac{10}{9}(1-10^{-50})$, 10^{-48} per cent. ; $\frac{10}{9}$.

11. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ ins. ; $\frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, 1-2^{-10}, 1-2^{-100}$ ins. ; $1, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$; $2^{-6}-2^{-46}$; 16.

12. (ii), (iv)   14. $\frac{43}{63}$.   15. 2500, $8\frac{1}{3}$, $\frac{a}{1-r}$.

16. 630, 23600.

XIX. d. (p. 388.)

1. 17·6 m.p.h., $2\frac{1}{3}$ m.   2. 0·0966.   3. 6·58, 8 sec.

4. £2400.   5. 3620.   6. 36 ft.

7. 119·5.   8. $\frac{2}{81}(10^9-82)$.   9. 65.

10. 541 yd.   11. 200°.   12. n^2.

13. £21. 16*s*. $4\frac{1}{2}$*d*.   14. n^2-n.   15. 15.

16. £9500, £14500.   17. 223 yd.   18. £163, 1490.

19. £61·4, 390.   20. £2500.

21. $\dfrac{1-(n+1)x^n+nx^{n+1}}{1-x}$; $\dfrac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}$.

22. Less than 2^{200}.   25. $\frac{5}{11}$.   26. $\frac{19}{63}$.

27. $\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$.   28. 1, 7, 19, 37.   29. Each = 225.

30. 2.3.4.5.6.7.8.9.10.11 − 1.

31. 265 ; $\frac{5}{8}$, $\left(\text{before } \frac{n}{m}, \text{ the number is } \frac{(n+m-1)(n+m-2)}{2}+m-1\right)$.

32. 2.

XIX. e. (p. 392.)

1. 1, 1·18, 1·38, 1·63, 1·91 ; 1·18 : 1 in each case.   2. 31·8 millions.

3. 46·9° C.   4. £101,000.   5. 40, 1·19.   6. 4·1 per cent.

XX. a. (p. 395.)

1. 30.   2. 120.   3. 12 ; 720.

4. 720.   5. 210.   6. 840 ; 840.

7. 90, 45.   8. 24.   9. 56.

10. $26 \times 99 \times 999999$.   11. $\frac{1}{2}\underline{|10}$.   12. 6, 24, 32.

13. 90.   14. $\underline{|8}$, 2880, 8640.   15. 192, 144.

16. 24, 12, 12.   17. 1440.   18. 14400.

19. 15 ; 719.   20. $\underline{|6}(\underline{|4})^2 = 414720$.   21. 36.

22. $(\underline{|6})^2 = 518400$.   24. $\frac{\underline{|20}}{\underline{|15}}, \frac{\underline{|20}}{\underline{|14}}, \frac{\underline{|30}}{\underline{|18}}, \frac{\underline{|n}}{\underline{|n-12}}, \frac{\underline{|n}}{\underline{|n-r}}$.

25. $n, (n-1)\underline{|n-1}$.   26. $2^n . \underline{|n}$.   27. $10^{22} \times 8{\cdot}1$.   28. 18, 8

XX. b. (p. 398.)

1. 10.
2. 364.
3. 22100.
4. $\dfrac{\underline{|52}}{\underline{|39}\,\underline{|13}}$, $10^{11} \times 6\cdot4$.
5. 66 ; 220.
6. 66 ; 132.
7. 31.
8. 13860.
9. 90, 15.
10. 462, 5775.
11. 720.
12. $\dfrac{\underline{|52}}{(\underline{|13})^4}$, $\dfrac{\underline{|52}}{24(\underline{|13})^4}$.
13. 15840.
14. 56.
15. 56.
16. 35.
19. $\dfrac{n-r+1}{r}$, $\dfrac{n}{r}$.
22. ${}^{n-1}C_{r-1}$, ${}^{n-1}C_r$.

XX. c. (p. 401.)

1. 1326, 1128, 198, $\frac{33}{221}$; 188 : 33 against.
2. 15, 6, $\frac{2}{5}$.
3. $\frac{11}{57}$.
4. 1365, 1001, $\frac{11}{15}$.
6. $\frac{1}{2}$, $\frac{1}{8}$, $\dfrac{1}{2^n}$.
7. $\frac{1}{4}$, $\frac{3}{4}$, $\frac{3}{8}$.
8. $\frac{1}{17}$.
9. $\frac{1}{36}$, $\frac{1}{6}$.
10. $\frac{3}{10}$, $\frac{1}{10}$.
11. $\frac{1}{3}$.
12. $\frac{11}{21}$.
13. 120, 60, 10.
14. $\underline{|15}$, $\dfrac{\underline{|15}}{\underline{|5}}$, $\dfrac{\underline{|15}}{\underline{|10}\,\underline{|5}}$, $\frac{1}{2}\underline{|14}$.
15. 12, 1728
16. 24, 64.
17. 340.
18. $3^{16} = 10^7 \times 4\cdot3$, $10^{-8} \times 2\cdot3$.
19. 3150.
20. 60.
21. 90.
22. $\dfrac{\underline{|2n}}{2^n \cdot \underline{|n}}$.
23. 11.
24. $\frac{11}{4165}$.
25. 4 ; one penny, two half-crowns ; $52\frac{1}{2}$ pence, say 4s. $4\frac{1}{2}d$.
26. $\frac{3}{8}$

XXI. a. (p. 406.)

1. $x^4 + x^3(a+b+c+d) + x^2(ab+ac+ad+bc+bd+cd)$
$+ x(abc+bcd+cda+dab) + abcd$;
$x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$; $x^4 + 4x^3 + 6x^2 + 4x + 1$;
$x^4 - 4x^3 + 6x^2 - 4x + 1$.
2. 64, 6, 15.
3. $abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde$; $10a^3$; -10.
4. 4, 35, 35.
5. $x^3 + 6x^2 + 11x + 6$; $x^4 + 10x^3 + 35x^2 + 50x + 24$.
6. $6x^3 + 11x^2 + 6x + 1$; $24x^4 + 50x^3 + 35x^2 + 10x + 1$.

XXI. b. (p. 407.)

1. $1+3x+3x^2+x^3$. **2.** $1+4x+6x^2+4x^3+x^4$.

3. $1-4x+6x^2-4x^3+x^4$. **4.** $x^3-6x^2+12x-8$.

5. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.

6. $x^5+5x^3+10x+\frac{10}{x}+\frac{5}{x^3}+\frac{1}{x^5}$. **7.** $8x^3-36x^2y+54xy^2-27y^3$.

8. $a^8-8a^6b+24a^4b^2-32a^2b^3+16b^4$. **9.** 35, $\frac{1}{6}n(n-1)(n-2)$.

10. $-15360x^3$. **11.** 24. **12.** 11.

13. $-20x^3y^3$. **14.** $120x^7y^3$, $120x^3y^7$.

15. ${}^nC_{r-1}\,.\,2^{r-1}x^{r-1}$; $(-1)^{r-1}\,.\,{}^nC_{r-1}\,.\,2^{r-1}x^{r-1}$.

16. ${}^nC_{r-1}x^{n-r+1}y^{r-1}$; ${}^nC_{n-r+1}x^{r-1}y^{n-r+1}$; Yes. **17.** x^3. **18.** x^4.

19. ${}^nC_1+{}^nC_2+\ldots+{}^nC_n=2^n-1$; $1-{}^nC_1+{}^nC_2-{}^nC_3+\ldots=0$.

20. 0. **21.** ${}^nC_r-{}^nC_{r-1}=\frac{(n-2r+1)\underline{|n}}{\underline{|r}\,\underline{|n-r+1}}$.

22. $1+nx+\frac{n(n+1)}{2}\,.\,x^2+\frac{n(n-1)(n+4)}{6}x^3$. **23.** 34.

24. 104060401, 9509900499. **25.** 1·1041, 0·83297.

XXI. c. (p. 409.)

2. $\frac{x^5}{1-x}$. **3.** $1+x+x^2+x^3+x^4$.

4. 0·01, 0·001. **5.** $\frac{x^3}{1+x}$, $-\frac{x^3}{1+x}$, $\frac{x^4}{1+x}$.

6. $1-x+x^2-x^3+x^4$; 0·0001, 0·000001. **7.** $\frac{3x^2-2x^3}{(1-x)^2}$, $\frac{3x^2+2x^3}{(1+x)^2}$.

8. $1+2x+3x^2+4x^3$, $1-2x+3x^2-4x^3$; 0·0003, 0·0003.

10. $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3$, 0·00001. **11.** $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$, 0·001.

12. $\frac{1}{x}-\frac{h}{x^2}$, 0·98. **13.** $\frac{1}{x}-\frac{h}{2x^3}$, 0·499, 0·332.

XXI. d. (p. 412.)

1. ab, 0·0002, 1·08. **2.** 1·06, $1+3h$. **3.** 1·000033 : 1, 1·000099 : 1.

4. 2 per cent. **5.** 6 per cent. **6.** 3, 2; 1·5 per cent.

7. $1+2h$, 1·02. **8.** 1·01, 0·94, 1·003, 1·004, 1·005, 0·1999.

9. $1+2h$, $1-5h$, 0·95. **10.** 0·028, $0{\cdot}555\frac{h}{\sqrt{l}}$.

11. 2 per cent. decrease, $-\frac{vp_1}{p}$.

12. 0, ±1 ; 1·005.

13. $3x^2$, $10x^9$, $-\frac{1}{x^2}$, $\frac{1}{2\sqrt{x}}$.

14. Decreased about 3 per cent.

15. 5 per cent. decrease.

16. $\frac{h^2}{2x}$ in.

17. $\theta + \frac{1}{3}\theta^3$.

18. 1 per cent. increase.

XXII. a. (p. 418.)

Note.—Some of the formulae below are given to a greater degree of accuracy than can be attained by drawing alone.

1. $y = 1\cdot62x - 0\cdot54$.

2. $l = 17\cdot95 + 0\cdot41W$.

3. $l = 0\cdot12W + 2\cdot1$.

4. $y = 9\cdot79 - 0\cdot033x$; 0·005.

5. $y = 6\cdot9x + 0\cdot1$; when $x = 3$, $y = 21$.

6. $P = 0\cdot0049v^2$.

7. $Q = 31000D^2$.

8. $R = 600 + 0\cdot9V^2$.

9. $k = \frac{1\cdot20}{d} - 0\cdot14$.

10. $S = 4\cdot09t^3$.

11. $y = \frac{12x}{x-7}$.

12. $t = 0\cdot0419d^{1\cdot168}$.

13. $p = 475v^{-1\cdot06}$.

14. $P = 18\cdot0 \times 10^{0\cdot051t}$, 16·0 (actually 15·91), 41·0.

15. $E = 10^{-14} \times 2\cdot185T^5$.

16. $\lambda = \frac{2960}{T}$.

17. $\delta = 10^{-8} \times 1\cdot24T^4 - 90$.

18. $y = 61 + 0\cdot068x + 0\cdot0024x^2$ (but better to write 61·1 for 61); 64·2 (actually 64·37).

19. $a = 10^{-4} \times 1\cdot5$; $b = 10^{-7} \times 9\cdot0$; 1·024.

20. Plot $\frac{d}{v}$ against $\sqrt{d}$; $a = 0\cdot21$, $b = 0\cdot25$.

XXIII. a. (p. 425.)

1. 51·8, 8·49, 6·76.

2. 1·58 in., 3·81 cm.

3. 11·2, $1\cdot405 \times 10^6$.

4. 23·6, 7·96 in.

6. 2·31.

7. 36.

XXIII. b. (p. 429.)

1. 32·7, 27·7 amp. ; 74·1, 62·7 amp.

2. 10, 24·2 amp. ; 0·52 mm.

3. 2·5, 5, 5, 3·3, 5·8 in.

4. 1·76, 6·99 in.

5. 1·2 cm.

6. 1·58, 3·54 in.

7. $y = 2x^2$, $y = 3x^2$.

8 28·6.

9. 62·4.

11. $-1\cdot06$.

12. 10·8.

13. 2850.

14. 0·254.

15. 3·29.

16. 1·44.

XXIII. c. (p. 435.)

1. £22. 8*s*., £34, £36.

2. 45·7, 55·7, 38·6 ; $Z=\frac{1}{7}(6X+10Y)$; 38·6, 48·6, 31·4 ;
$Z=\frac{1}{7}(6X+10Y-50)$.

3. $-11·4$, -30, 4·3 ; $Z=\frac{1}{7}(6X-10Y)$; 38·6, 20, 54·3 ;
$Z=\frac{1}{7}(6X-10Y+350)$.

4. 43, 21·5. **5.** 2·66, 4·28. **6.** 2, 3·75, $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$.

7. 60·8, 81·2. **8.** 7, 18. **9.** 40, 28.

10. 55, 45·5, -20 ; $Z=\dfrac{60(4X+Y)}{3X+2Y}$; $Z=\dfrac{60(4X+Y-80)}{3X+2Y-110}$.

11. $w=\dfrac{xz}{y}$, 6·67, 2·25, 4·79.

XXIII. d. (p. 438.)

1. $\log T=\frac{1}{4}\log H+\frac{3}{4}\log W$; 10, £10 ; £20 ; £27·1 ; £22.

2. 5·4, 191. **3.** 3·19, 5·18. **4.** 33·1, 17·6.

5. 11000, 38500.

6. (v) $t=3$. (vi) 3·45, 1, 1·45, imaginary. (vii) -1, $-1·45$, -3, $-3·45$.
(viii) put $x=10z$ and solve $z^2-2·3z-4·85=0$.

XXIV. a. (p. 443.)

1. Real +, − ; real +, − ; imaginary ; coincident + ; real −, − ; imaginary.

2. $x^2-x-2=0$, $4x^2-3x=0$, $x^2-2ax+a^2-b^2=0$;
$x^3-x^2(a+b+c)+x(ab+bc+ca)-abc=0$, $x^2-2=0$,
$x^2-2x\sqrt{2}+1=0$; $bdx^2+x(bc-ad)-ac=0$.

3. $\dfrac{-b\pm\sqrt{(b^2-ac)}}{a}$. **4.** $6\frac{1}{4}$. **5.** $\frac{25}{16}$, $-\frac{25}{16}$.

6. 2, $-0·05$. **8.** $\frac{1}{2}>x>-\frac{3}{4}$, $1\frac{1}{2}>x>-1$.

9. $x>1$ or $x<-4$.

10. $x>3$ or $1>x>-1$; $x>\frac{3}{2}$ or $\frac{1}{3}>x>-2$; $1>x>-\frac{1}{4}$ or $x<-4$;
$x>3$ or $1>x>-1$ or $x<-3$.

11. (i) $x>2$ or $-1>x>-3$. (ii) $2>x>-1$ or $x<-3$.

12. (i) $x>\frac{1}{2}$ or $\frac{1}{3}>x>-3$ or $x<-4$. (ii) $\frac{1}{2}>x>\frac{1}{3}$ or $-3>x>-4$.

13. 2, -3. **14.** $-0·6$, $-1·2$; 3·6. **15.** $\pm4\sqrt{3}$.

16. ± 8. **17.** $2q^2 = 9r$, $q = 0$, $6q^2 = 25r$, $r = 1$, $r = 0$, $q^2 = 4r + 1$.
27. 16 or 0. 28. $8\frac{7}{8}$. 29. $-\frac{5}{3}$, $18\frac{1}{4}$, $31\frac{1}{3}$.
30. $3b = 2q$, $9c = 4r$. 31. $6b^2 + c = 0$. 32. $y^2 - 40y = 900$, $2y^2 + 3y = 3$.
33. $4y^2 - 28y + 9 = 0$, $y^2 - 8y + 5 = 0$.
34. Roots of second equation are p times roots of first.
35. $b^2 = a^2 + 4ac$, $(a + b + c)^2 = b^2 - 4ac$. 36. Imaginary.
38. 6. 39. $\frac{11}{18}$ or 0. 40. $a = -2$.
41. $a + b + 2h = 0$ or $4a + b - 4h = 0$. 42. $-\frac{1}{3}$.
43. $b + c = 0$ or $b - c = 1$. 45. $4c > a > 0$. 46. $x = y = z$
47. Not between 2 and -2. 48. $\dfrac{1 \pm \sqrt{(1 - 4k^2)}}{2k}$, $\frac{1}{2} > k > -\frac{1}{2}$; $-\frac{1}{2}$.
49. Between $-\frac{1}{3}$ and $-\frac{1}{27}$. 50. 6 and 1.
52. $ry^2 + qy + 1 = 0$; $y^2 + y(4r - q^2) - r(q^2 - 4r) = 0$. 53. $q^2 - 4r = b^2 - 4c$.
54. $x = 2$, $y = -1$, no.
55. $(x - \alpha)(x - \beta)(x - \gamma) = 0$, $x^3 - x^2(\alpha + \beta + \gamma) + x(\alpha\beta + \beta\gamma + \gamma\alpha) - \alpha\beta\gamma = 0$;
$-\dfrac{b}{a}$, $\dfrac{c}{a}$, $-\dfrac{d}{a}$.
56. $p = -\Sigma\alpha$, $q = \Sigma\alpha\beta$, $r = -\Sigma\alpha\beta\gamma$, $s = \alpha\beta\gamma\delta$.
57. Sum is zero; sum = product; roots are of form $\pm a$, $\pm b$; one is zero.
58. -5. 59. -7.

XXIV. b. (p. 447.)

1. $(x - y)(x^2 + xy + y^2)$, $(a + 2b + c)(a^2 + b^2 + c^2 + bc - ca + ab)$
$(a - c)(a^2 + 3b^2 + c^2 + 3bc + ca + 3ab)$.
2. $(a^2 + 2a + 4)(a^2 - 2a + 4)$, $(3x^2 + y^2)(x^2 + 3y^2)$.
3. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$. 4. $x^2 + 5x + 5$, $x^2 + 5xy + 5y^2$.
7. $\sqrt{[xyz(x + y + z)]}$. 8. -1. 9. $\frac{3}{4}$.
10. $(x - 2y)(2x + y)(2x - y)$. 11. $\dfrac{1}{x+1} - \dfrac{1}{x+4}$.
12. $-3xyz(x + y)(y + z)(z + x)$. 13. $16(5b - 3a)(15a - 19b)$.
14. $(a - b + c)(a + b - c)(a - b - c)$. 15. Unchanged; divided by p.
16. $\dfrac{1}{n} \cdot 2^{n-1}$. 17. $(x - y)(x + y)(x^2 + y^2) \dots (x^{2^n} + y^{2^n}) = x^{2^{n+1}} - y^{2^{n+1}}$
18. (ii), (iv).
19. $(a^2 + x^2)(b^2 + y^2)$, $[c(ab + xy) + z(ay - bx)]^2 + [z(ab + xy) - c(ay - bx)]^2$.
20. 32; *acfhk*, *bdfgk*, *lgfcb*. 21. 7. 22. 0.

23. $n+3$; 1 ; 2 if n is even, 0 if n is odd ; 0 if $n \neq 0$. **24.** 0.

25. $7, n+1$. **26.** $2n+1$; 1 ; 0 if $n \neq 0$. **27.** $(-1)^n, (-1)^{\frac{n(n-1)}{2}}$.

28. $1+5x+10x^2+10x^3+5x^4+x^5$, $a^4+4a^3b+6a^2b^2+4ab^3+b^4$. **29.** 35.

30. $1+\frac{1}{2}x-\frac{1}{8}x^2$, $1-x+x^2$. **31.** $\sqrt{x}\left(1+\frac{h}{2x}-\frac{h^2}{8x^3}\right)$.

32. $1-2x+3x^2-4x^3+\ldots+(-1)^n(n+1)x^n+\ldots$,
$1+nx+\frac{n(n+1)}{1.2}x^2+\frac{n(n+1)(n+2)}{1.2.3}x^3+\ldots$.

34. Reflections in the y-axis. **35.** $\frac{x(x-1)}{(x+1)(x-2)} \equiv 1+\frac{2}{(x+1)(x-2)}$.

36. $[e(ac+pbd)+pf(ad+bc)]^2-p[f(ac+pdb)+e(ad+bc)]^2$.

37. $\frac{ar-cp}{bp-aq}$. **38.** $y^2(bl^2-2hlm+am^2)+2y(lh-am)+a=0$

39. $\frac{k}{(b-a)(b-c)}, \frac{k}{(c-a)(c-b)}$.

40. $\frac{(b+\lambda)(b+\mu)(b+\nu)}{(b-a)(b-c)}, \frac{(c+\lambda)(c+\mu)(c+\nu)}{(c-a)(c-b)}$.

XXIV. c. (p. 452.)

5. (i), (iii), (vii), (viii), (ix), (x). **8.** $x=2$, $y=1\frac{1}{2}$, $z=2\frac{3}{4}$, $t=\frac{1}{4}$.

10. 1·6 or 0. **11.** $-9:13:1$. **12.** $5:-1:8$; $\frac{5}{8}, -\frac{1}{8}$.

13. $(c-b):(1+ac):(1+ab)$. **14.** $\frac{br-cq}{aq-bp}, \frac{cp-ar}{aq-bp}$.

15. $\frac{x^2}{h_1b_2-h_2b_1}=\frac{2xy}{b_1a_2-b_2a_1}=\frac{y^2}{a_1h_2-a_2h_1}$,
$(a_1b_2-a_2b_1)^2=4(a_1h_2-a_2h_1)(b_2h_1-b_1h_2)$.

16. $4, 3, -2$ or $-4, -3, 2$. **18.** $\frac{4}{17}$. **20.** 17.

21. b^4+3d^3f ; $b^4+3ad^3f-5bdf^2$. **22.** $ab^4-c^2d^4$.

23. $\sqrt{(a^2-4b^2)}$. **24.** $\sqrt{\left(\frac{a+2b}{a-2b}\right)}$. **25.** $\sqrt{(a^2+b^2)}$.

XXIV. d. (p. 455.)

1. $0, 0, 0, n^2+2n$. **2.** $6, 7, 12, 12$.

3. $2\Sigma ab$, $3\Sigma a^2+2\Sigma ab$, 3, $3\Sigma abc$, $\log(n+1)$, $\log\left(\frac{n+1}{k}\right)$.

4. n, $1+2+3+\ldots+n$. **7.** $n(2n-1)$. **8.** $2^{2n}-4n-1$.

9. $\frac{n}{n+1}$; $\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{n(n+1)}$; $\frac{n}{n+1}$.

10. $\frac{1}{6}[q(q+1)(2q+1)-p(p-1)(2p-1)]$.

11. $\frac{1}{3}[q(q+1)(q+2)-p(p-1)(p+1)]$.

12. $\frac{n+1}{n(2n+1)}$.

13. $a\Sigma x^2+b\Sigma xy$, $3\Sigma x^2-2\Sigma xy$.

14. 2, 1.

15. $a\Sigma x^3+b\Sigma x^2y+cxyz$; $a=1$, $b=2$, $c=-12$.

16. $\frac{1}{2}(a^2-b^2)$, $\frac{1}{2}a(3b^2-a^2)$

17. $2a^2-6b^2$.

21. $c^2-\frac{a^2}{3}$.

22. $(x-y)^2+(y-z)^2+(z-x)^2$.

24. $\frac{1}{2}(x^2-3y^2-z^2)$.

XXIV. e. (p. 458.)

3. -2.

4. $(a+b)(2a^2-ab+b^2)$.

5. -2, 5.

6. Always, n odd, if $\frac{n}{2}$ is an odd integer.

7. $x+a+b+c$.

8. $-(a-b)(b-c)(c-a)$.

9. $3(x+y)(y+z)(z+x)$.

10. $(x+y)(y+z)(z+x)$.

11. $3(x-y)(y-z)(z-x)$.

12. $(x-y)(y-z)(z-x)(x+y+z)$.

13. $12abc(a+b+c)$.

14. $(x^2-yz)(y^2-zx)(z^2-xy)$.

15. $(xy-zw)(yz-wx)(zx-wy)$.

XXIV. f. (p. 459.)

Note.—Imaginary roots are omitted.

1. 0 or $\frac{1}{2}$ or $\frac{1}{3}$, 0 or $\frac{1}{3}$ or $\frac{1}{2}$.

2. $\pm\frac{ac}{b}$, $\pm\frac{ab}{c}$, $\pm\frac{bc}{a}$.

3. 0 or $\frac{ab+c}{a}$, $-\frac{c}{b}$ or a.

4. ± 2, ± 3.

5. 3 or -2 or $\pm\sqrt{7}$, 2 or -3 or $\pm\sqrt{7}$.

6. $\pm\frac{1}{2}$, $\pm\frac{1}{3}$.

7. $\pm\frac{a}{bc}$ or 0, $\pm\frac{b}{ca}$ or 0, $\pm\frac{c}{ab}$ or 0.

8. $\pm\sqrt{}$[illegible], $\pm\sqrt{}$[illegible], $\mp\sqrt{}$[illegible]

9. $1\frac{1}{3}$ or 3 or $\frac{-15\pm\sqrt{465}}{8}$, 4 or 2 or $\frac{-15\mp\sqrt{465}}{6}$.

10. $3c$ or $-2c$ or $-\frac{c}{2}(1\pm\sqrt{29})$, $-2c$ or $3c$ or $-\frac{c}{2}(1\pm\sqrt{29})$.

11. 2 or 1, 1 or 2.

12. 3, $\frac{1}{2}$.

13. 1 or 2 or -3, 1 or 2 or $\frac{1}{3}$.

14. $\frac{1}{2}$ or $\frac{1}{3}$, $\frac{1}{3}$ or $\frac{1}{2}$.

15. 3 or -5, $\pm\frac{4}{3}$, ± 2.

16. ± 2, ± 4.

17. ± 2, $\pm\frac{1}{2}$.

18. 9, 12.

19. 3 or -1, 3 or -7, -2 or -8

20. 12 or $\frac{18}{5}$, 2 or $-\frac{21}{5}$, 9 or $\frac{4}{3}$.

21. ± 6, ± 2.

22. 3 or -2, 2 or -3.

23. ± 2 or 0, ∓ 2 or 0.

24. 0 or ± 2 or $\pm\sqrt{6}$ or $\frac{1}{2}(\pm\sqrt{7}\pm\sqrt{3})$, 0 or ± 2 or $\mp\sqrt{6}$ or $\frac{1}{2}(\pm\sqrt{7}\mp\sqrt{3})$.

25. ± 3 or ± 1, ± 1 or ± 3. **26.** 0 or -2, -5 or 3, -3 or 1.

27. 4, not $2\frac{1}{4}$. **28.** 1, not 8. **29.** 3, none. **30.** 3.

31. -3 or $-2{\cdot}2$. **32.** 2. **33.** 1, not $-\frac{1}{2}$. **34.** -1 or -2.

35. 0. **36.** 3 or -1. **37.** $1, -5, -2\pm\sqrt{3}$.

38. 1 or 2. **39.** 0 or 2, -2 or 4, 2 or 0.

40. -1 or $-\frac{7}{4}$, 2 or -1, 5 or 26. **41.** 9 or 4, 6 or 6, 4 or 9.

42. $-4, -6, \pm\sqrt{19}-5$. **43.** $-1, \dfrac{7\pm3\sqrt{5}}{2}$.

44. $3, 2, \frac{1}{2}, \frac{1}{8}$. **45.** $x, y, z=1, 2, 3$ any order.

46. $2, \frac{2}{3}, \frac{6}{7}$. **47.** $a, b-a$.

48. $\dfrac{x}{p^2}=\dfrac{y}{q^2}=\dfrac{z}{r^2}=\pm\dfrac{1}{\sqrt{(p^2+q^2+r^2)}}$.

49. $\dfrac{x}{c+a-b}=\dfrac{y}{a+b-c}=\dfrac{z}{b+c-a}=\dfrac{1}{(b+c-a)(a+b-c)}$ or 0.

50. If $a=3$, $x=y=$ any number; if $a=1$, $(x+\frac{1}{2})(y+\frac{1}{2})=2\frac{1}{4}$; if $a=4$, $x=y=1$; if $a=2\frac{1}{2}$, $x=y=-2$; in all other cases $x=y=1$ or -2.

51. $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}=\dfrac{a+b+c}{abc}$ or 0.

52. $\dfrac{x}{b^2+c^2}=\dfrac{y}{c^2+a^2}=\dfrac{z}{a^2+b^2}=\pm\dfrac{1}{2\sqrt{(a^2+b^2+c^2)}}$.

XXIV. g. (p. 462.)

1. $a+c=2b$. **2.** $(a-b+c)(a-b-c)+2a+2b-2c=0$.

3. $y^2=4ax$. **4.** $x^2=y^2+2$. **5.** $(x+y)(3x-y)=16$.

6. $a+b=3c$ unless $x=y=0$. **7.** $(b+c)^2=(a+b)(a+c)$.

8. $5x+2y+z=0$. **9.** $x^2+y^2=\dfrac{2b^2}{a^2}$. **10.** $x^2+y^2=a^2$.

11. $a+2=bc$. **12.** $b+ac=0$ unless $x=y=0$.

13. $(a+c-b)(a+b-c)(b+c-a)=8$. **14.** $y^2-by-cx+ac=0$.

15. $a^3-3ab^2-c^3+3d^3=0$. **16.** $\lambda=a\pm b+c$.

17. $a^2b^2+b^2c^2+c^2a^2=abcd$. **18.** $(c_1a_2-c_2a_1)^2=(b_1c_2-b_2c_1)(a_1b_2-a_2b_1)$.

19. $2pqr+pq+qr+rp=1$, unless $x=y=z=0$. **20.** $ax-dx+3ad=0$.

REVISION PAPERS.

E. 1. (p. 465.)

1. No. 2. $2y$, $6x^2$, $-\frac{3x^3}{8}$, $\frac{a}{2}$. 3. $w=\frac{8P-W}{n}$.

4. $-2\frac{1}{2}$. 5. 2, 3, 4.

6. Hurdles 25, 20, 15, 10, 5; area 1000, 1600, 1800, 1600, 1000; 30.

E. 2. (p. 466.)

1. 2. 2. $\frac{l}{40}$; 13 : 28. 3. $3x$, $\frac{12}{x}$ pence; 2.

4. 1, x, a, 12^x. 5. 0, -1. 6. £$(0{\cdot}27x-74\frac{1}{4})$, £400

E. 3. (p. 466.)

1. £$(208-2{\cdot}6a-2{\cdot}6b-0{\cdot}6c-d-r)$. 2. 1·5, 5.

3. $V=\frac{tV_1}{t-RCL}$. 4. 27. 5. 6, 5; yes, -4, -3.

6. $V=\sqrt{\left(\frac{5Rh}{12d}\right)}$, 20 m.p.h.

E. 4. (p. 467.)

1. $6a+\frac{1}{3}b$., 1*s*. 3*d*. 2. 34, 30, 16.

3. $(4x+5)(4x-5)$, $(3x-2)(x+3)$, $x(3x+1)$.

4. 0·3, 12; -10; effort insufficient to start machine.

5. z^{15}. 6. 6.

E. 5. (p. 468.)

1. 32. 2. x, $3-x$ in.; $x=1\frac{3}{8}$.

3. $12x^2y^2(x^2-y^2)^2$, $x(x-3)(x+1)$, $x^2(x-6)(x+1)$, $x(x+1)$.

4. $1\frac{1}{3}$. 5. 4·8 in., 19·6 *ft.* lb. 6. $\frac{x}{3}$; $\frac{x}{6}$ hrs.; $2x$ mi, $\frac{x}{2}$ hrs.

E. 6. (p. 468.)

1. $l+b=43$.
2. $\frac{9}{8}$.
3. 7·40 in.
4. The tax on £x is £$\frac{x-160}{8}$, £10.
5. $-\frac{1}{x(x+h)}$, $-\frac{2x+h}{x^2(x+h)^2}$, -1, -2.
6. -1, $4\frac{1}{2}$.

E. 7. (p. 469.)

1. $2n-4$, $2n-2$, $2n+2$, $2n+4$; 12.
2. 2·6.
3. 0·5, 0·4.
4. 21, 100, 10, 16, 4; $a=7$, $b=3$.
5. $x=\pm 4s^3t^2$.
6. $10-x$; $8-x$; $18x-2x^2$; 4·5 in.

E. 8. (p. 470.)

1. The fewer there are out of work, the fewer felonies are committed, but the more cases of drunkenness occur.
2. 13, 1.
4. $\frac{8x}{x^2-1}$, 0.
5. $\frac{5}{3}$ or $-\frac{3}{2}$, 2 or $\frac{1}{2}$.
6. $V=30-5x$, $x=6-0{\cdot}2V$, $0\leqq x\leqq 3$, $30\geqq V\geqq 15$.

E. 9. (p. 471.)

1. $N=\frac{wvt}{2av+2aV-vt}$.
2. $\frac{15x}{16m-x}$ hrs.
3. 2, 26·8 sq. cm.
4. 2 or $\frac{1}{3}$.
5. 4.
6. $\frac{8}{\pi r^2}$, 1·08.

E. 10. (p. 472.)

1. $\frac{ab}{a+b}$ min., $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$.
2. 256 cu. in.
3. t^2-4, ± 2.
4. $5x(x+2)(x-2)$, $7(a-3)(a+1)$, $(b-6)(b+2)$.
5. 1 or -3, 3 or -1.
6. $4-x$; $6x-1{\cdot}5x^2$; 0, 4·5, 6, 4·5, 0

F. 1. (p. 473.)

2. 8*s.* 8*d.*
3. 25·2, 5·69.
4. $y=4t^{\frac{4}{3}}$, $x=2t^{\frac{2}{3}}$.
5. $\frac{1}{2}$, $\frac{1}{3}$.
6. 1,800,000, 0·018, 0·00018, 1800, $10^4\times 7{\cdot}38$, $10^{-5}\times 7{\cdot}38$.

F. 2. (p. 473.)

1. -16, 4.
2. $1\frac{1}{2}$ in.
3. $6x-x^2$; 4·5, 1·5 in
4. $\frac{1}{(1-x)^2}$, $\frac{kl}{t^2}$, $\sqrt{(2x)}$, $2\sqrt{x}$, $\frac{2}{\sqrt{x}}$.
5. 1·5.

$\frac{4a^2-2}{(a-1)(2a-1)}$, each $=4\frac{2}{3}$.

F. 3. (p. 474.)

1. 6, 8 in. 2. All, 1, 1·5, none. 3. $\frac{Wb}{W-2Pe}$. 4. 1401, 1·6721.

5. $\sqrt{(a^2+b^2)}$, $\sqrt[3]{a^2}+\sqrt[3]{b^2}$, $\frac{Wl}{bd^2}$, $K.\frac{\sqrt[4]{H^5}}{\sqrt{P}}$. 6. $P=\frac{K^2}{a^2}\cdot\sqrt{H^5}$.

F. 4. (p. 474.)

1. 1, 7, 1, 5, -2, -2. 3. $y^2+12y+9$, 5 or -23, 1 or -13.

4. 3, 1·5, 0, $\frac{1}{4}$. 5. 213, 7·08. 6. 1·40.

F. 5. (p. 475.)

1. $x(l-n)+n$ ft. 2. 2, $\frac{1}{512}$, 4, $\frac{1}{256}$, 1. 3. 12, 5 in.

4. $2\frac{1}{4}$, 1·31 or 0·19. 5. 12·4, 4·20. 6. $r=\sqrt[3]{\frac{3V}{4\pi}}$, 1·56 in.

F. 6. (p. 476.)

1. $\frac{1}{4}n^2(n+1)^2$. 2. $\mu=1+\frac{Rr}{f(R-r)}$. 3. 2 or $-\frac{4}{3}$, 1·90 or $-1·23$.

4. $2x^4$, $3x^{\frac{1}{3}}$, $12x^6$, $6x^2$, $8x^6$, $8x^{\frac{1}{2}}$. 5. 50,000. 6. 1200.

F. 7. (p. 476.)

1. $(10x+5)^2\equiv 100x(x+1)+25$. 2. 17*s.* 10*d.*

3. After 1 or 3 sec. 4. $\frac{1}{2}a^5$, $\frac{4}{a^3}$, $\sqrt{a^5}$, $\frac{2}{3}a^6$.

5. 0·205, 0·966. 6. 7·52.

F. 8. (p. 477.)

1. 0·1, 0·178, 0·1995. 2. 5, $3\frac{1}{2}$, $1\frac{1}{2}$.

3. $-\frac{1}{2}$ or $\frac{5}{4}$, 1·59 or $-0·420$. 4. $\sqrt{(10x-x^2)}$, a circle.

5. 23400. 6. $5\frac{1}{4}$, $7\frac{1}{4}$ in.

F. 9. (p. 478.)

1. $-\frac{1}{(a+1)(a+2)}$, each $=-\frac{1}{6}$, $\frac{x+2}{x+1}$.

2. $\sqrt{a}\times\sqrt{b}=\sqrt{(ab)}$, 8, 12, a^2b^2. 3. $\frac{a^2\sqrt{3}}{4}$.

4. 12, 9 in. 5. 2·12 in., $G=0·743d^{\frac{4}{5}}H^{\frac{1}{5}}$. 6. 0·229, 0·532.

F. 10. (p. 478.)

1. $(20x - 2x^2)$ sq. in., $AP = 5$ in.
2. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, 2, 3, $\frac{b}{a}$.
3. 3 or -2, -2 or 3.
4. 0·25, $\bar{1}$·75, $\bar{1}$·4, $\bar{1}$·6667.
5. 0·209.
6. 4 min. 14 sec.

F. 11. (p. 479.)

2. $\frac{3x^2 + 2xy + y^2}{(x+y)^3}$.
3. 0·36 per cent.
4. 0·319 or -1·57, $\frac{3}{4}$ or -2.
5. 20, 7·5 mm.
6. 10·6 ft.

F. 12. (p. 480.)

1. 554.
2. 16, $5\frac{1}{2}$
3. 2 or -1, $\frac{1}{2}$ or -1.
4. $\frac{x^{n-1}y}{(x+y)^m}$.
5. 2·83, 4·24, 1·41, 3·46.
6. 1·09.

F. 13. (p. 481.)

1. 52·6 in.
2. 25° with keel.
3. 189.
4. 4, $\frac{1}{4}$, -2, $-\frac{1}{2}$.
5. 4×10^{15}, 10^{18}.
6. $2\sqrt{5}$, 2, $\sqrt{3}$, 2.

F. 14. (p. 482.)

2. 2^{2n}, 2^{3n-3}, 8.
3. 2, 3, 4 in.
4. 3·065.
5. £4240.
6. 56900 tons.

F. 15. (p. 482.)

1. Logarithms 3, -2, 1·5, 0·25.
2. At the middle.
3. 0·4, 1·2.
4. $2x$ cars.
5. 3220, 0·136.
6. $100\left[\sqrt[n]{\frac{A}{P}} - 1\right]$, 5·2 per cent.

F. 16. (p. 483.)

1. 11.
2. $\frac{1-x}{1+x}$.
3. 10·2 in.
4. $k = (a-1)(1-b)$.
5. $\frac{81}{100}$, 9(8·7).
6. $v = 230$, 8·15 sec.

G. 1. (p. 484.)

1. $5\sqrt{x}$, $3\sqrt{x}$, 0·0000014, $\frac{6b}{a}$.
2. 39.
3. $t = \frac{v-u}{f}$, $s = vt - \frac{1}{2}ft^2$.
4. $4x - 2y = a$, $\frac{1}{2}(a-2y)(b-2y)y$
5. 405.
6. 7·2, 9 in.

G. 2. (p. 485.)

1. 2·25.
2. 2·2, $-3·3$.
3. £140, £110.
4. $\rho=8\times10^{27}V^{-\frac{3}{8}}$, $a=5·5$, $b=13$.
5. $\frac{1}{n+1}$, $n:1$, equal.
6. 2·5, 3·5 in.

G. 3. (p. 486.)

1. 60, 24.
2. $\sqrt{(ab^3)}\cdot\sqrt[4]{\frac{H^3}{G}}$, 0·000,000,000,014 ; $\frac{1}{49}$, $\frac{1}{3}$, 1.
3. $R=\sqrt{\left(\frac{Wr}{17·6P}\right)}$, 2·77.
4. $a-4b$, $a+7b$; 13 : 3.
5. 6750 tons.

G. 4. (p. 486.)

1. 8·485.
2. $6·91\times10^{-13}$.
4. $\frac{1+3x^2}{1-x^2}$.
5. $2\frac{1}{3}$ or $1\frac{1}{4}$, 2·41 or 1·18.
6. 1·8 sec.

G. 5. (p. 487.)

1. $\frac{1}{2}(pr+qs)$.
2. a, a^4, $a^{\frac{3}{2}}$, $a^{\frac{4}{3}}b^{\frac{7}{6}}$.
3. 84·75.
4. 2.
6. 3·56 in.

G. 6. (p. 488.)

1. £$5\frac{1}{2}$, £6 ; £$11\frac{1}{2}$.
2. $5\frac{10}{11}$ ft.
4. 2×3^n, $a^{\frac{3}{2}}$.
5. 171.
6. 245 amp.

G. 7. (p. 489.)

1. $\frac{a-x}{a}=\frac{x}{p}$.
2. $\frac{1}{2}x(p-x)$, $\frac{1}{2}x(a-x)$, x^2, $\frac{1}{2}pa$.
3. $v=\left(\frac{c}{p}\right)^{\frac{1}{7}}$, $v=\left(\frac{c}{p}\right)^{\frac{1}{r}}$.
4. $\frac{a^2+b^2}{a^2-2b^2}$.
5. $3y$, $10x$, $7xy$, $12xy$; $b+d$, $b-d$, $\sqrt{(bd)}$.
6. 5·9 in.

G. 8. (p. 490.)

1. $n>17\frac{1}{2}$.
2. 9 times.
3. 59·7.
4. 0·6645, 3, 40·98, $\frac{1}{49}=0·0204$.
5. 4 or $-\frac{84}{41}$, 2·4 or $-\frac{743}{203}$.
6. 6·912.

G. 9. (p. 490.)

1. 40, 50 m.p.h.
2. 6, $\frac{4}{25}$.
3. -4, $-\frac{1}{2}$.
4. 37·3 c.c.
5. 4500 days.
6. 8640 lb.

G. 10. (p. 491.)

1. $x=0{\cdot}8, 0{\cdot}6, 0{\cdot}12$; $y=12, 24, 120$. 2. 5·12 : 1. **3.** 1·6, 4·4.

4. 0, ±1, all, none. **5.** $\frac{a+bk}{1+k}$. **6.** 9·65 in.

H. 1. (p. 492.)

1. 5, 6; (*a*) and (*b*) large and positive; $6 > x > 5$; > -1 (actually $\geq -\frac{1}{4}$).

2. 1007, −995; 6·0009, 5·9991. **3.** ±1.

4. 0·542. **5.** 120 gr. **6.** £3000.

H. 2. (p. 492.)

1. −2, 3, 6, 7, 6, 3, −2; 7; 2; $4{\cdot}65 > x > -0{\cdot}65$; 3·73 or 0·27; 1 or 3; 4·83 or −0·83.

2. $\frac{a^2}{2h}+\frac{h}{2}$, 130 ft. **3.** 6, 0·6, −0·5. **4.** 1·7, 0·29.

5. $-\frac{17}{3}, \frac{27}{4}$. **6.** 2·5 dynes.

H. 3. (p. 493.)

1. 0, no, large. **2.** 321 cu. in. **4.** £1500, £1200.

5. 2·36, 0·17, −2·53. **6.** 0·018, 0·022, 2·2.

H. 4. (p. 494.)

1. $\frac{x}{x+1} > \frac{x-1}{x}$ if $x > 0$, 1000. 2. 1, 1, $\rightarrow +\infty$, $\rightarrow -\infty$, $x=0$.

3. 33, 31. **4.** −1, no value. **5.** 35800. **6.** 10 mi., 1 mi.

H. 5. (p. 495.)

2. 2·154. **3.** 2, −6, $4a^2+14a-6$, x^2+9x+2.

4. $h=\frac{12}{x^2}$, $S=x^2+4xh$, 2·885 in. 5. $9x^2+9xh+3h^2$, 9·09

6. 0·003, 0·3 per cent.

H. 6. (p. 496.)

1. −9, 99, −99·99; no; 100·01, 1000·0001.

2. 3, $-3 > x > -3{\cdot}006$, $-3 < x < -2{\cdot}995$, 1, 1.

3. $\frac{1}{25}, \frac{9}{100}, \frac{3}{25}$. **5.** 6, 2·5 in.

6. 2, 3, $\sqrt{5}=4{\cdot}47$, $6\sqrt{5}=13{\cdot}4$.

H. 7. (p. 497.)

1. 37·5, 20·25, 45, 30 ft. per sec.
4. $x=\sqrt[3]{175}\,.\,y^{\frac{2}{15}}$, 3·23.
5. 9, 6 in.
6. $\frac{x^3}{y^3}>$ or $<\frac{x}{y}$ as $x>$ or $<y$.

H. 8. (p. 497.)

1. 2, 3·9, $4-h$, 4.
2. 7, 7; $2x+h$ if $h\neq 0$; $2x$.
3. £$(15+4n)$.
5. 20 in.
6. $a=\frac{1+x}{1-x}=\frac{1+y}{2(1-y)}$, $xy-3x+3y=1$.

H. 9. (p. 498.)

1. -4; -1; $1>x>-3$; 1·83 or $-3{\cdot}83$; 1·24 or $-3{\cdot}24$; 0·73 or $-2{\cdot}73$.
2. -4; -3; $-h-2$ if $h\neq 0$; -2.
4. 0·932 or $-1{\cdot}60$.
5. 3, $7+2\sqrt{10}=13{\cdot}3$, $\frac{1}{3}(\sqrt{5}-\sqrt{2})=0{\cdot}274$.

H. 10. (p. 499.)

1. 1·15; 0·308; $2>x>0$ and $x<-2$; 2·214, $-0{\cdot}539$, $-1{\cdot}675$; (*a*) and (*b*) 2·115, $-0{\cdot}254$, $-1{\cdot}861$; (*c*) 1·861, 0·254, $-2{\cdot}115$.
2. 0·8.
4. 1, $\sqrt{2}$, 2^x.
6. 7, 7.

K. 1. (p. 500.)

1. $\frac{17}{500}$, $\frac{1}{25}$, about 0·03.
2. $2x+4$, $-\frac{1}{(x-1)^2}$.
3. -2, $3a^2-5$.
5. $t=x-1$, $y=x^2-x$, $t=\pm 1$.
6. $y^2=2ax+a^2$, $y=\sqrt{(2ax+a^2)}$.

K. 2. (p. 501.)

1. -1, -1, 0.
2. $4-6x$, $4-6x$, $4-6x$, $\frac{3}{5}+\frac{4}{5}x$, $-\frac{6}{x^3}$, $\frac{4}{\sqrt{x}}$.
3. 18; $-\frac{5}{6}$, $4\frac{1}{12}$.
4. $y=3x-8$.
6. $0{\cdot}026l$ at distance $0{\cdot}578l$ from A; gradient $\frac{1}{16}$.

K. 3. (p. 501.)

1. $8x-\frac{1}{x^2}$, $48x^2-8x+2$, $\frac{5}{3}x^{-\frac{2}{3}}-\frac{1}{3}x^{-\frac{4}{3}}$.
2. $3a^2-12a-15$; 5, -1.
4. 18, 66 ft. per sec.
5. $\frac{dx}{dt}=\frac{k}{x}$.
6. $1\frac{2}{3}$.

K. 4. (p. 502.)

1. $-\frac{2x}{(x^2-2)^2}$, $\frac{1}{2}$.   2. $\sqrt{\frac{3}{2x}}, \frac{3}{\sqrt{x}}, -\frac{12}{x^4}, 2x-\frac{2}{x^3}$.

3. 0, $\pm 2\sqrt{3}$; ± 2.   4. $\frac{d\theta}{dt}=k(15-\theta)$.   6. 214 : 1.

K. 5. (p. 503.)

1. $4-\frac{4}{t^2}, 9t^2+\frac{4}{t^2}-\frac{15}{t^4}, 12{\cdot}5t^{\frac{3}{2}}$.   2. $3x^2-27$; ± 3; 54, -54.

3. Grows lighter and weight vanishes when $x=7{\cdot}5$.

4. 0·00649, 0·00736, 0·000087.   5. $2x+3+h$, 2, $2x+3$, 2.

6. 4800, 29500 yds.

K. 6. (p. 504.)

1. $x=1$ (min.), $x=-3$ (max.).   3. $4x-x^2$, $x=2$.

4. $(2p+q)^2+(p-2q)^2$, 20^2+15^2 or 24^2+7^2.   5. 14·4.   6. $s=k\,.\,t^3$

K. 7. (p. 504.)

1. $-3-6x^2$.   2. $\frac{dx}{dt}=2(x+5)$.

3. $h=\frac{1200}{\pi r^2}$, $S=\sqrt{\left(\frac{144\times 10^4}{r^2}+\pi^2r^4\right)}$, 230 sq. ft., 6·46 ft.

4. 81, 0.   5. $\frac{17}{22}, \frac{85}{18}$.   6. $19\frac{5}{9}$ lb.

K. 8. (p. 505.)

1. 2 (min.), -2 (max.).   2. $\frac{5}{6}$.

3. $9x^2-7, 18x$; $10x, 10$; 7, 0; 0, 0; $-\frac{6}{x^2}, \frac{12}{x^3}$.

4. ± 125.   5. $2\frac{1}{3}$ in.   6. $60x-\pi x^2$, 9·55 in.

K. 9. (p. 506.)

1. Not if $x=1$, no, 2.

2. 3400, 3564, 3417·84; 164, 178·4; $180-16h$; 180 ft. per sec.

3. $\delta y=4x\,.\,\delta x+2\,.\,(\delta x)^2$; $\delta x=3\,.\,\delta z$; $\delta y=12(3z+1)\,.\,\delta z+18\,.\,(\delta z)^2$; $12(3z+1)$

4. 1 in.   5. 40, 16875.   6. $\frac{b}{a}$.

K. 10. (p. 507.)

1. $ab : a^2 : bc$.
2. $x\sqrt{(25-x^2)}$, $12\frac{1}{2}$ sq. cm
3. Less than 13·75 in.
4. $\frac{1}{100}$, $\frac{7}{1000}$, $\frac{2+h}{300}$, $\frac{1}{150}$.
5. $\frac{dy}{dx}$ +, −, −, + ; $\frac{d^2y}{dx^2}$ −, −, +, +.
6. 0·87 cu. ft.

K. 11. (p. 507.)

1. 165.
3. 25 yd.
4. 0·00101, 0·00107, 0·00112, 0·00107.
5. $2x+\frac{1}{x^2}$, $\frac{2}{3}\sqrt[3]{\frac{2}{x}}$.
6. 12·3.

K. 12. (p. 508.)

1. Decreased in ratio 1 : 16, 18 c.p.
2. $\frac{1}{a}$.
3. 548.
4. 102, →∞.
6. 8 in.

K. 13. (p. 509.)

1. $x=2{\cdot}94$.
3. $b=60+0{\cdot}3r$, 72.
4. 6·955 in., $y=2\sqrt{(\pi x)}$
5. Positive if $0<x<4{\cdot}76$, increasing if $x<3$; $x=3$.
6. 8 ft., 48 sq. ft., 256 cu. ft.

K. 14. (p. 509.)

2. 0·73, 11·9, 84·0.
3. $2(1+\sqrt{3})=5{\cdot}46$, -2
4. Each $\simeq 0{\cdot}43$, $\frac{d}{dx}\log x \simeq \frac{0{\cdot}43}{x}$.
6. $\frac{1}{2}$.

K. 15. (p. 510.)

1. 79·6.
2. 60, 240, 540 ; 30, 120, 270 ; 20, 80, 180.
3. $\frac{x+y}{xy}$.
4. Each $\simeq 2{\cdot}3$, $\frac{d}{dx}(10^x) \simeq 2{\cdot}3 \times 10^x$.
5. $-\frac{5}{x^2}$, $-\frac{x^2}{5}$, 1.
6. $\frac{1}{25}$ when $y=+10$.

K. 16. (p. 511.)

1. $\frac{a^2}{2b^3}$; 16, 64, 2, 0·001.
2. $x : (y-x)$.
3. $A=\frac{10a(a-10)}{a-20}$, 17·2 sq. cm.
4. $2x+h-3$, $2x-3$, 1·5.
5. $\frac{2x}{(1-x^2)^2}$.
6. 75 ; 1·2, −1·2 ; 30 (min.), −30 (max.)

L. 1. (p. 512.)

1. $W=\frac{114bld^2}{x(l-x)}$ (in tons), where $AP=x$ feet and all measurements are in feet.

2. $27:8$. 3. (ii) -20000. (iii) No. (iv) 3·000. (v) $\frac{9}{(x-1)(x-5)}$, 3.

4. 0·26. 5. $y=\frac{1}{6}(3x^2-4x^{\frac{3}{2}}+7)$, $y=4x+1+\frac{1}{2x}$.

6. $\frac{5}{3}x^3-\frac{4}{x}+c$; $\frac{5}{11}x^{2\cdot 2}-\frac{5}{x^{0\cdot 2}}+c$.

L. 2. (p. 512.)

1. $d+h-\sqrt{(2dh)}$. 2. $y:x$, $(1+y):1$, $1:y$.

3. $\frac{3}{4}$, $\frac{1}{4}$. 4. $ky=hs$, $\frac{s}{\sqrt{(s^2-10000)}}$, 2 in.

5. 24. 6. $135:64$.

L. 3. (p. 513.)

1. 1·5, -1, 10^x. 2. $\sqrt{b}:\sqrt{a}$. 3. ± 3, ± 4; 5, -5; $4\sqrt{2}$, $-4\sqrt{2}$

4. $\frac{x}{2}\sqrt{(400-x^2)}$, $10\sqrt{2}$, 100. 5. $\frac{3}{4}h$. 6. $\frac{dx}{dt}=-0\cdot 4x$, $0\cdot 16x$.

L. 4. (p. 514.)

1. $x>15$, $0<x<3$. 2. 51 lb., $R=\frac{d^2v^3}{10^6\times 288}$.

3. 4, -4; $x=3$; $10\frac{2}{3}$. 4. $6t-t^2$ lb.-sec.

5. $\frac{1024\pi}{125}=25\ 7$. 6. Yes; 6160, 18480, 21560 yd

L. 5. (p. 515.)

1. $(x-1)^3-3(x-1)+2$. 2. $y=\frac{4x^2}{x+12}$, 9·21. 3. 6·17.

4. $0<x<3$, -27. 5. $\frac{1}{2}h$. 6. A square.

L. 6. (p. 516.)

1. 15·6, 62·2 cu. ft. per hr. 2. No change. 3. $z=t^{\frac{1}{4}}$.

4. $\frac{2r}{3}$, $\frac{rh}{2(h-r)}$ in. 5. $\frac{47\pi}{16}=9\cdot 23$. 6. 1·5, $\frac{1}{4}$.

L. 7. (p. 516.)

1. 1, 2; 2, 2; 3, 3. 2. Equal.

3. Indices are 0·0791, 2·3980, $\bar{2}\cdot 1761$, 0·8060. 4. $\pm\frac{2\sqrt{3}}{9}=\pm 0\cdot 385$.

5. 170 ft.-lb. 6. 5·75, 5·79 mi.

L. 8. (p. 517.)

1. $T=\dfrac{x^2-160x}{10x+5400}$; 0, 3, 9·335, 9·93 per cent.

2. 441 mm. 3. $1{\cdot}408p$. 4. $\frac{155}{103}=1{\cdot}50$.

5. $\frac{1}{3}\sqrt{21}=1{\cdot}53$. 6. $y=\frac{1}{6}\left(\dfrac{1}{x^2}+8x+3\right)$.

L. 9. (p. 518.)

1. $x>4$, yes. 2. $\dfrac{p^2-q^2}{2p}$. 3. (Decreased) 0·97 : 1.

4. $\dfrac{3a}{8}$. 5. $-\frac{1}{4}$, 3. 6. $\dfrac{1}{2}, \dfrac{7\pi}{24}=0{\cdot}916$.

L. 10. (p. 519.)

1. $\dfrac{b-c}{2}+\dfrac{a^2}{8(b-c)}$, $60\frac{5}{8}$ ft. 2. 0, 0, 0; $\log x+\log y=\log(xy)$.

3. $z=\dfrac{60x}{y}$. 4. $\dfrac{2(a^2+3ah+3h^2)}{3(a+2h)}$.

5. 0·232 in. per sec. 6. 0·8, $y=\frac{1}{20}(16x+10x^3-x^5)$.

M. 1. (p. 520.)

1. 1, 46; 6, 132; 0, 99. 2. 11, 20, 26, 29; 44; $3n-1$.

3. 26, 1326. 4. 2·34 in. 5. $\frac{1}{2}$. 6. £$\dfrac{9A}{4}$.

M. 2. (p. 520.)

1. 1050. 2. 15 m.p.h., $10\frac{1}{2}$ sec. 3. $\frac{1}{50}, \frac{8}{15}$.

4. 19, $\frac{181}{19}$. 5. 21, 3 sq. in.

M. 3. (p. 521.)

1. 49 lb. 2. $3t^2+\dfrac{1}{t^3}$ 3. -2, $5n-2$, $\dfrac{(n+1)(5n-4)}{2}$.

4. 2·90, $-0{\cdot}60$, $-2{\cdot}30$. 5. 7, 23, 144. 6. $2r-1$, n^2.

M. 4. (p. 522.)

1. 2·50. 2. $c=0{\cdot}16a^2d^{1\cdot5}$. 3. 1·23 or $-4{\cdot}23$.

4. 2^n+n. 5. $2\times3^{n-1}$, 3^n-1. 6. £5940, £5940.

M. 5. (p. 523.)

1. $F=\dfrac{W}{m}+\dfrac{Wr}{2240}+\dfrac{2Ws}{gt^2}$. 2. -6, $-\frac{1}{24}$. 3. 3·59.

4. 4, 3, 2·62, 0·631. 6. $(9+\sqrt{x_1})^2$, $(n-1+\sqrt{x_1})^2$.

M. 6. (p. 523.)

1. $V=\left(\frac{2gH}{ml}\right)^{\frac{1}{n}}\times d^{\frac{2}{n}}$.
2. 16·0.
3. 30.
4. 2, 8, 14.
5. 3^{4-n}, $10^{-8}\times 2{\cdot}3$.
6. 1050, 802 yd.

M. 7. (p. 524.)

1. £600.
2. 0·00632.
3. 6, 56.
4. 3×2^n, $n+2+2^n$.
5. 2·47.
6. 24 in.

M. 8. (p. 524.)

1. 2·24 p.m.
2. $\frac{x^3+x^2+4x}{x^2-1}$.
3. 5·38.
4. $\frac{7}{8}$.
5. $6\frac{1}{2}$, 2.
6. Red 348, blue 372 sq. in.

M. 9. (p. 525.)

1. 1973 ft. per sec.
2. $\frac{3a+5b}{8}$.
3. $d=\frac{1}{8}G^{0{\cdot}4}\,.\,L^{0{\cdot}2}\,.\,H^{-0{\cdot}2}$.
4. 157.
5. 24 yd.
6. Rate of flow about $\frac{1}{2}$ cu. in. per sec., 7998 cu. ft.

M. 10. (p. 526.)

1. $(k-1)x^2+12x+9(k-5)=0$.
2. $d=0{\cdot}059L^{1{\cdot}5}$, 75·6.
3. 7, 4.
4. 108, 198, 1683.
5. $\frac{4}{5}[1-(-\frac{1}{4})^n]$, 0·8.
6. 13·6 yrs.

N. 1. (p. 526.)

1. $y=\frac{24}{15}(x-29)$, 104.
2. ${}^nP_r=\frac{\underline{|n}}{\underline{|n-r}}$, 380.
3. $a+b+c+d$, $b^3(d-a-c)$.
4. 1·435.
5. Readings for $w=35$, 50; $l=21+\frac{w}{3}$.
6. $2n(n-1)$, 36.

N. 2. (p. 527.)

1. $R=\frac{Av^2}{800}$, $v\propto\frac{1}{\sqrt{A}}$.
2. $\frac{375\pi}{16}=73{\cdot}6$.
3. 30.
4. x^3, 45.
5. 0, ± 10; 10·005.
6. 5 per cent.

N. 3. (p. 528.)

2. $18A^3 = 24389V^2$. 3. $3\frac{3}{8}$ pints. 4. 15.

5. 12, $\frac{3252h}{l\sqrt{l}}$, down. 6. 40, 8.

N. 4. (p. 529.)

1. $a = 1\cdot115$, $n = 0\cdot5$ 2. $3\frac{3}{4}$, $\frac{xy}{z-y}$. 3. 28.

4. 1·049. 5. $16y^4 - 80y^2 + 64$; 1, ± 3, -5. 6. $\frac{1}{88}$

N. 5. (p. 529.)

2. 13·6 ft.-lb. 3. 0, 0, 0 or 4, 6, 9. 4. 12. 5. 105.

N. 6. (p. 530.)

1. $\log_a b$, $\frac{1}{8}$. 2. $\sqrt{(abcd)}$, $\frac{1}{4}a^2\sqrt{3}$. 3. 1260.

4. Increased 7 per cent. 5. 5·0067, 5·027. 6. £16. 13*s*. 4*d*., $\frac{2}{27}$.

N. 7. (p. 531.)

1. $V = 168d^2\sqrt{H}$, reduced in ratio 1 : 4. 2. 160,000.

3. $\frac{h^2}{2x}$. 4. $\left(1-\frac{1}{N}\right)^2$, $\left(1-\frac{1}{N}\right)^p$, 7.

5. $\frac{1}{6}$. 6. $\frac{4}{3}\pi a^3$, $\frac{4}{3}\pi a^3$.

N. 8. (p. 532.)

1. Increased in ratio 8 : 1.

2. The lines $y = \pm 1$; everywhere discontinuous. 3. 64, 15.

4. $1 - y + y^2$; each is $1 + \frac{x}{n} - \frac{(n-1)x^2}{2n^2}$. 5. 462, $\frac{6}{11}$.

N. 9. (p. 533.)

1. $P = 1\cdot94 \times 10^{-11} \times d^{5\cdot5} \,.\, n^{3\cdot5}$. 2. 7019298. 3. $a\sqrt{\frac{2}{3n}}$.

4. $\frac{1}{4}$. 5. 3, 315, $\frac{\lfloor 16}{2^{15}} = 638512875$. 6. 10, 1, $\frac{n(n-1)}{2}$, $\frac{1}{2}(n-3)(n-4)$.

N. 10. (p. 533.)

1. $M \propto \frac{lr^4}{t^3}$. 2. 20, 10, 13. 3. 28.

4. $c^3 + c^2(a+b) + c(a^2 + ab + b^2) + a^3 + a^2b + ab^2 + b^3$, 10. 6. $\frac{1}{4}$.

P. 1. (p. 534.)

1. $\frac{100h}{a}$, 0·05 per cent.
2. $\frac{a}{100-x} > \frac{b}{100+y}$, $\frac{b}{100-y} > \frac{a}{100+x}$.
3. $P = 0{\cdot}183W + 0{\cdot}32$.
5. $2x-1$, $-\frac{5}{x^2}$, $\frac{x^3}{3}+\frac{1}{x}+c$, $2\frac{1}{3}$.
6. $\pi x^2\left(a-\frac{x}{3}\right)$, $\frac{4}{21\pi} = 0{\cdot}0606$ in. per sec.

P. 2. (p. 535.)

1. $x = 10^7 \times 2{\cdot}51t^{3\cdot2}$.
2. $W = 40 + \frac{1}{5}I$, $P = \frac{40}{I} + \frac{1}{5}$.
3. 2·80.
4. $10^5 \times 7{\cdot}03$, $10^5 \times 7{\cdot}09$.
5. $\frac{4}{15}$.
6. $333\frac{1}{3}$ ft.

P. 3. (p. 536.)

1. $2 \log x$, 3, 1, $\frac{3}{2}$.
2. $ac : ab : bc$; $a^2 - ab$, $ab - b^2$.
3. $d = 0{\cdot}11V^2$.
4. 459 : 53.
5. 4·52 in.
6. 1·12 in. per sec., 4·47 in., $2\int_{-5}^{5}\sqrt{\frac{20}{25-x^2}}\,dx$.

P. 4. (p. 537.)

1. 2·50.
2. $v = 8\sqrt{h}$.
3. $\pi(2a^2 + 2al + h^2)$.
4. 0; +, +; no; −, +.
5. $\frac{1}{30}$.
6. 65·1 ft.-lb.

P. 5. (p. 537.)

1. $(2x+100)$ per cent., $(100+x) : 100$.
2. $m = 1{\cdot}276d^2$.
3. 64·4° F., 36·7° C.
4. 12300.
5. $\delta z \simeq -\frac{\delta y}{2y^{\frac{3}{2}}}$, $\delta y \simeq -2x\,\delta x$, $\frac{x}{(49-x^2)^{\frac{3}{2}}}$.
6. $4\frac{1}{2}$.

P. 6. (p. 538.)

1. 0·8155.
2. $h = \frac{P}{0{\cdot}0092P + 0{\cdot}29} = \frac{3{\cdot}4P}{1 + 0{\cdot}031P}$.
4. 3·73.
6. Area of quadrant of circle, 0·787, 0·783.

P. 7. (p. 539.)

1. $\frac{x}{r^2}$, $AP = 6{\cdot}21$ ft.
4. $t = 0{\cdot}28d^{1\cdot16}$, 232 sec.
5. $\frac{3}{2}$, $-\frac{3}{4}$; $\frac{3}{4}$, $-\frac{5}{16}$.
6. $\frac{x}{12}(2x^2 - 9x + 12)$.

P. 8. (p. 540.)

1. $\frac{b^2}{4\pi^2}(\pi a-b)$, 4·58 in. 2. $x^3-x^2-4=0$, one. 3. $d=11{\cdot}4\sqrt{H}$.
4. 93·7. 5. (2, 4); (−2, −4). 6. 25 lb., 16·8 in.

Q. 1. (p. 541.)

3. $(4a+3b)^2-(a+7b)^2$. 4. 7.
5. $2a+a^2$, $4a+2a^2+a^3$, $\frac{a(a^n-2^n)}{a-2}$. 6. 6, 12, 5.

Q. 2. (p. 541.)

1. $a+b=q\neq 0$ and $p\neq 0$; $q=0$ or if $a+b=q$ and $p=0$. 4. 3, 5.
5. 10·067, 10·07. 6. £12·3.

Q. 3. (p. 542.)

1. $-1\frac{1}{3}$. 2. xy, $\frac{x^3+y^3}{x^3-y^3}$. 3. $\frac{2}{3}$, 2, $\frac{1}{3}$. 4. 3. 5. $-\frac{3}{2}$.

Q. 4. (p. 542.)

1. $n=\log\left(\frac{p_1}{p_2}\right)\div\log\left(\frac{p_1T_2}{p_2T_1}\right)$. 2. $-b$ or $a-1$, $-a$ or $b-1$.
3. a, $\frac{a-1}{a}$. 5. $24a$, $15a$, $20a$.

Q. 5. (p. 543.)

3. $-(l+m+n)^2$. 4. $r\left(\frac{\sqrt{3}}{2}\right)^n$.
6. $(3x-y)(3y-x)(x+y)$; $1+A(x-1)(x-2)(x-3)(x-4)$.

Q. 6. (p. 544.)

4. $f^2=bc$, $g^2=ca$, $h^2=ab$.
5. $3(y-z)(z-x)(x-y)$; $(a-b)(b-c)(c-a)(a+b+c)$.

Q. 7. (p. 544.)

4. 1 or −9; 1, $1\frac{1}{2}$, 2. 5. $2\frac{1}{2}\geq m\geq 1$.
6. $(x-\alpha)(x-\beta)(x-\gamma)=0$, q, $-2p$, $-3q$.

Q. 8. (p. 545.)

2. $\frac{2z^3}{1+z+z^2}$. 4. −2, 19, 19, 4.
5. $x=\beta(x_1+vt_1)$, $t=\beta\left(t_1+\frac{vx_1}{c^2}\right)$. 6. $-a$, $-b$; 2 or $\frac{1}{2}$.

Q. 9. (p. 545.)

1. $(m^2+mn+n^2)^2$. 3. $2y^2+y(4c-1)+2c^2-c-5=0$, $\frac{1}{4}$.

4. $(a+b+c)(bc+ca+ab)$, $24abc$. 5. ± 3 or $\pm\frac{1}{3}$, ± 3 or $\pm\frac{1}{3}$, ∓ 1 or $\pm 4\frac{1}{3}$.

6. (i) $\dfrac{1}{(2-x)^2}+\dfrac{2}{2-x}-\dfrac{2}{3-x}$; (ii) $4x-2$.

Q. 10. (p. 546.)

3. $t^2(b^2-ac)-t(a^2+bc-a)+c^2+ab-b=0$.

5. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$. 6. $(x+y-z)(y+z-x)(z+x-y)$.